Violent fluctuations of the fracton density of states on the percolation cluster and its backbone

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We report the fracton density of states (DOS) and its fluctuation properties on the infinite two-dimensional critical percolation cluster and its backbone. The fracton DOS fluctuations, as expressed by the number variance $\langle[\delta N(E)]^2\rangle$ in an energy width E, follow the quadratic law $\langle[\delta N(E)]^2\rangle \propto \langle N(E)\rangle^2$, instead of the usual linear Poissonic behavior normally expected for localized states. We also find that the average DOS for the percolation backbone obeys the one-parameter fracton scaling theory with a spectral dimension $d_s^{\rm BB} = 1.23 \pm 0.02$. This kind of violent DOS fluctuation cannot be understood in the context of random matrix theories and is discussed in connection to intermittency and multifractal localization.

Percolation represents the simplest example of a geometrical phase transition with numerous applications in diverse fields, ranging from the spread of a disease to transport in disordered media.[1–4] A great advance in percolation theory was achieved via the recognition that the percolation cluster is a random fractal object[1–6] having statistical self-similarity properties up to the percolation correlation length ξ_p. This is made explicit via the law $M \propto L^{d_f}$, which describes how the average total mass M within a linear distance L scales with L, defining the fractal dimension d_f. Attention has also been focused on the percolation backbone which is obtained if we remove from the percolation cluster the non-current-carrying dangling ends. Then the averaged mass drastically reduces when compared to the percolation cluster but still scales as $\propto L^{d_f^{\rm BB}}$, defining the smaller fractal dimension of the backbone, $d_f^{\rm BB}$. In two dimensions $d_f = 91/48$ and $d_f^{\rm BB} = 1.62 \pm 0.05$.[1]

During the last decade most of the attention has been focused on important questions concerning the elastic dynamics of the percolation cluster in view of real fractal systems, such as polymers, rubers, and gels. Within what is known as the dynamical scaling approach the presence of anomalously slow diffusion[1] at $p = p_c$ and the concept of fractons[7] have been introduced. In the case of spin-wave spectra at $p = p_c$ one obtains, for small E, the power law[7,8]

$$\langle N(E)\rangle \propto E^{d_s/2}, \tag{1}$$

where $\langle N(E)\rangle$ is the averaged cumulative density of states (IDOS) counting the fracton states from zero energy up to E. The exponent d_s is the fracton dimension,[7] which in the case of the percolation backbone is replaced by $d_s^{\rm BB}$, and for ordered systems both reach the space dimension value d so that the usual magnon DOS is recovered. Moreover, since the percolation cluster behaves as an ordinary continuous medium for length scales $L \gg \xi_p$, being fractal only for $L < \xi_p$, a crossover occurs in the corresponding DOS from an ordinary magnon or phonon very-low-E regime to the rest of the low E's which is known as the fracton regime described via Eq. (1).[8–14] The long-wavelength dynamics for fractons is summarized by the dispersion law

$$E \propto k^{d_w} \qquad \text{for } k\xi_p \to 0, \tag{2}$$

which is different from the quadratic $E \propto k^2$ magnon dispersion. The spectral dimensions are subsequently defined as $d_s = 2d_f/d_w$ and $d_s^{\rm BB} = 2d_f^{\rm BB}/d_w^{\rm BB}$, respectively. Moreover, Alexander and Orbach (AO),[7] using an assumption of a single characteristic length ξ_p, proposed that for percolation a mean field value $d_s = 4/3$ should approximately hold in any dimension greater than 2. Moreover it is expected that the transport properties of the excitations on fractal clusters will be affected by the topological "disorder," in a way similar to what disorder does to electron in metals. In fact, the fracton states turn out to be superlocalized[15] with amplitudes decaying faster than exponential, at least for typical samples before averaging. This phenomenon implies the presence of a variety of characteristic lengths and has also been described as multifractal localization.[16]

Our aim is to study the localization properties of the fractons by considering the DOS fluctuations in percolating clusters and backbones in two dimensions within the appropriate statistical random matrix ensembles. The corresponding matrices are short ranged and sparse, i.e., with most of the matrix elements being identically zero, with the "disorder" due to the random positions of the

nonzero matrix elements while in the usual Gaussian random matrix ensembles,[17–21] instead, all the matrix elements are random variables. We consider the question of fracton DOS and the corresponding DOS fluctuations for the percolating cluster and the percolation backbone at p_c by using the Gaussian elimination eigenvalue counting algorithm.[22] We compute $\langle N(E)\rangle$ and $\langle[\delta N(E)]^2\rangle = \langle N(E)^2\rangle - \langle N(E)\rangle^2$ numerically for our matrix ensembles and display the unambigous presence of violent fracton spectral fluctuations by establishing the law

$$\langle[\delta N(E)]^2\rangle \propto \langle N(E)\rangle^2, \tag{3}$$

which is intimately related to intermittency and the unusual fracton localization. For a better presentation of our results we focus on the following questions: (i) How do the results for the spectral dimension change, the AO conjecture in particular, when one considers instead of the critical percolation cluster the percolation backbone. (ii) What is the magnitude of the spectral density fluctuations and are these results compatible from what is known from random matrix theories? (iii) What are the localization properties of fractons in the absence of the dangling ends?

The percolation backbone is generated following techniques reported in the literature.[23–27] First, the percolation cluster is constructed at the critical percolation threshold $p_c = 0.593$ for the two-dimensional $L \times L$ square lattice. Then the largest cluster is isolated using the labeling technique of Refs. 28, 29 and eventually, the backbone is isolated from the percolation cluster via the so-called "burning algorithm."[26] The choice of two end points on the percolation cluster, lying on diagonally opposite corners, is the essential feature of the algorithm which proceeds in three steps: (i) Starting at the first point the cluster burns by examining all its neighbor sites and assigning specific index values increasing by +1, which denote the distance of the new site from the point of origin. This step ends once the opposite point is found and assigned its index value. (ii) In the second step burning starts at the second end point, and only these sites can be burned which have a smaller index value than the value of the site burned in the previous time step. Similarly, this burning ends once the point the first point is reached. (iii) In the final step one burns all that have been formed in the previous two steps. Again, as in the second step, a site can only be burned if its index value is smaller than the value of the site burned in the previous time step. For a given loop, it becomes part of the backbone only if the backbone can be reached in more than one way. If it is reached only via one path, this means that the loop leads to a dangling end, and it is not included. When reached in more than one way, then all the sites burned are added to the growing backbone. This last step must be repeated several times until no more parts can be added to the backbone. Once all three steps are finished the backbone is isolated. Moreover, cyclic boundary conditions are used at the ends of the lattice, so that if two sites which belong to the backbone are at opposite ends, they can communicate.

In order to check our construction of the backbone we have calculated its fractal dimension which we find to be $d_f^{BB} = 1.62 \pm 0.05$. Calculations for the end-to-end distance and the number of sites visited for random walkers evolving in the backbone can be found in Ref. 30. The random walk dimension was $d_w^{BB} = 2.70 \pm 0.02$, about 6% smaller than $d_w = 2.87 \pm 0.02$, which is obtained for the percolation cluster.[1] The spectral dimension calculated from counting the number of sites visited for random walks on the percolation cluster is found to be $d_s = 1.31 \pm 0.02$,[1] and for the backbone $d_s^{BB} = 1.23 \pm 0.02$.[30] The reduction of the spectral dimension on the backbone has been assigned there due to the fact that in the backbone there are no new sites to visit on isolated branches and the particle spends more time revisiting the same sites again and again.

The question of dynamics on random fractal structures can be dealt with in a straightforward manner via the solution of a tight-binding equation on the lattice defined by the percolation cluster or the percolation backbone. The equations of motion corresponding to a single spin deviation are

$$\left(E - \sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\right)\psi_{\mathbf{r}} = \sum_{\mathbf{r}'} J_{\mathbf{r},\mathbf{r}'}\psi_{\mathbf{r}'}, \tag{4}$$

where $\mathbf{r}$ labels the percolating cluster sites on the lattice and when E is an eigenvalue $\psi_{\mathbf{r}}$ is the corresponding wave function amplitude on site $\mathbf{r}$. J is the exchange constant, conveniently chosen to be 1, and the summations are performed over all available sites $\mathbf{r}'$, nearest neighbors of $\mathbf{r}$, on the incipient percolating cluster or backbone. From Eq. (3) the statistical matrix ensemble is created which consists of random matrices defined in the orthogonalized site basis representation. The random matrices are real, symmetric, and sparse with five, at most, nonzero elements per matrix row. The off-diagonal matrix elements (bond strengths) are 1 or 0, when nearest-neighbor sites are present or absent, respectively, while the diagonal elements (site energies) are equal to the number of nearest-neighbor sites of $\mathbf{r}$ present, ranging from 1 to 4 in two dimensions. The resulting gapless spectrum consists of strictly positive energies E.

The calculation of the averaged IDOS at p_c proceeds as follows: We collect all the eigenvalues at energy windows for many different, randomly generated clusters, so that the average $\langle N(E)\rangle$ can be determined. The sources of error in this type of calculation are twofold: First, due to the finite number of samples making up our statistical ensemble, a statistical error exists. This can be estimated from the scatter of the number of eigenvalues in a given energy window and is used below to define the transport properties. Another kind of error is due to the finite size of the chosen sample which results in a limited number of sites in a given energy window and may cause pronounced discreteness in the spectrum when the matrix size is not large enough. This kind of error also limits us to extend the calculation down to very small energies and it has been monitored in our calculations by choosing large enough samples.

We report our results for the IDOS for the $d = 2$ percolation backbone in Fig. 1. In the double logarithmic plot

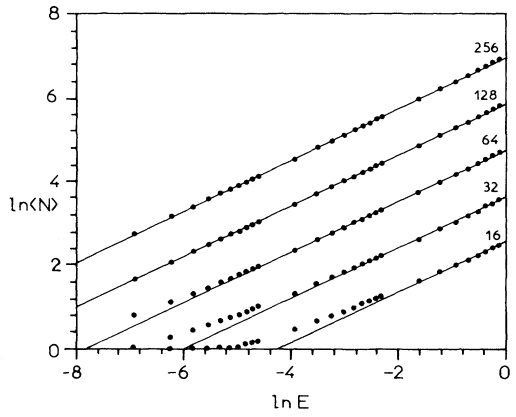

FIG. 1. A log-log plot of the numerically computed averaged cumulative DOS $N(E)$ versus the energy E on the two-dimensional percolation bacbone at $p = p_c$ generated on five different sizes $L = 16$, 32, 64, 128, and 256 and for 2000, 1000, 500, 100, and 10 random runs, respectively. From the slopes of the solid lines $d_s/2$ is extracted which gives the spectral or fracton exponent as $d_s^{\mathrm{BB}} = 1.23 \pm 0.02$.

the fracton law of Eq. (1) implies that the data should be in straight lines with gradient $d_s/2$. Despite the sources of error the data of Fig. 1 lie rather accurately on straight lines. A least-squares fit gave gradients from which the exponent d_s^{BB} is estimated as 1.23±0.02. A similar calculation for the percolation cluster gives the well-known values $d_s = 1.31 \pm 0.02$. These results confirm the dynamical scaling approach of AO,[4] as summarized by Eq. (1), also for the percolation backbone but with a significantly lower critical exponent. The small deviations from the straight lines can be understood as arising from the numerical difficulties due to the chosen finite sizes.

In Fig. 2 we display the number variance $\langle[\delta N(E)]^2\rangle$ as a function of the energy E and in Fig. 3 we establish the quadratic fluctuation law of Eq. (3) both for the critical percolation cluster and the backbone. In

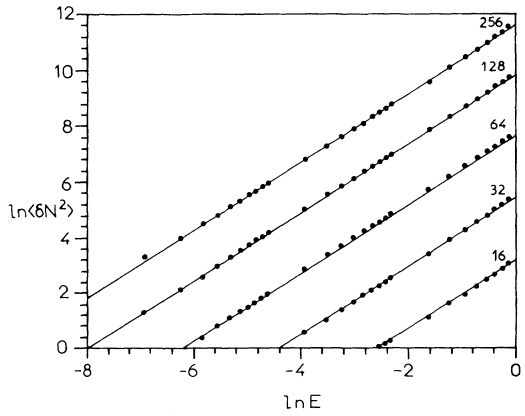

FIG. 2. The same as Fig. 1 but for the number variance $\langle[\delta N(E)]^2\rangle$ as a function of the energy E. The solid lines are the best fits to the data and their slopes are found to be equal to d_s, establishing the law: $\langle[\delta N(E)]^2\rangle \propto \langle N(E)\rangle^2$.

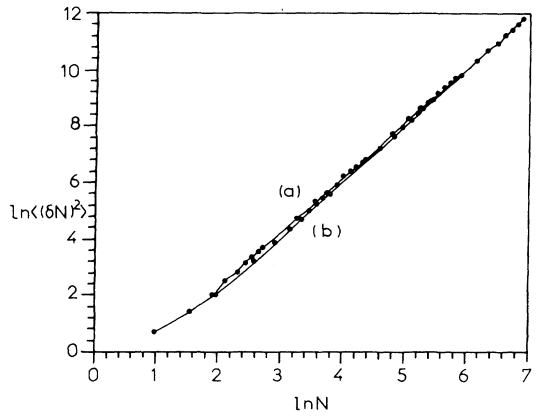

FIG. 3. The novel relation $\langle[\delta N(E)]^2\rangle \propto \langle(N(E)\rangle^2$ is shown to be valid both (a) for the percolation cluster and (b) for the percolation backbone. The small difference in the slopes lies in the slightly reduced value of d_s for the percolation backbone.

Fig. 4 we display the full probability distribution of the $N(E)$ values for different random configurations. The broadness of this distribution is in accordance with the fact that the fluctuations are very large. This kind of DOS fluctuation is not expected from the random matrix theories because for the Gaussian ensembles the number variance is weakly dependent (logarithmically) on $\langle N(E)\rangle$, which implies strongly correlated eigenvalues repelling their closest neighbors.[17] Moreover, the logarithmic $\langle[\delta N(E)]^2\rangle \propto \ln\langle N(E)\rangle$ behavior and the level repulsion can be associated with delocalized states as observed in small metallic samples.[17–20] Our results cannot be understood by ordinary Poissonian statistics[17–20] either, since in this case the eigenstates are localized and the eigenvalues are randomly distributed according to the usual statistical law $\langle[\delta N(E)]^2\rangle \propto \langle N(E)\rangle$. The change of the number variance from a logarithmic to linear dependence has been exploited[21] for the identification of

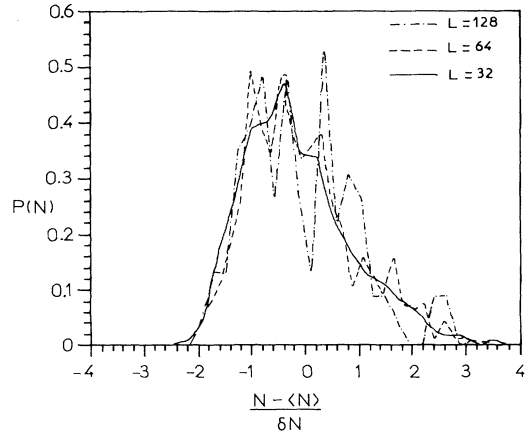

FIG. 4. The distribution of the $N(E)$ values for a specific energy form zero to $E = 0.10$ and three different sizes (as shown in the figure). The x axis is the rescaled quantity $(N - \langle N\rangle)/\delta N$ so that different sizes should coincide.

the Anderson transition and the determination of mobility edges separating delocalized from localized states in the spectrum. At the mobility edge an intermediate $\langle[\delta N(E)]^2\rangle \propto \langle N\rangle^{2/d}$ law was recently obtained[32] in $d = 2 + \epsilon$ dimensions.

The fracton fluctuations found [Eq. (3)] are much stronger than all the above kinds and it seems natural to relate them to the superlocalized fracton decay,[15,16] which is due to the long-range fractal potential correlations. Superlocalization concerns only individual wave function amplitude realizations and in order to see it one must average the logarithm of the wave function amplitude. This contrast between individual realizations and averaged characteristics of the wave functions is precisely what is implied by our DOS sample-to-sample fluctuation study. The violent DOS fluctuations found should be due to the multifractal properties of the spectral measure and, therefore, are a typical characteristic of multifractality[16] and intermittency.[31] There is no reporting of such a spectrum of dimensions for the DOS at the moment which could eventually be related to our unusual DOS fluctuation results. Intermittency is a widespread phenomenon which appears as enhancement of rare deviations in multiplicative random quantities in multiparticle production and hydrodynamics.

The main results obtained are as follows: (i) The averaged fracton DOS obeys the AO scaling, both for the percolation cluster and its backbone, and we have directly determined the corresponding spectral dimensions. (ii) The fracton DOS fluctuations found are unususally large and can be cast neither onto the weak logarithmic nor the linear dependences corresponding to extended and localized states. The quadratic law obtained is, instead, compatible with a stronger kind of localization. (iii) The presence of strong DOS fluctuations for the backbone shows that even in the absence of dangling ends fractons display strong localization properties. Moreover, the violent fluctuations found in this paper should be, at least in principle, observed by neutron and Raman scattering experiments in fractal polymers and aerogels.[33] Our results could also be relevant for the electronic and localization properties in dendritic aromatic molecules.

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