Electromagnetic waves in a Josephson junction in a thin film

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We consider a one-dimensional Josephson junction in a superconducting film with a thickness that is much less than the London penetration depth. We treat an electromagnetic wave propagating along this tunnel contact. We show that the electrodynamics of a Josephson junction in a thin film is nonlocal if the wavelength is less than the Pearl penetration depth. We find the integrodifferential equation determining the phase difference between the two superconductors forming the tunnel contact. We use this equation to calculate the dispersion relation for an electromagnetic wave propagating along the Josephson junction. We find that the frequency of this wave is proportional to the square root of the wave vector if the wavelength is less than the Pearl penetration depth.

I. INTRODUCTION

The electromagnetic properties of tunnel Josephson junctions have been the subject of intensive studies over the past three decades.[1] Considerable attention has been, in particular, attracted to the investigation of SIS-type Josephson contacts. In this case the tunnel junction is formed by a thin layer of an insulator. This dielectric layer between two superconducting plates can be treated as a transmission line or a parallel plate resonator when the electromagnetic properties are concerned. It follows from this approach that an electromagnetic wave with a specific dispersion relation may propagate along the SIS-type Josephson junction.[2]

The existence of this Swihart electromagnetic wave results, in particular, in self-induced resonances, usually referred to as Fiske steps.[3] These specific resonances are observed as peaks in the current-voltage curves of the tunnel Josephson junctions and arise when the frequency $2eV/\hbar$, corresponding to the voltage across the Josephson junction, V, becomes equal to the frequency of a standing Swihart wave.[4,5] The study of Fiske steps is one of the methods to treat the electromagnetic properties of the SIS-type Josephson contacts. Recently, it was successfully applied to investigate the electromagnetic properties of the grain boundaries in $YBa_2Cu_3O_{7-\delta}$ high-temperature superconductors.[6]

Usually, the dispersion relation of a Swihart electromagnetic wave, $\omega(k)$, is determined by the sine-Gordon equation that leads to[1]

$$\omega = \omega_J \sqrt{1 + k^2\lambda_J^2}, \tag{1}$$

where

$$\omega_J = \sqrt{\frac{2ej_c}{\hbar C}} \tag{2}$$

is the Josephson frequency, C is the specific capacitance of the tunnel junction, and

$$\lambda_J = \sqrt{\frac{c\Phi_0}{16\pi^2\lambda j_c}} \tag{3}$$

is the Josephson penetration depth.

The Swihart electromagnetic wave corresponds to the limiting case $1 \ll k\lambda_J$. It follows then from Eq. (1) that $\omega \approx \omega_J\lambda_J k$; i.e., this wave is propagating along the Josephson junction with a constant velocity

$$c_s = \omega_J\lambda_J = \frac{c}{\sqrt{8\pi\lambda C}}. \tag{4}$$

The linear dispersion relation $\omega = c_s k$ results in an equidistant set of peaks in the current-voltage curves of SIS-type Josephson tunnel junctions.[4,5]

We can determine the phase difference between the two superconductors forming the tunnel contact $\varphi(y,t)$ in the mainframe of the local Josephson electrodynamics, i.e., by the sine-Gordon equation, as long as $k\lambda \ll 1$.[1] It means, in particular, that for an electromagnetic wave propagating along a Josephson junction with $\lambda \ll \lambda_J$ the dispersion relation $\omega = c_s k$ is valid in the region $\lambda_J^{-1} \ll k \ll \lambda^{-1}$.

Let us now discuss the general case, i.e., the case when restrictions on the wave vector k are given by the inequalities $kd_0 \ll 1$ and $k\xi \ll 1$, where d_0 is the thickness of the insulating barrier and ξ is the coherence length. We treat here a SIS-type Josephson junction formed by two superconducting plates. In this case the space distribution of φ is one dimensional and the relation between the phase difference $\varphi(y,t)$ and the magnetic field in the superconductors is nonlocal if $1 \ll k\lambda$. As a result the function $\varphi(y,t)$ is determined by an integro-differential equation,[7,8] i.e., the electrodynamics of a Josephson junction is nonlocal as far as the region of wave vectors, $1 \ll k\lambda$, is concerned.

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Using this equation it was shown[9,10] that the dispersion relation for an electromagnetic wave with $1 \ll k\lambda$ takes the form

$$\omega = c_s \sqrt{\frac{k}{\lambda}}. \tag{5}$$

The phase velocity of this electromagnetic wave is inversely proportional to the square root of the wave vector k. Note that it results in a nonequidistant set of the values of the voltage V corresponding to self-induced resonances in the current-voltage curve. (We present the theory of self-induced Fiske resonances in the mainframe of the nonlocal Josephson electrodynamics elsewhere.)

The nonlocality of Josephson electrodynamics is most pronounced when considering a SIS-type Josephson junction in a thin superconducting film with the thickness $d \ll \lambda$. In this case the space scale of the magnetic field variation in the superconductors forming the contact is given by the Pearl penetration depth[11]

$$\lambda_{\text{eff}} = \frac{\lambda^2}{d} \gg \lambda. \tag{6}$$

Thus, the electrodynamics of a Josephson junction in a thin film is nonlocal if $1 \ll k\lambda_{\text{eff}}$. This region of wave vectors is much wider than the one given by the inequality $1 \ll k\lambda$.

In this paper we consider an infinite one-dimensional Josephson tunnel junction in a superconducting film with a thickness $d \ll \lambda$. We show that the electrodynamics of a Josephson contact is nonlocal if the space scale of variation of φ is less than the Pearl penetration depth, i.e., if $1 \ll k\lambda_{\text{eff}}$. We derive the integro-differential equation determining the phase difference $\varphi(y,t)$. We use this equation to calculate the dispersion relation $\omega(k)$ for a electromagnetic wave propagating along SIS-type Josephson junction.

The paper is organized in the following way. In Sec. II, we consider the electrodynamics of a long one-dimensional Josephson junction in a superconducting film with a thickness that is much less than the London penetration depth. We treat the general case of an arbitrary relation between the Josephson penetration depth and the effective Pearl penetration depth. We derive the integro-differential equation determining the phase difference distribution along the Josephson junction. In Sec. III, we apply this equation to calculate the dispersion relation for an electromagnetic wave propagating along the Josephson junction. In Sec. IV, we summarize the overall conclusions.

II. BASIC EQUATIONS

Let us consider a thin superconducting film (xy plane) with an infinitely long SIS-type Josephson junction parallel to the y axis as shown in Fig. 1. (Note, that a Josephson junction in a thin strip was considered by Humphreys and Edwards,[12] while treating the critical current dependence on the external magnetic field.) We treat here the case when $\lambda \gg \xi$, in which the London equations govern

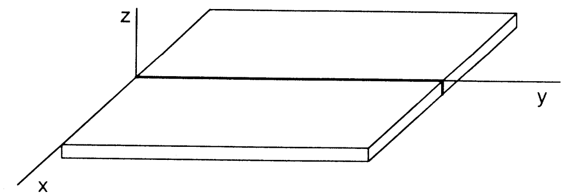

FIG. 1. A thin superconducting film with a Josephson junction (thick line).

the fields and currents inside the superconductor. Thus, outside the tunnel contact the relation between the current density $\mathbf{j}$ and the magnetic field $\mathbf{h}$ is given by[13]

$$\mathbf{h} + \frac{4\pi\lambda^2}{c}\,\text{rot}\,\mathbf{j} = 0. \tag{7}$$

Introducing the vector potential $\mathbf{A}$ and combining Eq. (7) with the equation

$$\mathbf{h} = \text{rot}\,\mathbf{A}\ , \tag{8}$$

we express the current density $\mathbf{j}$ in the form

$$\mathbf{j} = \frac{c}{4\pi\lambda^2}\,(\mathbf{S} - \mathbf{A}), \tag{9}$$

where outside the tunnel junction the vector field $\mathbf{S}$ is given by the formula

$$\mathbf{S} = \frac{\Phi_0}{2\pi}\,\nabla\theta \tag{10}$$

and θ is the phase of the order parameter.

The quantities $\mathbf{j}$, $\mathbf{A}$, and $\mathbf{S}$ are nearly independent on the z coordinate in the limiting case of a thin film, i.e., for $d \ll \lambda$. Therefore, in order to find the fields and currents we replace the superconducting film with the thickness $d \ll \lambda$ by an infinitely thin current-carrying sheet in the plane $z = 0$. The current density $\mathbf{j}$ in this plane is determined then by the averaging of Eq. (9) over the thickness d,[11,14] which results in

$$\mathbf{j} = \frac{c}{4\pi\lambda_{\text{eff}}}\,(\mathbf{S} - \mathbf{A})\,\delta(z). \tag{11}$$

Let us now choose the London gauge; i.e., let us assume that $\text{div}\,\mathbf{A} = 0$. Then, substituting Eq. (11) into the Maxwell equation

$$\text{rot}\,\mathbf{h} = \frac{4\pi}{c}\,\mathbf{j}\ , \tag{12}$$

we find the equation describing the vector potential $\mathbf{A}$ in the form

$$-\Delta\mathbf{A} + \lambda_{\text{eff}}^{-1}\mathbf{A}\,\delta(z) = \lambda_{\text{eff}}^{-1}\mathbf{S}\,\delta(z). \tag{13}$$

The vector field $\mathbf{S}$ is related to the phase difference

$$\varphi(y) = \theta(+0,y) - \theta(-0,y). \tag{14}$$

This relation is given by the equation

$$\operatorname{rot}\mathbf{S} = \frac{\Phi_0}{2\pi}\,\varphi'(y)\,\delta(x)\,\hat{\mathbf{z}}, \tag{15}$$

following from Eq. (10) and taking into account the singularity of the function $\theta(x,y)$ at $x=0$. The vector $\hat{\mathbf{z}}$ is here for the unit vector along the z axis.

Applying the continuity equation $\operatorname{div}\mathbf{j}=0$ to Eq. (11) we find that $\operatorname{div}\mathbf{S}=0$. Thus, we can present the vector field $\mathbf{S}$ as a curl of a certain vector field $\mathbf{F}$, namely,

$$\mathbf{S}(\boldsymbol{\rho}) = \operatorname{rot}\mathbf{F}, \tag{16}$$

where $\boldsymbol{\rho}=(x,y)$ and

$$\mathbf{F} = F(\boldsymbol{\rho})\,\hat{\mathbf{z}}. \tag{17}$$

Substituting Eq. (17) into Eq. (15) we find the equation describing the function $F(\boldsymbol{\rho})$ in the form

$$\Delta F = -\frac{\Phi_0}{2\pi}\,\varphi'(y)\,\delta(x). \tag{18}$$

Note that the scalar function $F(\boldsymbol{\rho})$, determines both components of the vector field $\mathbf{S}(\boldsymbol{\rho})$ reducing by this way the complexity of the problem.

The current density across the Josephson junction $j_x(0,y)$ is a sum of two terms, namely, the tunnel and the displacement current densities,

$$j_x(0,y) = \left[j_c\sin\varphi + \frac{\hbar C}{2e}\,\frac{\partial^2\varphi}{\partial t^2}\right] d\,\delta(z). \tag{19}$$

At the same time it follows from Eq. (9) that the current density $j_x(0,y)$ can be written as

$$j_x(0,y) = \frac{c}{4\pi\lambda_{\text{eff}}}\,[S_x(0,y) - A_x(0,y,0)]\,\delta(z). \tag{20}$$

Equating the expressions for the quantity $j_x(0,y)$ given by Eqs. (19) and (20) we find that

$$j_c\sin\varphi + \frac{\hbar C}{2e}\,\frac{\partial^2\varphi}{\partial t^2} = \frac{c}{4\pi\lambda^2}\,[S_x(0,y) - A_x(0,y,0)]\ . \tag{21}$$

Thus, to derive the closed form of the equation describing the phase difference $\varphi(y,t)$ it is necessary to find the functional relation between

$$\Delta_x(y) = S_x(0,y) - A_x(0,y,0) \tag{22}$$

and $\varphi(y,t)$. We use here a Fourier transformation in order to do it and defining the Fourier transforms for $\mathbf{A}(\mathbf{r})$ and $\mathbf{S}(\boldsymbol{\rho})$ as

$$\mathbf{A}(\mathbf{r}) = \int \mathbf{A}_{\mathbf{q},p}\,\exp(i\mathbf{q}\boldsymbol{\rho} + ipz)\,\frac{d^2\mathbf{q}dp}{(2\pi)^3} \tag{23}$$

and

$$\mathbf{S}(\boldsymbol{\rho}) = \int \mathbf{S}_{\mathbf{q}}\,\exp(i\mathbf{q}\boldsymbol{\rho})\,\frac{d^2\mathbf{q}}{(2\pi)^2}. \tag{24}$$

Using $\mathbf{A}_{\mathbf{q},p}$ and $\mathbf{S}_{\mathbf{q}}$ we can present the value of Δ_x by the integral

$$\Delta_x = \frac{1}{4\pi^2}\int_0^\infty q\,dq\int_{-\pi}^{\pi} d\vartheta\,(S^x_{\mathbf{q}} - A^x_{\mathbf{q}})\,\exp(iqy\sin\vartheta), \tag{25}$$

where ϑ is the polar angle in the (q_x,q_y) plane and

$$\mathbf{A}_{\mathbf{q}} = \int_{-\infty}^{\infty}\mathbf{A}_{\mathbf{q},p}\,\frac{dp}{2\pi}. \tag{26}$$

The next step is to apply Fourier transformation to Eq. (13), which results in

$$(q^2+p^2)\,\mathbf{A}_{\mathbf{q},p} + \lambda_{\text{eff}}^{-1}\,\mathbf{A}_{\mathbf{q}} = \lambda_{\text{eff}}^{-1}\,\mathbf{S}_{\mathbf{q}}. \tag{27}$$

It follows from Eq. (27) that the relation between $\mathbf{A}_{\mathbf{q}}$ and $\mathbf{S}_{\mathbf{q}}$ has the form

$$\mathbf{A}_{\mathbf{q}} = \frac{\mathbf{S}_{\mathbf{q}}}{1+2q\lambda_{\text{eff}}} \tag{28}$$

and thus

$$S^x_{\mathbf{q}} - A^x_{\mathbf{q}} = \frac{2q\lambda_{\text{eff}}}{1+2q\lambda_{\text{eff}}}\,S^x_{\mathbf{q}}. \tag{29}$$

To calculate the Fourier transform $S^x_{\mathbf{q}}$ we take the derivative of Eq. (18) with respect to y and substitute S_x instead of $\partial F/\partial y$. As a result it comes out that

$$\Delta S_x = -\frac{\Phi_0}{2\pi}\,\varphi''(y)\,\delta(x). \tag{30}$$

It follows from Eq. (30) that the Fourier transform $S^x_{\mathbf{q}}$ is given by the formula

$$S^x_{\mathbf{q}} = \frac{\Phi_0}{2\pi q^2}\int_{-\infty}^{\infty} dy\,\varphi''(y)\exp(-iqy\sin\vartheta)\ . \tag{31}$$

Combining now Eqs. (31), (29), (25), and (21) we find the integro-differential equation describing the phase difference $\varphi(y,t)$ in the form

$$\frac{1}{\omega_J^2}\,\frac{\partial^2\varphi}{\partial t^2} + \sin\varphi = l_J\int_{-\infty}^{\infty} dy'\,K\Big(\frac{y-y'}{2\lambda_{\text{eff}}}\Big)\,\varphi''(y'), \tag{32}$$

where

$$K(u) = \frac{1}{\pi}\int_0^\infty \frac{J_0(v)}{v+|u|}\,dv, \tag{33}$$

$J_0(v)$ is the zero-order Bessel function, and

$$l_J = \frac{c\Phi_0}{16\pi^2\lambda^2 j_c}. \tag{34}$$

Note that Eq. (32) can be rewritten as[8]

$$\frac{1}{\omega_J^2}\,\frac{\partial^2\varphi}{\partial t^2} + \sin\varphi = \frac{l_J}{\pi}\int_{-\infty}^{\infty}\frac{dy'}{y'-y}\,\frac{\partial\varphi}{\partial y'} \tag{35}$$

in the limiting case when the characteristic space scale of the phase difference variation is much less than λ_{eff}.

III. DISPERSION RELATION

Let us now consider a small amplitude electromagnetic wave propagating along the Josephson junction. The corresponding solution of Eq. (32) then reads

$$\varphi = \varphi_0 \exp(iky - i\omega t), \qquad |\varphi_0| \ll 1. \tag{36}$$

Substitution of Eq. (36) into Eq. (32) results in the following dispersion relation

$$\omega = \omega_J \sqrt{1 + 2k^2 \lambda_{\rm eff} l_J \mathcal{K}(2k\lambda_{\rm eff})}, \tag{37}$$

where

$$\mathcal{K}(x) = \int_{-\infty}^{\infty} K(u) \exp(ixu)\, du. \tag{38}$$

The function $\mathcal{K}(x)$ has the explicit form

$$\mathcal{K}(x) = \begin{cases} \dfrac{1}{\pi\sqrt{1-x^2}} \ln \dfrac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} & \text{if } x < 1, \\ \dfrac{1}{\sqrt{x^2-1}} \left[1 - \dfrac{2}{\pi} \arctan \dfrac{1}{\sqrt{x^2-1}}\right] & \text{if } x > 1. \end{cases} \tag{39}$$

Using Eqs. (37) and (39) we find, in particular, the dispersion relation $\omega(k)$ in the limiting cases $k\lambda_{\rm eff} \ll 1$ and $k\lambda_{\rm eff} \gg 1$,

$$\omega = \begin{cases} \omega_J \sqrt{1 - \dfrac{4k^2\lambda_{\rm eff} l_J}{\pi} \ln(k\lambda_{\rm eff})} & \text{if } k\lambda_{\rm eff} \ll 1, \\ \omega_J \sqrt{1 + k l_J} & \text{if } k\lambda_{\rm eff} \gg 1. \end{cases} \tag{40}$$

Thus, for $k\lambda_{\rm eff} \gg 1$ and $kl_J \gg 1$ the frequency of an electromagnetic wave propagating along the SIS-type tunnel Josephson junction in a thin film is proportional to the square root of the wave vector.

The dispersion relation $\omega \propto \sqrt{k}$ leads, in particular, to a nonequidistant set of the self-induced Fiske resonances in the current-voltage curves for the voltages $V > V_c$, where

$$V_c = \frac{\hbar\omega(\lambda_{\rm eff}^{-1})}{2e} \approx \frac{c_s}{\lambda}\sqrt{\frac{d}{\lambda}}. \tag{41}$$

Note that for a thin superconducting film the relation $k\lambda_{\rm eff} \sim 1$ corresponds to a wave length that is λ/d times bigger than the London penetration depth.

IV. SUMMARY

To summarize, we have found the integro-differential equation describing the phase difference in case of the SIS-type tunnel Josephson junction in a thin superconducting film. We apply this equation to calculate the dispersion relation for an electromagnetic wave propagating along the Josephson contact. We have shown that if the wavelength is small compared with the Pearl penetration depth $\lambda_{\rm eff}$, the frequency is proportional to the square root of the wave vector.

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