Two extended versions of the continuous two-dimensional Heisenberg model

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We analyze two extended versions [the Ishimori model (IM) and a related system, which will be called the modified Ishimori model (MIM)] of the continuous Heisenberg model in (2+1) dimensions within the complex Hirota scheme. The IM is an integrable (2+1)-dimensional topological spin-field model that has been studied in many theoretical frameworks. The MIM has been introduced quite recently by some of the present authors [Phys. Rev. B **49**, 12 915 (1994)]. Using the same stereographic variable in the Hirota formulation, we build up some exact solutions both for the IM and the MIM in the compact and noncompact case. For the IM, the configurations are a class of static solutions related to a special third Painlevé equation, time-dependent solutions linked to another kind of the third Painlevé transcendent, and asympotic time-dependent solutions whose energy density behaves as a Yukawa potential. For the MIM, configurations are a class of exact solutions expressed in terms of elliptic functions, and a class of time-dependent solutions related to a particular form of the double sine-Gordon and the double sinh-Gordon equations with variable coefficients. We discuss the configurations and certain known solutions which clarify the different possible phenomenological roles played by the considered topological spin-field models.

I. INTRODUCTION

Topological field models, usually relevant to string theory,[1] have been applied successfully in the last years to handle many problems pertinent to condensed matter physics.[2]

Here we consider the topological spin-field model in two-space and one-time dimensions:

$$S_t = \frac{1}{2i}[S, S_{xx} + \alpha^2 S_{yy}] + \beta^2 S_y \phi_x - S_x \phi_y, \qquad (1.1a)$$

$$\phi_{xx} + \alpha^2 \beta^2 \phi_{yy} = 2\alpha^2 \beta^2 Q, \qquad (1.1b)$$

where $\alpha^2 = \pm 1$, $\beta^2 = \pm 1$, subscripts denote partial derivatives, $S = S(x, y, t)$ is a 2×2 matrix defined by

$$S = \begin{pmatrix} S_3 & \kappa S_+^* \\ \kappa S_+ & -S_3 \end{pmatrix}, \qquad (1.2)$$

$S_+ = S_1 + iS_2$, the asterisk means complex conjugation, $Q = \frac{1}{2}\{\mathrm{Tr}(i/2)S[S_x, S_y]\}$ is a conserved topological charge density, and ϕ is a real scalar field. The functions $S_j(x, y, t)$ (j=1,2,3), which are real-valued components of a classical unit "spin" vector $\mathbf{S}(x, y, t)$, belong to the two-dimensional (2D) sphere S^2 ($\kappa^2 = 1$) or the pseudosphere $S^{1,1}$ ($\kappa^2 = -1$), i.e.,

$$S_3^2 + \kappa^2 (S_1^2 + S_2^2) = 1. \qquad (1.3)$$

For $\beta^2 = -1$, Eqs. (1.1) describe the Ishimori model (IM), which can be regarded as an integrable version (it has a Lax pair formulation[3,4]) of the continuous 2D Heisenberg model.[5] Both the compact ($\kappa^2 = 1$) and the noncompact ($\kappa^2 = -1$) IM's admit exact solutions classified by an integer topological charge (localized solitons[6] and vortexlike[3] and closed stringlike configurations[7]). Furthermore, the IM possesses an infinite-dimensional symmetry algebra of the Kac-Moody type with a loop algebra structure.[8,9] This feature characterizes other nonlinear field equations in 2 + 1 dimensions of physical significance having a Lax pair formulation, such as the Kadomtsev-Petviashvili equation,[10] the Davey-Stewartson equation,[11] and the three-wave resonant system.[12] Apart from these nice properties, at present it is not known whether the IM is a Hamiltonian system.

Conversely, for $\beta^2 = +1$, Eqs. (1.1) describe a spin-field system endowed with a Hamiltonian structure.[13] We shall call this system a modified Ishimori model (MIM). Similarly to what happens for the IM, the MIM allows a symmetry algebra of the Kac-Moody type with a loop algebra structure. However, this does not imply that the MIM is surely integrable. In fact, so far a Lax pair has been found only for $\phi_{xy} = 0$.[13] The question of the integrability of the MIM for $\phi_{xy} \neq 0$ remains open.

Just as it occurs for the IM, the MIM provides similar solutions; a few of them turn out to be of the helical and the rotontype, and meronlike configurations provided by a fractional topological charge.[13] These results indicate that the IM and the MIM may refer to different physical situations. This appears mostly evident in relation to the configurations endowed with a nonvanishing topological charge. In fact, the vortices found in the compact IM (Ref. 3) and the stringlike configurations allowed by its noncompact version[7] have an *integer* topological charge, while, as we shall see later, the meronlike excitations in MIM are characterized by a *fractional* topological charge.

The above considerations suggest that it should be interesting to pursue a comparative study of the IM and the MIM. Keeping in mind this idea, in the following we apply the Hirota representation to look for a special class of configurations by choosing the same form of

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the stereographic variable in terms of which one can express the spin-field components and the auxiliary field ϕ. In doing so, for the (compact and noncompact) IM we obtain some static and dynamical configurations which can be expressed in terms of certain special forms of the third Painlevé transcendent. Other interesting time-dependent configurations lead to asymptotic expressions for the spin-field variables which are associated with an energy density of the Yukawa type.

On the other hand, for the noncompact MIM we find, as static exact configurations, a class of solutions expressed in terms of elliptic functions. (A corresponding class of configurations for the compact MIM has been already determined in Ref. 13.) Furthermore, for both the compact and the noncompact MIM another result is constituted by a class of time-dependent solutions related, respectively, to a particular form of the double sine-Gordon and the double sinh-Gordon equations with variable coefficients.

For our purposes, let us recall the Hirota scheme.[14] This consists essentially in writing Eqs. (1.1) by using the stereographic projection representation

$$S_+ = \frac{2\zeta}{1+\kappa^2|\zeta|^2}, \quad S_3 = \frac{1-\kappa^2|\zeta|^2}{1+\kappa^2|\zeta|^2}, \tag{1.4}$$

and putting $\zeta = \frac{g}{f}$, where $f = f(x,y,t)$ and $g = g(x,y,t)$ are two arbitrary differentiable complex functions. Then, Eqs. (1.1) take the form

$$(|f|^2 - \kappa^2|g|^2)(iD_t - D_x^2 - \alpha^2 D_y)(f^* \cdot g)$$

$$-f^* g(iD_t - D_x^2 - \alpha^2 D_y^2)(f^* \cdot f - \kappa^2 g^* \cdot g) = 0, \tag{1.5a}$$

$$\phi_{xx} + \alpha^2\beta^2\phi_{yy} = \frac{4i\alpha^2\beta^2\kappa^2}{\Delta^2}[D_y(g\cdot f)D_x(g^*\cdot f^*)$$

$$-D_y(g^*\cdot f^*)D_x(g\cdot f)], \tag{1.5b}$$

with $\Delta = |f|^2 + \kappa^2|g|^2$, where the operators D_t, D_x, and D_y stand for the "antisymmetric derivatives," i.e., $D_t(a\cdot b) = a_t b - a b_t$, and so on.

A particular solution to Eq. (1.5b) valid for any value (± 1) of the parameters α^2 and β^2 is given by

$$\phi_x = \frac{-2i\beta^2\alpha^2}{\Delta} D_y(f^*\cdot f + \kappa^2 g^* \cdot g),$$

$$\phi_y = \frac{2i}{\Delta} D_x(f^*\cdot f + \kappa^2 g^*\cdot g). \tag{1.6}$$

However, the compatibility condition $\phi_{xy} = \phi_{yx}$ is not identically satisfied. Therefore, this is a constraint which has to be taken into account in order to solve Eq. (1.5a).

Interesting phenomenological aspects of the IM and the MIM can be evidenced assuming first that f and g are (complex) functions of $z = x + iy$ and its conjugate, namely, $f = f(z, z^*, t)$ and $g = g(z, z^*, t)$. Consequently, with the help of the operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{z^*} = \frac{1}{2}(\partial_x + i\partial_y)$, Eqs. (1.5), (1.6) and the related compatibility condition can be written in complex form. We shall call the full set of these equations complex Hirota's formulation (CHF) of the spin-field model (1.1) (see the Appendix). Second, we are looking for special solutions to the CHF by setting $f = [a^*(z^*)]^{1/2}$ and $g = [a(z)]^{1/2}\psi(|z|)$, where $a(z)$ and $\psi(|z|)$ are, respectively, a complex and a real function to be determined. This choice corresponds to the stereographic variable

$$\zeta = [a(z)/a^*(z^*)]^{1/2}\psi(|z|). \tag{1.7}$$

To be precise, below we shall limit ourselves to the cases $\alpha^2 = 1$, $\kappa^2 = \pm 1$.

II. CASE $\beta^2 = -1$

Let us put Eqs. (1.5), for $\beta^2 = -1$ (IM), in the CHF. Then, the compatibility condition $\phi_{zz^*} = \phi_{z^*z}$ [see (1.6)] entails

$$a(z) = a_0 \exp[(\lambda/2)z^2], \tag{2.1}$$

where a_0 and λ are, respectively, an arbitrary complex and a real constant. On the other hand, the complex form of (1.5a) furnishes the nonlinear ordinary differential equation

$$(1+\kappa^2\psi^2)\left(\psi_{rr} + \frac{1}{r}\psi_r\right) + (1-\kappa^2\psi^2)\left|\frac{a_z}{a}\right|^2 = 2\kappa^2\psi\psi_r^2, \tag{2.2}$$

where $a(z)$ is given by (2.1) and $z = re^{i\theta}$. Using the transformation

$$u = \ln r, \qquad \text{(i) } \psi = \tan\frac{\gamma}{4}, \quad \text{for} \quad \kappa^2 = 1;$$

$$\text{(ii) } \psi = \tanh\frac{\gamma}{4}, \quad \text{for} \quad \kappa^2 = -1, \tag{2.3}$$

Eq. (2.2) takes, correspondingly, the form

$$\gamma_{uu} + \lambda^2 e^{4u}\sin\gamma = 0 \tag{2.4}$$

and

$$\gamma_{uu} + \lambda^2 e^{4u}\sinh\gamma = 0. \tag{2.5}$$

Equations (2.4) and (2.5) are related to a special case of the third Painlevé transcendent, defined by[15]

$$\frac{d^2W}{dz^2} = \frac{1}{W}\left(\frac{dW}{dz}\right)^2 - \frac{1}{z}\frac{dW}{dz} + (\alpha_0 W^2 + \alpha_1) + \alpha_2 W^3 + \frac{\alpha_3}{W}, \tag{2.6}$$

where $W = W(z)$, and α_j $(j = 0,1,2,3)$ are arbitrary constants.

This can be seen by putting in (2.4) and (2.5): (i) $e^{2u} = \sigma$, $W = e^{i\frac{\gamma}{2}}$, and (ii) $e^{2u} = \sigma$, $W = e^{\frac{\gamma}{2}}$, respectively. We get

$$W_{\sigma\sigma} = \frac{W_\sigma^2}{W} - \frac{1}{\sigma}W_\sigma - \frac{\lambda^2}{16}\left(W^3 - \frac{1}{W}\right). \tag{2.7}$$

Thus, Eq. (2.7) corresponds to the particular case of the third Painlevé equation (2.6) where $\alpha_0 = \alpha_1 = 0$ and $\alpha_2 = -\alpha_3 = -\frac{\lambda^2}{16}$.

Dynamical configurations to the IM yielding (2.7) when the time is switched off can also be obtained. Indeed, starting from

$$\zeta' = \zeta\rho(t), \tag{2.8}$$

where ζ is given by (1.7) and $\rho(t)$ is a function of the time to be found, the CHF provides $\rho = \exp[-i(Et+D)]$ with E, D real constants, and

$$(1+\kappa^2\psi^2)\left(\psi_{rr}+\frac{1}{r}\psi_r\right) - 2\kappa^2\psi\psi_r^2 + \lambda^2 r^2(1-\kappa^2\psi^2) + E(1+\kappa^2\psi^2)\psi = 0. \tag{2.9}$$

By means of the substitution $\psi = \tan\frac{\gamma}{4}$ (for $\kappa^2 = 1$) or $\psi = \tanh\frac{\gamma}{4}$ (for $\kappa^2 = -1$), Eq. (2.9) becomes

$$\gamma_{uu} + \lambda^2 e^{4u}\sin\gamma + 2Ee^{2u}\sin\frac{\gamma}{2} = 0 \quad (\kappa^2 = 1) \tag{2.10}$$

or

$$\gamma_{uu} + \lambda^2 e^{4u}\sinh\gamma + 2Ee^{2u}\sinh\frac{\gamma}{2} = 0 \quad (\kappa^2 = -1). \tag{2.11}$$

The change of variables $W = e^{i\frac{\gamma}{2}}$ and $e^{2u} = \sigma$ transforms Eq.(2.10) into the third Painlevé equation

$$W_{\sigma\sigma} = \frac{W_\sigma^2}{W} - \frac{1}{\sigma}W_\sigma - \frac{\lambda^2}{16}\left(W^3 - \frac{1}{W}\right) - \frac{E}{8\sigma}(W^2-1), \tag{2.12}$$

which corresponds to the choice $\alpha_0 = -\alpha_1 = -\frac{E}{8}$ and $\alpha_2 = -\alpha_3 = -\frac{\lambda^2}{16}$ of the free parameters α_j present in (2.6).

By rescaling the independent variable σ, namely, by setting $\xi = i\frac{\lambda}{4}\sigma$, Eq. (2.12) takes the form

$$W_{\xi\xi} = \frac{W_\xi^2}{W} - \frac{1}{\xi}W_\xi + W^3 - \frac{1}{W} + \frac{2\nu}{\xi}(W^2-1), \tag{2.13}$$

with $\nu = \frac{iE}{4\lambda}$.

On the other hand, by taking $\widetilde{W} = e^{\frac{\gamma}{2}}$ with $e^{2u} = \sigma$, Eq. (2.11) reduces formally to Eq. (2.12), where now $\widetilde{W}$ is a real function. Assuming $\tau = \frac{\lambda}{4}\sigma$, we are led to the equation

$$\widetilde{W}_{\tau\tau} = \frac{\widetilde{W}_\tau^2}{\widetilde{W}} - \frac{1}{\tau}\widetilde{W}_\tau - \widetilde{W}^3 + \frac{1}{\widetilde{W}} - \frac{2\widetilde{\nu}}{\tau}(\widetilde{W}^2-1), \tag{2.14}$$

with $\widetilde{\nu} = -\frac{E}{4\lambda}$.

Equations (2.13) and (2.14) are invariant under the transformations

$$W \to \frac{1}{W}, \quad \widetilde{W} \to \frac{1}{\widetilde{W}}, \tag{2.15}$$

respectively.

At this stage some comments are in order.

(i) Equation (2.13) coincides formally with Eq. (1.31) of Ref. 16 for the scaling limit of the spin-spin correlation function of the two-dimensional Ising model. To be precise, in Ref. 16 a one-parameter family of solutions $\eta(\tau;\nu,\mu)$ to the above mentioned Eq. (1.31) was found by the request that these remain bounded as the independent variable τ approaches infinity along the positive real axis. Furthermore, the authors of Ref. 16 built up the large and the small-τ behavior of $\eta(\tau;\nu,\mu)$ under certain conditions for the parameters μ and ν. For example, as $\tau \to \infty$ one has

$$\eta \sim 1 - \mu\Gamma\left(\nu+\frac{1}{2}\right)2^{-2\nu}\tau^{-\nu-\frac{1}{2}}e^{-2\tau}, \tag{2.16}$$

where Γ denotes the gamma function. This expansion will be used later to provide explicit asymptotic solutions to the Ishimori model.

(ii) Equation (2.14) resembles Eq. (1.31) of Ref. 16, but it is really different from the latter because the term $\widetilde{W}^3 - 1/\widetilde{W}$ in (2.14) has opposite sign. At present, the role of Eq. (2.14) in the context of spin-field models seems unknown. Its possible physical meaning could be explored following a procedure similar to that exploited in Ref. 16.

Now, by substituting

$$\zeta' = e^{i(\lambda xy - Et + \delta)}\psi(r) \tag{2.17}$$

into (1.4), where δ is a constant, we get the spin-field components S_j in terms of ψ [see (2.8), (1.7), and (2.1)]:

$$S_1 = 2\cos(\lambda xy - Et + \delta)\frac{\psi}{1+\kappa^2\psi^2}, \tag{2.18a}$$

$$S_2 = 2\sin(\lambda xy - Et + \delta)\frac{\psi}{1+\kappa^2\psi^2}, \tag{2.18b}$$

$$S_3 = \frac{1-\kappa^2\psi^2}{1+\kappa^2\psi^2}. \tag{2.18c}$$

Limiting ourselves, for simplicity, to consider the compact case ($\kappa^2 = 1$), the quantities (2.18) become

$$\begin{aligned} S_1 &= \cos(\lambda xy - Et + \delta)\sin\gamma, \\ S_2 &= \sin(\lambda xy - Et + \delta)\sin\gamma, \\ S_3 &= \cos\gamma, \end{aligned} \tag{2.19}$$

where $\sin\gamma = \frac{1}{2i}(W - W^*)$, $\cos\gamma = \frac{1}{2}(W + W^*)$, and $W = W(\xi)$ satisfies Eq.(2.13). The auxiliary field ϕ can be derived from (1.6) keeping in mind that

$$g = e^{-i(Et+D)}[a(z)]^{1/2}\psi(r) \text{ and } f = [a^*(z^*)]^{1/2},$$

$a(z)$ being expressed by (2.1). We get

$$\phi_x = -2\lambda x S_3, \quad \phi_y = -2\lambda y S_3, \tag{2.20}$$

which furnishes the topological charge density [see (1.16)]

$$Q = \frac{1}{2}(\phi_{yy} - \phi_{xx}) = \lambda(xS_{3x} - yS_{3y}).$$

The total topological charge,

$$Q_T = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q\,dx\,dy \tag{2.21}$$

can be evaluated, in principle, from the properties of the third Painlevé transcendent W defined by Eq. (2.13) [see (2.19)].

Another interesting, more explicit example of solution to the IM related to the third Painlevé equation arises from (2.11) by choosing $\lambda = 0$ and $E < 0$. In fact, in this case Eq. (2.11) can be written as

$$W_{\rho\rho} = \frac{1}{W}W_\rho^2 - \frac{1}{\rho}W_\rho + W^3 - \frac{1}{W}, \tag{2.22}$$

where $e^{\frac{\gamma}{4}} = W, \quad e^u = r = 2\rho/(|E|)^{\frac{1}{2}}$, and

$$\zeta = e^{i(|E|t-D)}\psi(r). \tag{2.23}$$

We remark that (2.22), where W and ρ are real quantities, is exactly Eq. (1.3) (for $\nu = 0$) studied in Ref. 16. Therefore, we can exploit (2.16) to provide an explicit asymptotic solution to the IM. In doing so, from (1.4) and (2.23) we find

$$S_1 = \cos(|E|t - D)\sinh\frac{\gamma}{2}, \tag{2.24a}$$

$$S_2 = \sin(|E|t - D)\sinh\frac{\gamma}{2}, \tag{2.24b}$$

$$S_3 = \cosh\frac{\gamma}{2}, \tag{2.24c}$$

where $\sinh\frac{\gamma}{2} = \frac{1}{2}(W^2 - W^{-2})$ and $\cosh\frac{\gamma}{2} = \frac{1}{2}(W^2 + W^{-2})$.

By resorting to (2.16) with $\nu = 0$ and identifying τ with ρ, as $\rho \to \infty$ we have

$$W(\rho; 0, \mu) \sim 1 - \mu(\pi)^{\frac{1}{2}} e^{-2\rho}\rho^{-\frac{1}{2}}. \tag{2.25}$$

Then, the spin-field components (1.4) become

$$S_1 \sim -2\mu(\pi)^{\frac{1}{2}} \cos(|E|t - D)e^{-2\rho}\rho^{-\frac{1}{2}}, \tag{2.26a}$$

$$S_2 \sim -2\mu(\pi)^{\frac{1}{2}} \sin(|E|t - D)e^{-2\rho}\rho^{-\frac{1}{2}}, \tag{2.26b}$$

$$S_3 \sim 1, \tag{2.26c}$$

where the parameter μ is real. The auxiliary field ϕ related to (2.23) turns out to be a constant. This can be seen from (1.6) with $f = 1$ and $g = \zeta$ [see (2.23)]. The topological charge density vanishes. On the other hand, the energy density $\mathcal{E}$ carried by the spin components (2.16) is

$$\mathcal{E} = \frac{1}{2}\sum_{j=1}^{3} S_{jr}^2 = \frac{1}{8}\gamma_r^2\cosh\gamma = \frac{1}{4}|E|W_\rho^2(W^2 + W^{-6}), \tag{2.27}$$

where W obeys the special Painlevé equation of the third kind (2.14), $S_{jr} = \frac{\partial S_j}{\partial r}$, $W_\rho = \frac{\partial W}{\partial \rho}$. With the help of (2.25), we obtain the asymptotic value

$$\mathcal{E} \sim 2|E|\mu^2(\pi)e^{-4\rho}\rho^{-1}. \tag{2.28}$$

The expression (2.28) tells us that, for large values of ρ, the energy density of the spin configuration (2.24) is of the Yukawa type.

III. CASE $\beta^2 = 1$

By using the stereographic variable (1.7) in the CHF of the spin-field model (1.1) for $\beta^2 = 1$ (MIM), from the compatibility condition $\phi_{zz^*} = \phi_{z^*z}$ [see (1.6)] we obtain

$$a(z) = a_0 z^\lambda, \tag{3.1}$$

where a_0 is an arbitrary complex constant, and λ is a real number. On the other hand, with the aid of (3.1) the complex version of (1.5a) yields

$$(1+\kappa^2\psi^2)\left(\psi_{rr} + \frac{1}{r}\psi_r\right) + \frac{\lambda^2}{r^2}(1-\kappa^2\psi^2)\psi = 2\kappa^2\psi\psi_r^2. \tag{3.2}$$

By way of change of variables $u = \ln r$, and (i) $\psi = \tan\frac{\gamma}{4}$, for $\kappa^2 = 1$, (ii) $\psi = \tanh\frac{\gamma}{4}$, for $\kappa^2 = -1$, Eq. (3.2) can be written as

$$\gamma_{uu} + \lambda^2 \sin\gamma = 0, \tag{3.3}$$

and

$$\gamma_{uu} + \lambda^2 \sinh\gamma = 0, \tag{3.4}$$

respectively. The first is the equation for the pendulum, which admits the solution

$$\gamma = 2\arcsin[k\,\mathrm{sn}(\lambda u, k)], \tag{3.5}$$

where $\mathrm{sn}(\cdot)$ denotes the Jacobian elliptic function of modulus k ($0 \le k \le 1$). The spin-field components, the auxiliary field ϕ, the topological charge density, and the energy associated with (3.5) and their limit cases ($k = 0$ and $k = 1$) have been already discussed in Ref. 13.

However, for the reader's convenience, below we report the main results. Let us take $\lambda = 1$ for simplicity. Then, by taking $g = \left(\frac{1-\mathrm{dn}(u,k)}{z^*}\right)^{1/2}$ and $f = \left(\frac{1+\mathrm{dn}(u,k)}{z}\right)^{1/2}$, where $\mathrm{dn}^2(\cdot) - 1 = k^2\mathrm{sn}^2(\cdot)$, the variable (1.7) reads

$$\zeta = \left(\frac{z}{z^*}\right)^{1/2}\left[\frac{1 - \mathrm{dn}(u,k)}{1 + \mathrm{dn}(u,k)}\right]^{\frac{1}{2}}. \tag{3.6}$$

Introducing (3.6) into (1.4) gives the radially symmetric spin-field configuration

$$\begin{aligned} S_1 &= k\,\mathrm{sn}(u,k)\cos\theta, \\ S_2 &= k\,\mathrm{sn}(u,k)\sin\theta, \\ S_3 &= \mathrm{dn}(u,k), \end{aligned} \tag{3.7}$$

while the auxiliary field ϕ turns out to be

$$\phi = 2\arcsin[\mathrm{sn}(u,k)] + \phi_0, \tag{3.7$'$}$$

ϕ_0 being a constant of integration. The topological charge density is

$$Q = \frac{1}{r^2}\frac{d}{du}\mathrm{dn}(u,k), \tag{3.8}$$

which implies a vanishing total topological charge $Q_T = 0$. Now, we recall that the MIM is a constrained Hamiltonian system described by the Hamiltonian density[13]

$$H = H_M + H_\phi = \frac{1}{2}\sum_{j=1}^{3}(S_{jx}^2 + \alpha^2 S_{jy}^2) + \frac{1}{4}(\alpha^2\phi_x^2 + \phi_y^2), \tag{3.9}$$

where

$$H_M = \frac{1}{2}\sum_{j=1}^{3}(S_{jx}^2 + \alpha^2 S_{jy}^2), \qquad H_\phi = \frac{1}{4}(\alpha^2\phi_x^2 + \phi_y^2), \tag{3.9$'$}$$

$$\begin{aligned} \phi_x &= 2\alpha^2(qp_y - \Lambda_y), \\ \phi_y &= -2(qp_x - \Lambda_x), \end{aligned} \tag{3.10}$$

q and p are a pair of canonical variables defined by

$$q = -\arctan\frac{S_2}{S_1}, \quad p = S_3, \tag{3.11}$$

and $\Lambda = \Lambda(x, y, t)$ is a differential function determined by the compatibility condition $\phi_{xy} = \phi_{yx}$, namely,

$$\Lambda_{xx} + \alpha^2\Lambda_{yy} = \partial_x(qp_x) + \alpha^2\partial_y(qp_y). \tag{3.12}$$

The quantities (3.10) obey Eq. (1.1b) ($\beta^2 = 1$). It is noteworthy that for $\alpha^2 = 1$ (the case under consideration), Eq. (3.12) takes the form

$$\nabla \cdot \mathbf{v} = 0, \tag{3.13}$$

where

$$\mathbf{v} = \nabla\Lambda - q\nabla p. \tag{3.14}$$

Therefore, the MIM (for $\alpha^2 = 1$) can be regarded as an incompressible "spin fluid," in which the velocity is given by (3.14). Formula (3.14) enables us to find the expression for the velocity of the configuration (3.7), (3.7$'$). In doing so, from Eqs. (3.14), (3.10), and (3.11) we obtain

$$\begin{aligned} v_1 &= \frac{\phi_y}{2} = \frac{1}{r}\sin\theta\,\mathrm{dn}(u, k^2), \\ v_2 &= -\frac{\phi_x}{2} = -\frac{1}{r}\cos\theta\,\mathrm{dn}(u, k^2), \end{aligned} \tag{3.15}$$

where v_1 and v_2 are the components of $\mathbf{v}$ along the x and y axes, respectively.

We note that $|v|^2 \equiv H_\phi$ [see (3.9$'$)]. Then, the contribution to the total energy density due to the field ϕ can be interpreted essentially as the kinetic energy density of the configuration (3.7), (3.7$'$). The nonlinear excitation (3.7) allows us to build up a configuration endowed with a fractional topological charge. This can be done for $k = 1$. In fact, in this case $\mathrm{sn}(u, k) \to \tanh u$ and $\mathrm{dn}(u, k) \to \mathrm{sech} u$. Thus, from (3.7) we have

$$S_1 = \frac{r^2 - 1}{r^2 + 1}\cos\theta, \quad S_2 = \frac{r^2 - 1}{r^2 + 1}\sin\theta, \quad S_3 = \frac{2r}{r^2 + 1}. \tag{3.16}$$

The auxiliary field ϕ corresponding to (3.6) can be derived from (1.6). It reads[13]

$$\phi = 4\arctan r, \tag{3.17}$$

apart from a constant of integration.

The Hamiltonian density (3.9) related to the configuration (3.16) or (3.17) becomes

$$H = \frac{1}{2r^2} + \frac{4}{(1 + r^2)^2}, \tag{3.18}$$

where the terms on the right-hand side are the contribution of the magnetic part and of the field ϕ, respectively.

The topological charge density is

$$Q = \frac{1}{2}(\phi_{xx} + \phi_{yy}) = 2\frac{1 - r^2}{r(1 + r^2)^2}. \tag{3.19}$$

Therefore, the total topological charge

$$Q_T = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Q\,dx\,dy \tag{3.20}$$

vanishes. Anyway, starting from (3.16), we can construct a static solution endowed with a fractional topological charge $Q_T = +\frac{1}{2}$, i.e.,

$$\mathbf{T} = \sigma(1 - r)\mathbf{S} + \sigma(r - 1)\mathbf{S}_0, \tag{3.21}$$

where σ stands for the step function, $\mathbf{S}$ is given by (3.16), and $\mathbf{S}_0 \equiv (0, 0, 1)$.

A configuration having an opposite topological charge, $Q_T = -\frac{1}{2}$, can also be found. This reads

$$\mathbf{T}' = \sigma(r - 1)\mathbf{S} + \sigma(1 - r)\mathbf{S}_0. \tag{3.22}$$

The static solutions (3.21) and (3.22) bear some analogies with other field configurations provided by a fractional topological charge ($Q_T = \pm\frac{1}{2}$), such as, for instance, the merons discovered in the two-dimensional O(3) nonlinear σ model and in four-dimensional non-Abelian gauge theory.[17]

Now let us deal with Eq. (3.4). This can be considered as an equation of the sinh-Gordon (sinh-Poisson) type in the variable $u = \ln r$. It is related to the description of negative-temperature configurations in the theory of vortex filaments in ^{4}He. It affords the solution

$$\gamma = 4\,\mathrm{arctanh}[\sqrt{k}\,\mathrm{sn}(v, k^2)], \tag{3.23}$$

where

$$v = \sqrt{\frac{c - \lambda^2}{8k}}(u - u_0), \tag{3.24}$$

c and u_0 are constants of integration, $c > \lambda^2$, and k is a positive number such that

$$k = \frac{c + 3\lambda^2 - \sqrt{8\lambda^2(c + \lambda^2)}}{c - \lambda^2} < 1. \tag{3.25}$$

The condition $c > \lambda^2$ ensures the reality of (3.23).

The spin-field components can be obtained with the aid of (3.23) by replacing into (1.4) the stereographic variable

$$\zeta = e^{i(\theta+\theta_0)\lambda} \sqrt{k}\,\mathrm{sn}(v,k^2) \tag{3.26}$$

[see (1.7), where $a(z)$ is given by (3.1) and $a_0 = |a_0|e^{i\theta_0}$]. Putting for simplicity $\theta_0 = 0$, these are

$$S_1 = \frac{2\sqrt{k}\,\mathrm{sn}(v,k^2)\cos(\lambda\theta)}{1 - k\,\mathrm{sn}^2(v,k^2)}, \tag{3.27a}$$

$$S_2 = \frac{2\sqrt{k}\,\mathrm{sn}(v,k^2)\sin(\lambda\theta)}{1 - k\,\mathrm{sn}^2(v,k^2)}, \tag{3.27b}$$

$$S_3 = \frac{1 + k\,\mathrm{sn}^2(v,k^2)}{1 - k\,\mathrm{sn}^2(v,k^2)}. \tag{3.27c}$$

On the other hand, by choosing $g(z) = \sqrt{a(z)}\,\psi(r)$ and $f(z^*) = \sqrt{a^*(z^*)}$, Eq. (1.6) yields

$$\phi_r = \frac{2\lambda}{r}\left(\frac{1 + k\,\mathrm{sn}^2(v,k^2)}{1 - k\,\mathrm{sn}^2(v,k^2)}\right) \tag{3.28}$$

and $\phi_\theta = 0$.

By integrating (3.28), we get

$$\phi(r) = 2\lambda\sqrt{\frac{8k}{c-\lambda^2}}\,[2\Pi(k,v;k^2) - F(v;k^2)] + \text{ const}, \tag{3.29}$$

where $F(v;k^2)$ and $\Pi(k,v;k^2)$ denote the elliptic integral of the first and the third kinds, respectively, i.e.,

$$F(v;k^2) = \int_0^{\mathrm{sn}(v,k^2)} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{3.30}$$

and

$$\Pi(k,v;k^2) = \int_0^{\mathrm{sn}(v,k^2)} \frac{dt}{(1-kt^2)\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{3.31}$$

Looking at (3.28) and (1.1b) the topological charge density is given by

$$Q = \phi_{rr} + \frac{1}{r}\phi_r = \frac{\lambda}{r}\frac{d}{dr}\left(\frac{1 + k\,\mathrm{sn}^2(v,k^2)}{1 - k\,\mathrm{sn}^2(v,k^2)}\right), \tag{3.32}$$

which leads to a vanishing total topological charge. With the aid of Eqs. (3.27) and (3.9), we can easily evaluate the energy of the spin-field configuration (3.28), (3.29). We shall omit here its explicit expression.

Concerning the limit cases $k = 0$ and $k = 1$, only the latter will be considered, because the former leads to a complex value of γ. The case $k = 1$, which corresponds to $\lambda = 0$, yields [see (3.27)]

$$S_1 = \cos\theta_0 \sinh[2(au + b)], \tag{3.33a}$$

$$S_2 = \sin\theta_0 \sinh[2(au + b)], \tag{3.33b}$$

$$S_3 = \cosh[2(au + b)], \tag{3.33c}$$

where a, b, θ_0 are real constants and $u = \ln r$. The auxiliary field ϕ turns out to be a constant and the topological charge density Q is zero [see (3.28) and (3.32)], while for the energy density we obtain

$$H = 2a^2e^{2u}\cosh[4(au + b)]. \tag{3.34}$$

In analogy to the IM, using Eq. (2.8), we can introduce special dynamical solutions for the MIM as well. In this case the CHF furnishes

$$\rho(t) = e^{-i(Et+D)} \tag{3.35a}$$

and

$$\psi_{rr} + \frac{\psi_r}{r} + \frac{\lambda^2}{r^2}\psi\left(\frac{1-\kappa^2\psi^2}{1+\kappa^2\psi^2}\right) + E\psi = 2\frac{\kappa^2\psi_r^2\psi}{1+\kappa^2\psi^2}. \tag{3.35b}$$

By means of the transformations $\psi = \tan\frac{\gamma}{2}$ (for $\kappa^2 = 1$) or $\psi = \tanh\frac{\gamma}{2}$ (for $\kappa^2 = -1$) and $u = \ln r$, we get

$$\gamma_{uu} + \frac{\lambda^2}{2}\sin 2\gamma + e^{2u}E\sin\gamma = 0 \qquad (\kappa^2 = 1), \tag{3.36}$$

$$\gamma_{uu} + \frac{\lambda^2}{2}\sinh 2\gamma + e^{2u}E\sinh\gamma = 0 \qquad (\kappa^2 = -1). \tag{3.37}$$

These equations look as, respectively, a double sine-Gordon and a double sinh-Gordon equation with variable coefficients. Indeed, they resemble formally those obtained from (2.10) and (2.11) by setting $\lambda = 0$. Therefore, in this case both the IM and the MIM allow configurations having the same characteristics. Finally, we notice that for small γ Eqs. (3.36) and (3.37) can be linearized to give equations of the Bessel type.[18]

IV. CONCLUSIONS

We have investigated two extended versions of the continuous Heisenberg model in 2+ 1 dimensions using the Hirota technique. The first system is the Ishimori model, while the second one has been introduced in Ref. 13 and can be regarded as a modified version of the former. The basic motivations for a comparative study of these models are the following: (i) the IM allows a Lax pair representation, but it seems to be not endowed with an Hamiltonian structure; (ii) the MIM admits a Lax pair only for special values of the auxiliary field (conversely, it can be described by a Hamiltonian); (iii) the models can be formulated in an unified manner. To the aim of clarifying the possible phenomenological aspect of the systems, we have looked for a class of solutions starting from the same ansatz for the stereographic variable ζ involved in the Hirota representation. We have found exact configurations for the two models under consideration both in the compact and in the noncompact cases. For the IM, these are static and time-dependent solutions connected with certain particular forms of the third

Painlevé transcendent and a class of time-dependent solutions whose asympotic behavior allows an energy density of the Yukawa type. For the MIM, we have obtained a class of static solutions expressed in terms of elliptic functions and time-dependent configurations related to a particular form of the double sine-Gordon and the double sinh-Gordon equations with variable coefficients.

On the basis of these results, it turns out that the IM and the MIM may describe quite different physical situations. This emerges in part from the comparison of Eqs. (2.4), (2.5), (2.10), and (2.11) with Eqs. (3.3), (3.4), (3.36), and (3.37). Furthermore, while the IM possesses vortex configurations labeled by an integer topological charge,[3,7] the MIM has meronlike solutions which can be interpreted as vortices[19] characterized by a fractional topological charge. We remark also that for the MIM, which can be regarded as a constrained Hamiltonian system, via (3.14) one can determine explicitly (for $\alpha^2 = 1$) the velocity of the allowed excitations.

Finally, we notice that recently the continuous 2D Heisenberg model has been analyzed within the anyon theory.[20] It has been shown that static magnetic vortices correspond to the self-dual Chern-Simons solitons described by the Liouville equation. The related magnetic topological charge is associated with the electric charge of anyons. This result is a challenge for scrutinizing, in this direction, both the Ishimori model and its modified version.

APPENDIX

Using the operators $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{z^*} = \frac{1}{2}(\partial_x + i\partial_y)$, Eq. (1.5a) and the compatibility condition $\phi_{xy} = \phi_{yx}$ [see (1.6)] can be written, respectively as

$$\begin{aligned}(|f|^2 - \kappa^2|g|^2)\{i(f_t^* g - f^* g_t) - 2(1+\alpha^2)(g f_{zz^*}^* + f^* g_{zz^*} - f_z^* g_{z^*} - f_{z^*}^* g_z)\\ -(1-\alpha^2)[g(f_{zz}^* + f_{z^*z^*}^*) + f^*(g_{zz} + g_{z^*z^*}) - 2f_z^* g_z - 2f_{z^*}^* g_{z^*}]\}\\ -f^* g\{i[f f_t^* - f^* f_t - \kappa^2(g g_t^* - g^* g_t)] - 2(1+\alpha^2)[f f_{zz^*}^* + f^* f_{zz^*} - f_z^* f_{z^*} - f_{z^*}^* f_z\\ -\kappa^2(g g_{zz^*}^* + g^* g_{zz^*} - g_z^* g_{z^*} - g_{z^*}^* g_z)] - (1-\alpha^2)[f(f_{zz}^* + f_{z^*z^*}^*) - 2f_z^* f_z - 2f_{z^*}^* f_{z^*} + f^*(f_{zz} + f_{z^*z^*})\\ -\kappa^2(g g_{zz}^* + g g_{z^*z^*}^* - 2g_z^* g_z - 2g_{z^*}^* g_{z^*} + g^* g_{zz} + g^* g_{z^*z^*})]\} = 0 \quad \text{(A1)}\end{aligned}$$

and

$$\begin{aligned}2(\alpha^2+\beta^2)\{\Delta(f f_{zz^*}^* + \kappa^2 g g_{zz^*}^* - \text{c.c.}) - [(f f_z^* + \kappa^2 g g_z^*)(f f_{z^*}^* + \kappa^2 g g_{z^*}^*) - \text{c.c.}]\}\\ -(\alpha^2-\beta^2)\{[f(f_{zz}^* + f_{z^*z^*}^*) + \kappa^2 g(g_{zz}^* + g_{z^*z^*}^*) - \text{c.c.}]\Delta + [(f^* f_z + \kappa^2 g^* g_z)^2 + (f^* f_{z^*} + \kappa^2 g^* g_{z^*})^2 - \text{c.c.}]\} = 0, \quad \text{(A2)}\end{aligned}$$

with

$$\Delta = (|f|^2 + \kappa^2|g|^2).$$

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