# Nonuniversality in random-matrix ensembles with soft level confinement

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Two families of strongly non-Gaussian random-matrix ensembles (RME's) are considered. They are statistically equivalent to a one-dimensional plasma of particles interacting logarithmically and confined by the potential that has the long-range behavior $V(\epsilon) \sim |\epsilon|^\alpha$ $(0 < \alpha < 1)$, or $V(\epsilon) \sim \ln^2 |\epsilon|$. The direct Monte Carlo simulations on the effective plasma model show that the spacing distribution function (SDF) in such RME's can deviate from that of the classical Gaussian ensembles. For power-law potentials, this deviation is seen only near the origin $\epsilon \sim 0$, while for the double-logarithmic potential the SDF shows the crossover from the Wigner-Dyson to Poisson behavior in the bulk of the spectrum.

The classical theory of random matrices (RMT) developed by Wigner, Dyson, and Mehta[1] provides a statistical description of the energy levels in a variety of quantum chaotic systems. In this way, one of the simplest statistical characteristics is the probability distribution $P(s)$ of the spacing between nearest-neighbor eigenvalues. In the framework of the classical RMT, the spacing distribution function (SDF) follows very closely a universal curve known as the Wigner surmise.[1] Its most important characteristic is the vanishing of $P(s)$ at $s = 0$, which demonstrates level repulsion. In contrast, for classically nonchaotic systems, the random energy levels are described by another universal distribution, the Poisson statistics, which assumes all levels to be uncorrelated.

Both universal statistics are realized in a disordered system of noninteracting electrons. The metal phase that exists for relatively weak disorder, was proved[2] to be described by the Wigner-Dyson statistics, while the level statistics in the insulator phase is close to the Poisson distribution.

The transition between these two phases, known as the Anderson transition, has much in common with the critical phenomena in second-order phase transitions and can be described by the scaling approach.[3] Using scaling arguments, one can show that in the critical region near the Anderson transition there should exist a third universal statistics.[4] The detailed scaling analysis done recently[5] showed this statistics to be drastically different from both the Wigner-Dyson and the Poisson statistics, the corresponding spectral correlation functions being characterized by nontrivial exponents related to the correlation length exponent $\nu$.

It is of great interest to see if a description of the critical statistics in terms of random matrices is still possible. Clearly, if this is the case, the corresponding matrix ensembles must be of a completely different kind from the ones belonging to the Wigner-Dyson universality class or from those leading to the Poisson statistics. Our interest in this direction was prompted by another recent discovery[6] of a new family of random matrices, a one-parameter solvable model, that displays a crossover in the spacing distribution from a highly correlated Wigner-Dyson distribution to a completely uncorrelated Poisson distribution when the parameter is varied.

In this paper we have tried to establish whether the nonclassical behavior of level correlations is a generic feature shared a by broader class of random matrix models. We will consider two strongly non-Gaussian ensembles of random matrices. We will show that the first one breaks the Wigner-Dyson universality only locally, in the center of the spectrum. It will nevertheless allow us to understand better the second class of models, which is similar to the exactly solvable model studied in Ref. 6. Here we will show that this class of random matrices indeed breaks the Wigner-Dyson universality globally and displays a crossover to a Poisson-like distribution.

Let us consider a physical system described by an $N \times N$ random matrix $H$ whose eigenvalues $\{\epsilon_n\}, n = 1, ..., N$ will also be randomly distributed. Within the maximum entropy ansatz[7,8] for describing the eigenvalue distribution at a given mean level density $\rho(\epsilon)$, we can use the effective plasma model introduced by Dyson,[1] where the joint probability density function $P(\{\epsilon\})$ is mapped onto the Gibbs distribution of a classical one-dimensional plasma of fictitious particles with a pairwise logarithmic repulsion $-\ln|\epsilon_n - \epsilon_m|$ and a one-particle potential $V(\epsilon)$ to keep the system confined:

$$P(\{\epsilon\}) = Z^{-1} \exp[-\beta \mathcal{H}(\{\epsilon_n\})], \tag{1}$$

$$\mathcal{H}(\{\epsilon_n\}) = -\sum_{i<j} \ln|\epsilon_i - \epsilon_j| + \sum_i V(\epsilon_i). \tag{2}$$

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Here $Z$ is the partition function and $\beta$, which plays the role of an inverse temperature in the corresponding Gibbs ensemble, is related to the symmetry of the original random matrix ensemble and is equal to 1, 2, or 4 for orthogonal, unitary, and symplectic ensembles, respectively.[1]

When the confining potential is quadratic, the model is the Gaussian ensemble studied by Wigner, Dyson, and Mehta and the corresponding SDF is very close to the Wigner surmise. The inclusion of higher powers of $\epsilon^2$ was shown[9] to have no effect on the SDF in the limit $N \to \infty$. Until very recently the confining potential was believed to be irrelevant for level correlations in the thermodynamic limit $N \to \infty$.

However, a recent work[6] has demonstrated that this is not the case. For some specific one-parameter family of confining potentials $V_p(\epsilon)$, the exact solution in terms of the non-classical $q$-orthogonal polynomials was found. It showed the level correlations to deviate from the conventional Wigner-Dyson form as the parameter $p$ increases, the SDF approaching the Poisson distribution for large values of $p$. This was associated with the asymptotics of the confining potential $V_p(\epsilon) \sim \ln^2|\epsilon|$ for $|\epsilon| \gg 1$ that is an extremely "soft" confinement as compared to the Gaussian confining potential $V(\epsilon) = \epsilon^2$.

In this paper two main questions will be addressed. The first one is how soft should the confining potential be in order to see deviations from the classical Wigner surmise. The second question is whether the exact solution found in Ref. 6 represents the generic features of *all* models with the double-logarithmic long-range behavior of the confining potential.

In order to answer the first of these questions, we consider a family of power-law potentials:[10]

$$V(\epsilon) = \frac{A}{2}|\epsilon|^\alpha, \tag{3}$$

where $A > 0$ and $\alpha > 0$ are two constant parameters. For $\alpha = 2$, Eq. (3) reduces to the Gaussian quadratic confinement, while in the limit $\alpha \to 0$ the combination of such power-law potentials reproduces the double-logarithmic potential

$$V(\epsilon) = \ln^2|\epsilon| = \lim_{\alpha\to 0}[\alpha^{-2}(|\epsilon|^\alpha - 1)^2]. \tag{3$'$}$$

We will see that the level statistics exhibits *two* sharp transitions when the parameter $\alpha$ decreases. The first one occurs at $\alpha = 1$, and it is connected with the breakdown of translational invariance in the eigenvalue space that is present for $\alpha \geq 1$ in the limit $N \to \infty$. For $\alpha < 1$, the SDF shows a nonclassical, $\alpha$-dependent behavior only near the center of the spectrum $\epsilon = 0$. The second critical value is $\alpha = 0$. For confining potentials that increase only logarithmically, the SDF turns out to deviate from the Wigner-Dyson form everywhere in the bulk of the spectrum.

The first transition is seen already within the mean-field (MF) approximation suggested by Dyson.[1] Let us define $\rho(\epsilon) = \sum_i \delta(\epsilon - \epsilon_i)$. By substituting this definition into Eq. (2), one obtains the continuous version of the energy functional $\mathcal{H}[\rho]$ in terms of $\rho(\epsilon)$. The extremum of this functional corresponds to an equilibrium of the effective plasma expressed by the equation

$$\int d\epsilon' \langle\rho(\epsilon')\rangle \ln|\epsilon - \epsilon'| = V(\epsilon) + c, \tag{4}$$

where $\langle\rho(\epsilon)\rangle$ is the mean density, and the Lagrange multiple $c$ is to be found from the normalization condition $\int\langle\rho(\epsilon)\rangle\, d\epsilon = N$.

The solution $\rho_{\mathrm{MF}}(\epsilon)$ to the MF, Eq. (4), confined to the region $-D < \epsilon < D$, can be found using the Cauchy method[11] and is given by

$$\rho_{\mathrm{MF}}(\epsilon) = \frac{1}{\pi^2}\sqrt{D^2 - \epsilon^2}\,\mathrm{Re}\int_0^D \frac{dV/d\xi}{\sqrt{D^2 - \xi^2}}\frac{\xi d\xi}{\xi^2 - \epsilon_+^2}, \tag{5}$$

where $\epsilon_+ = \epsilon + i0$ and the band edge $D$ is to be found from the normalization condition.

For $\alpha \geq 1$ (strong confinement) the main contribution to the integral in Eq. (5) is made by the region $\xi \sim D$. In the thermodynamic limit $N \to \infty$ the band edge is also divergent, $D \to \infty$. Therefore, for any fixed $|\epsilon| \ll D$, one can neglect the $\epsilon$ dependence in the integrand of Eq. (5). Then the mean level density tends to a constant $\rho \sim N^{1-1/\alpha}$, signaling the translational invariance in the $\epsilon$ space.

However, for $\alpha < 1$ (weak confinement), the integral in Eq. (5) is convergent even in the limit $D \to \infty$. The corresponding limiting function $\rho_{\mathrm{MF}}^\infty(\epsilon) \propto |\epsilon|^{\alpha-1}$ can be easily found as the limit $z = \epsilon/D \to 0$ of the exact solution $\rho_{\mathrm{MF}}(\epsilon)$ to Eq. (5):

$$\rho_{\mathrm{MF}}(\epsilon) = AC_\alpha \frac{\sqrt{1-z^2}}{2\pi|\epsilon|^{1-\alpha}}\, F\left(\frac{1}{2}, \frac{1+\alpha}{2}; \frac{3}{2}; 1 - z^2\right). \tag{6}$$

Here $C_\alpha = \frac{\alpha^2 2^{-\alpha}\Gamma(\alpha)}{\Gamma(\alpha/2)\Gamma(1+\alpha/2)}$, $F(a,b;c;x)$ is a hypergeometric function, and the band edge is given by $D = 2\left(\frac{N\Gamma^2(\alpha/2)}{2A\Gamma(\alpha)}\right)^{1/\alpha}$.

Thus for $\alpha < 1$ the mean density, Eq. (6), shows the lack of translational invariance in the large-$N$ limit and is singular at $\epsilon = 0$.

This singularity, however, appears only in the MF approximation. An exact treatment for $\beta = 2$, which is based on the representation in terms of orthogonal polynomials, shows the value $\langle\rho(0)\rangle$ to be finite:[12]

$$\langle\rho(0)\rangle = \frac{(A/\pi)^{1/\alpha}}{(2/\alpha)\Gamma(1/\alpha)} \sum_{i=0}^{\infty}\left[\frac{\Gamma(i+1/2)}{\Gamma(i+1)}\right]^{2/\alpha}. \tag{7}$$

Thus, in the case of weak confinement the MF approximation fails to describe the mean level density near the origin. It is natural to suppose that all the level correlation functions will also have a nonclassical form in this region.

In order to study the correlation functions and, in particular, the SDF, we have exploited the Coulomb plasma analogy and carried out systematic Monte Carlo (MC) simulations on the one-dimensional classical system whose probability distribution is given by Eqs. (1)

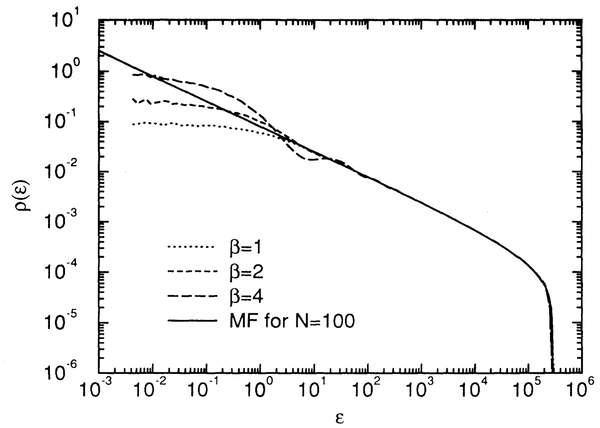


FIG. 1. Particle density for power-law potential with $\alpha = 0.5$: The Monte Carlo results for $\beta = 1, 2, 4$ are plotted vs the MF density.

and (2). As a check that this method works and is numerically accurate, we have first studied the three Gaussian ensembles whose density, two-point correlation functions, and spacing distribution are exactly known.[1] For these systems the MC simulations turned out to work extremely well for each of these quantities. For the power-law potential, Eq. (3), for $\alpha < 1$, we have carried out simulations of systems up to $N = 200$ particles. The simulations are very stable even for smaller $N$, and we have typically worked with $N = 100$. The evaluation of the mean density is straightforward. In Fig. 1 we plot this quantity for $\alpha = 0.5$ and $\beta = 1, 2, 4$. The Monte Carlo result agrees very well with $\rho_{\rm MF}$ found from Eq. (6) except around the origin, where the simulation is more accurate and correctly gives a finite density at $\epsilon = 0$. For $\beta = 2$ the Monte Carlo value coincides with that found from Eq. (7). The simulations with different numbers of particles illustrate another important property of the particle-density for weak confinement ($\alpha < 1$), that is the "incompressibility" of the core of the particle-density distribution. In contrast to the $\alpha \geq 1$ case, for $\alpha < 1$ the confining potential is so weak that it does not "compress" particles in the core region near the origin. On adding more particles to the system, these get positioned about the wings of the distribution, rather than distribute themselves homogeneously throughout the spectrum, as in the case of strong confinement ($\alpha \geq 1$). The particle density $\rho(\epsilon)$ in the core region is almost independent of the number of particles but depends on the inverse temperature $\beta$.

The latter dependence is also a characteristic feature of the weak confinement. For strong confinement, the $\beta$ dependence is present only in $1/N$ corrections to the mean density and thus is negligible. It leads, in particular, to the independence of the mean level density of the symmetry of the Hamiltonian. For random matrix ensembles with weak confinement considered here, all of the $\beta$ dependence is "accumulated" in the core region near the origin that contains a few levels on the average.

The MC evaluation of the SDF is, in principle, also straightforward. However, in order to compare it with the Wigner surmise we need to rescale the particle positions $\epsilon$ so that the average spacing between two adjacent particles is one. This is known as an "unfolding procedure" and is always used in numerical calculations of spectral correlations.[13] It consists in introducing the new variable $\sigma$ instead of $\epsilon$ according to a map:

$$\sigma(\epsilon) = \int_0^{\epsilon} \langle \rho(\epsilon') \rangle \, d\epsilon'. \tag{8}$$

The mean density is trivially unity as a function of this variable.

In order to study the SDF in the *bulk*, we use the MF solution Eq. (6) for unfolding according to Eq. (8). We checked that the obtained unfolded mean density is consistently equal to one, except close to the origin and the band edge. The unfolded spacing $\tilde{P}(\sigma)$ turned out, within our numerical accuracy, identical to the Wigner surmise for any $\alpha$. Therefore, in the *bulk* of the spectrum, the Wigner-Dyson universality holds for the power-law weakly confining potentials.

However, this universality is broken around the origin. To show this, we consider a reference particle fixed at the origin. The unfolded spacing must be evaluated in a different way here, since the MF density is not accurate. Therefore, we perform the unfolding by computing

$$\tilde{P}(\sigma) = \left[\frac{P(\epsilon)}{\rho(\epsilon)}\right]_{\epsilon=\epsilon(\sigma)}, \tag{9}$$

where the function $\epsilon(\sigma)$ is obtained by numerically inverting Eq. (8) and using for $\rho(\sigma)$ the density evaluated by MC simulations. The result is shown in Fig. 2, where we plot the unfolded SDF for few values of $\alpha < 1$ and $\beta = 1$ in comparison with the classical spacing of the Gaussian ensemble. We can clearly see that $\tilde{P}(\sigma)$ for small $\sigma$ does not follow the Wigner-Dyson universal behavior $\sigma^{\beta}$ and starts out roughly like $\sigma^{\beta/\alpha}$. If we assume that the new variable $\sigma$ is proportional to $\epsilon^{\alpha}$, as obtained from Eq. (8) using $\rho(\epsilon) \propto \epsilon^{\alpha-1}$, this behavior would correspond to

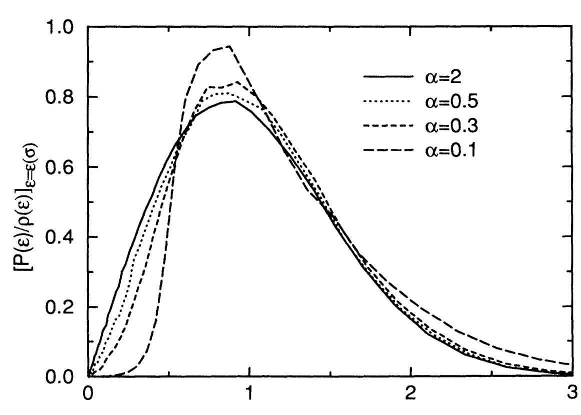


FIG. 2. Nearest-neighbor spacing distribution in the middle of the spectrum for $\beta = 1$ and different values of $\alpha$. The $\alpha = 2$ case corresponds to the Gaussian orthogonal ensemble. Similar results are obtained for $\beta = 2, 4$

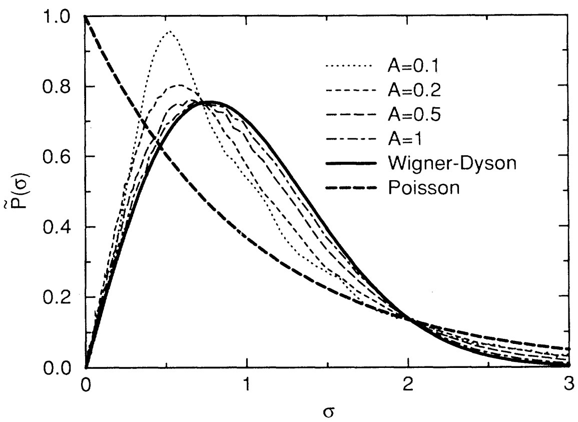


FIG. 3. Nearest-neighbor spacing distribution for the logarithmic confining potential, measured in the bulk of the spectrum at $\beta = 1$. By decreasing the parameter $A$, the spacing deviates from the universal Wigner-Dyson distribution approaching the Poisson distribution (both also plotted).

$P(\epsilon)/\rho(\epsilon) \propto \epsilon^{\beta}$. Notice also that the decay of the SDF for $s \gg 1$ depends on $\alpha$ and is slower than that for the Wigner-Dyson distribution. We conclude that for the power-law weak confining potential, the Wigner-Dyson universality is broken only locally around $\epsilon \sim 0$. This conclusion is also reached for $\beta = 2$, using the independent method of orthogonal polynomials.[12]

Now we consider the second class of random matrices, with the confining potential that behaves asymptotically like $V(\epsilon) \propto \ln^2 |\epsilon|$. Since our goal is to study the eigenvalue correlations *in the bulk* of the spectrum, we choose for numerical simulations the regularized confining potential that is equal to zero at the origin:

$$V(\epsilon) = \frac{A}{2} \ln^2(1 + B|\epsilon|), \tag{10}$$

where $A$ and $B$ are parameters of order 1.

In Fig. 3 we show the *bulk* SDF for $A = 1, 0.5, 0.2, 0.1$, for the orthogonal symmetry ($\beta = 1$), together with the spacing distribution of the corresponding Gaussian ensemble for comparison. We can clearly see that for small enough $A$ the spacing distribution departs from the Wigner distribution and shows an incipient tendency to become more Poisson-like when $A$ is further reduced. Similar deviations from the Gaussian ensemble occur also for the unitary and symplectic case ($\beta = 2, 4$).

This is very similar to the crossover found analytically in Ref. 6 for the exactly solvable model with the double-logarithmic long-range behavior of the confining potential. We can conclude, therefore, that the crossover is indeed not an exclusive property of the exactly solvable model and is more likely a generic feature shared by all the random matrix ensembles with the double-logarithmic asymptotics of the confining potential.

Moreover, the crossover in the spacing distribution displayed by this family of random matrices is remarkably similar to the transition observed for $\beta = 1$ in exact numerical calculations[4,14,15] on disordered tight-binding models going through the Anderson transition. Upon increasing disorder, the function $P(s)$ moves away from the Wigner surmise, yet displays a remarkable fixed point at $\sigma = 2$, where all curves meet. This property is clearly obeyed also by the family $P(\sigma, A)$ of this random matrix model. Thus it is really possible that random matrix theory is able to describe the energy level statistics at the Anderson transition.

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