Conductance distribution in quantum dots with point contacts

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By extending the supersymmetry formalism we develop a statistical description for the conductance through a quantum dot coupled to external leads with point contacts. The electronic states inside the dot are formed by both the random and regular parts of confinement potential and have a finite lifetime due to the presence of the external contacts. The explicit form of the distribution function for the one-channel conductance is obtained. This function depends in a universal way on the transmission coefficients between the dot and the leads. The generalization of the results for the low-frequency impedance of the dot is suggested.

A number of problems concerning quantum transport in small electronic device of nanometer dimensions attracted recently a lot of attention.[1] The first problem is that of the Coulomb interaction between electrons inside the dot, which becomes more and more important as the size of the dot diminishes. However, its role for a still large system is reduced primarily to charging effects and, therefore, the Coulomb interaction can rather simply be incorporated into the theory of noninteracting particles.

Another problem is that the shape or the host potential of the small electron devices is quite irregular. In the presence of such irregularities, the electron behavior inside the dot becomes chaotic and its description requires considerable effort. One can, for example, use the same statistical approaches developed in nuclear physics for studying properties of complex nuclei.[2] These statistical theories are based on the hypothesis that a complex system can be described by random matrices.[3] Therefore, in Refs. 2 and 4–6 formulas from random matrix theory (RMT) were used to describe transport through the dots.

However, it is always desirable to start from a microscopic Hamiltonian and to perform direct calculations. Using the supersymmetry technique,[7] one can calculate starting from a Hamiltonian with disorder different correlation functions,[8] as well as distributions of such quantities as local density of states[9,10] or conductance.[11] These quantities are very important because they give some information about statistics of wave functions and can be compared directly with the experimental results.

The present work addresses the conductance statistics in quantum dots. In Ref. 11, the conductance of a dot with nonfluctuating level widths was studied. Level widths can be finite due to either the coupling of the dot to leads or to different types of inelastic scattering. In real devices at low temperature, the level widths are primarily due to the measurement contacts. Therefore, they fluctuate, following the fluctuation of the value of eigenfunctions near the contacts. In this work we give results for conductances in quantum dots with point contacts by taking into account the above-mentioned fluctuations of level widths. The size of the contact area is assumed to be less than the wavelength of Fermi electrons. The tunneling properties of the point contact are described by the transmission coefficients, which can be experimentally controlled by changing the height of separation barrier. In such a system, the level widths strongly fluctuate and we expect to get a new universal distribution for the conductance.

The present model of a quantum dot with nonideal point contacts was already studied in Ref. 6 by using the RMT. Here we find the distribution of conductance for this model within the supersymmetry method. In comparison with Ref. 6 we consider a more general case, where the contacts are not equal. Additionally, we find dependence of the distribution function on an external parameter, which corresponds to the low frequency of an applied potential or to the frequency of phase-breaking processes inside a dot.

We start with the following Hamiltonian for the dot

$$H_d = H_{d0} + U_1(\mathbf{r}), \tag{1}$$

$$H_{d0} = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 - \epsilon + U_0(\mathbf{r}), \tag{2}$$

where $U_0(\mathbf{r})$ is the regular part of a confining potential, $U_1(\mathbf{r})$ stands for a random potential, and ϵ is the Fermi energy. Here we will present the complete calculation only for the unitary case (broken T invariance by a magnetic field 12).

The point contacts between the dot and the external drain and source leads are described by the following extra term H_c in the Hamiltonian

$$H_c = H_{c0} + H_{cL}, \tag{3}$$

where

$$H_{c0} = \frac{i\alpha}{2\pi}\Delta \tag{4}$$

and

$$H_{cL} = \frac{i}{2\pi\nu}\left[\alpha_1\delta\left(\mathbf{r}-\mathbf{r}_1\right)+\alpha_2\delta\left(\mathbf{r}-\mathbf{r}_2\right)\right]. \qquad (5)$$

Equation (5) can be derived starting with the term

$$H_t = \sum_{j=1,2}\left[J_j\psi_d^+(\mathbf{r}_j)\psi_l(\mathbf{r}_j)+\text{H.c.}\right],$$

which describes tunneling between the dot and the external leads. Integrating over the electron degrees of freedom of the lead [operator $\psi_l^+(\mathbf{r})$] provided that the lead is just a bulk metal, we come to Eq. (5). The dimensionless parameters $\alpha_{1,2}$ in Eq. (5) are then equal to

$$\alpha_{1,2} = 2\left(\pi J_{1,2}\right)^2 N_{1,2}(\epsilon)\,\nu\,, \qquad (6)$$

where ν is the mean density of states in the dot, $\Delta = (\nu V)^{-1}$ is the mean level spacing, V is the volume of the dot, and $N_{1,2}(\epsilon)$ is the density of states in the leads $1,2$.

The first term in the Hamiltonian H_c [Eq. (4)] is introduced to take into account the effect of incoherent scattering by phonons and the electron exchange between the dot and the host matrix. An extended contact whose dimensions are comparable with the dot size can also be described by α, with α being controlled by the barrier height of this contact.

In the neglect of the discrete nature of electron levels inside the dot, the parameters $\alpha_{1,2}$ in Eq. (5) determine directly the transmission coefficients $t_{1,2}$ of point contacts

$$t_{1,2} = 2\alpha_{1,2}/\left(1+\alpha_{1,2}/2\right)^2\,. \qquad (7)$$

Notice that the transmission coefficient takes the maximal value $t_{1,2}=1$ at an intermediate coupling $\alpha_{1,2}=2$, but not in the limit $\alpha_{1,2}\to\infty$.

For the discrete states, switching of the tunneling term (5) leads to broadening the levels. At small $\alpha_{1,2}\ll 1$, the nth level width is

$$\gamma_n = \left[\alpha_1|\psi_n(r_1)|^2+\alpha_2|\psi_n(r_2)|^2\right]/(\pi\nu)\,, \qquad (8)$$

where $\psi_n(r)$ is the wave function. Now the level width fluctuates from one level to another and these fluctuations correlate with wave function fluctuations and for not too small $\alpha_{1,2}$ also with energy spectrum. It makes the calculation of the conductance more difficult.

We consider the following generalization of the Landauer-type formula for the frequency-dependent conductance:

$$G(\omega) = \frac{2e^2}{h}\,\frac{\alpha_1\alpha_2}{(\pi\nu)^2}\,G^R_{\epsilon+\omega/2}(\mathbf{r}_1,\mathbf{r}_2)\,G^A_{\epsilon-\omega/2}(\mathbf{r}_2,\mathbf{r}_1)\,, \qquad (9)$$

where $G^{R,A}$ are the exact retarded and advanced Green functions for the total Hamiltonian of the dot connected to the leads. Having integrated over the degrees of freedom of the leads we imply $G^{R,A}$ in Eq. (9) the Green functions of the Hamiltonian

$$H = H_d + H_c\,. \qquad (10)$$

The external frequency ω in Eq. (9) corresponds to the frequency of oscillations of the chemical potentials in the leads with respect to the dot levels.[13,14] According to Ref. 15, the applied ac voltage in the limit $\omega t_L \ll 1$ ($t_L = L^2/D$ is the diffusive time across the dot) is expected to be equivalent to the above oscillations of the chemical potential. The appearance of ω in Eq. (9) is effectively reduced to the replacement $\alpha\to\alpha - i\pi\omega/\Delta$ in the Green functions. Therefore, we can first consider the static conductance $G(\alpha)$ and get the results for dynamic conductance of type (9) from the static one by the analytical continuation over the variable α.

The conductance distribution function $P(g)$ is defined as

$$P(g) = \langle\delta(g - Gh/2e^2)\rangle\,, \qquad (11)$$

where the angular brackets stand for averaging over irregularities in the system described by the term U_1 in Eq. (1). In order to evaluate the function $P(g)$, one has to find the eigenenergies and the eigenfunctions for a fixed disorder, to express the conductance in Eq. (11) in terms of these quantities, and then average over a disorder ensemble. As was shown in Refs. 9–11 for a closed system this task can be carried out within the supersymmetry formalism of Ref. 7. The problem is then reduced to the zero-dimensional supersymmetric σ model, which has the same form as the one derived earlier in Ref. 7. In the present case of an open system due to the locality of the second term H_{cL}, Eq. (5), the form of the appropriate σ model differs essentially from that used previously.

To calculate Eq. (11), we expand the δ function in Eq. (11) in a series in G, use Eq. (9), and write the products $[G^R_\varepsilon(r_1,r_2)G^A_\varepsilon(r_2,r_1)]^m$ in the form of integrals over supervectors ψ. This can be done without increasing the size of the supervector ψ because the Green functions contain only two different energies $\epsilon+i\delta$ and $\varepsilon-i\delta$. For the unitary ensemble the calculations are most simple. By proceeding in the standard way (Ref. 7), we reduce the averages of products of the Green functions to integrals over the supervectors ψ with a Lagrangian containing "interaction" $\left(\bar\psi\psi\right)^2$. This interaction can be decoupled by an integration over 8×8 supermatrices Q. This step allows us to integrate over the supervectors ψ. By collecting now the series, we come to the following expression for the distribution function:

$$P(g) = \langle\delta[g+\alpha_1\alpha_2 g^{12}_{33}(r_1,r_1)\,g^{21}_{33}(r_2,r_2)]\rangle_Q\,, \qquad (12)$$

where $\langle\cdots\rangle_Q$ stands for the functional integral with the weighting factor $\exp(-F)$;

$$F = \frac{1}{2}\int \text{STr}\left[\ln\left(\tilde H + iQ/2\tau\right)\right]_{r,r} dr\,. \qquad (13)$$

In Eq. (13), the symbol STr means the supertrace, and Q is the above-mentioned supermatrix with the additional constraint $Q^2=1$. The notations for the elements in Eq. (12) are the same as in Ref. 7. The Hamiltonian $\tilde H$ in Eq. (13) is defined as

$$\tilde H = H_{d0} + \Lambda H_c\,, \qquad (14)$$

where Λ is the diagonal matrix with elements $\Lambda^{11} = -\Lambda^{22} = 1$, and H_{d0} and H_c are determined by Eqs. (2) and (4). The supersymmetric Green function $g(r,r')$ in Eq. (12) is the solution of the following equation:

$$(\tilde{H} + iQ/2\tau)g(r,r') = i\delta(r-r') \ . \tag{15}$$

When deriving Eqs. (12) and (13), we assumed that the distance between the points r_1 and r_2 much exceeds the mean free path l. Below we consider the limit $\alpha \ll E_c/\Delta$, where $E_c = D/L^2$ is the Thouless energy. In this case we can neglect all spatial fluctuations of Q and retain only the zero spatial harmonics.

Equations (13) and (15) can be further simplified by expanding in $H_c = H_{c0} + H_{cL}$. The result of the expansion is very different for H_{c0} and H_{cL}. As concerns H_{c0}, it is sufficient to take only the first term in the expansion of Eq. (13) for the free energy F and neglect its contribution to the function $g(r,r)$. This approximation is the same that was used in previous works.[7,9,11] When expanding in H_{cL}, one should take into account all terms of the expansion. But the fact that H_{cL} contains only δ functions gives the possibility to write the results in a compact form. For the function $g(r_1,r_1)$, we obtain

$$g(r_1,r_1) = \nu\pi Q(1+\alpha_1\Lambda Q/2)^{-1} \tag{16}$$

and analogously for $g(r_2,r_2)$. The same procedure gives for the free energy F, the equation

$$F = F_0 + \frac{1}{2}\mathrm{STr}\sum_{n=1,2}\ln(1+\alpha_n\Lambda Q/2) \ , \tag{17}$$

where F_0 describes the part of the free energy originated from H_{c0} in Eq. (13),

$$F_0 = (\alpha/4)\mathrm{STr}(\Lambda Q) \ . \tag{18}$$

Equations (12)–(18) are general, but below we present their solution only for the unitary ensemble. In this case the parameterization of the supermatrix Q is most simple[7]

$$Q = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} Q_0 \begin{pmatrix} \bar{u} & 0 \\ 0 & v \end{pmatrix} \ , \tag{19}$$

where

$$Q_0 = \begin{pmatrix} \cos\hat{\theta} & i\sin\hat{\theta} \\ -i\sin\hat{\theta} & -\cos\hat{\theta} \end{pmatrix} \ , \tag{20}$$

and the 4×4 matrix $\hat{\theta}$ has the form

$$\hat{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & i\theta \end{pmatrix} \ , \tag{21}$$

with $0 < \theta < \pi$ and $0 < \theta_1 < \infty$. In Eq. (19), $u = u_1u_2$, and v are the 4×4 unitary submatrices, which are parametrized as

$$u_1 = \exp\begin{pmatrix} 0 & 2\eta \\ -2\bar{\eta} & 0 \end{pmatrix}, \quad u_2 = \exp\begin{pmatrix} i\varphi & 0 \\ 0 & i\chi \end{pmatrix} \ , \tag{22}$$

$$v = \exp\begin{pmatrix} 0 & 2\,i\kappa \\ -2\,i\bar{\kappa} & 0 \end{pmatrix} \ . \tag{23}$$

Above η, κ are anticommuting variables and φ, χ are angles ranging from 0 to 2π.

Let us present a transformation that allows us the evaluation of the integral in Eq. (12). First, we rewrite Eqs. (16) and (17) using the identity

$$[Q_0(1+\alpha_n\Lambda Q_0/2)^{-1}]^{12} = i\sin\hat{\theta}\hat{D}_n^{-1} \ , \tag{24}$$

where

$$\hat{D}_n = \begin{pmatrix} D_{nF} & 0 \\ 0 & D_{nB} \end{pmatrix} \ . \tag{25}$$

The elements D_{nF} and D_{nB} in Eq. (25) are equal to

$$D_{nF} = (1+\alpha_n/2)^2 + \alpha_n(\cos\theta - 1) \ ,$$

$$D_{nB} = (1+\alpha_n/2)^2 + \alpha_n(\cosh\theta_1 - 1) \ .$$

Then the function F in Eq. (17) can be written as

$$F = F_0 + \sum_{n=1,2}\ln(D_{nB}/D_{nF}) \ . \tag{26}$$

With Eqs. (19)–(26), calculations become rather lengthy although straightforward. Finally, we obtain the following expression for $P(g)$

$$P(g) = \frac{d}{dg}\frac{D_1D_2}{A\lambda+B}\left[(\lambda\sinh\alpha+\cosh\alpha)\exp(-\alpha\lambda) + \frac{\sinh\alpha}{\alpha}(\lambda^2-1)\frac{d}{d\lambda}\frac{A}{A\lambda+B}\exp(-\alpha\lambda)\right] \ , \tag{27}$$

where

$$\begin{aligned} g &= t_1t_2(\lambda^2-1)/(D_1D_2) \ , \\ D_{1,2} &= 2+(\lambda-1)t_{1,2} \ , \\ A &= -(\lambda-1)(t_1+t_2-t_1t_2)-2 \ , \\ B &= (\lambda-1)(t_1+t_2-t_1t_2)-2\lambda \ . \end{aligned} \tag{28}$$

Equation (27) is the main result of the present work. We see that the conductance distribution function for the dot with point contacts can be written in an explicit form and even in terms of elementary functions. For $t_1 = t_2 = t$ and $\alpha = 0$, Eq. (27) gives the distribution of Ref. 6, where $P(g)$ was found only in an integral form.

The function $P(g)$ monotonously decreases in the region $0 \le g \le 1$ (λ varies from 1 to ∞). Outside this region $P(g) = 0$, which means that the reduced one-channel conductance (in units $2e^2/h$) cannot be larger than 1. For $\alpha \gg 1$, the distribution function transforms into the one obtained in Ref. 11. In the limit of weak symmetric coupling to external reservoirs ($t_1 = t_2 = t \ll 1$ and $\alpha = 0$), the distribution function can be represented in the form

$$P(g) = \begin{cases} 4/t^2 \ , & g \ll t^2 \\ (t/8)g^{-3/2} \ , & t^2 \ll g \ll 1 \\ t/4 \ , & (1-g) \ll 1 \ . \end{cases} \tag{29}$$

The asymptotic Eq. (29) can be interpreted in terms of resonance tunneling:[11] Since in the limit $t \to 0$ the level width is much smaller than the level separation, the conductance is determined by the level, which is the nearest to the Fermi energy ϵ. The distribution is then formed by fluctuations in the position of this conducting level and by fluctuations of its wave function near the contacts.

In the other limit $t_1 = t_2 = 1$ ($\alpha = 0$), we see from Eq. (27) that

$$P(g) = 1 \tag{30}$$

over the whole interval $(0, 1)$. In this case the level widths are comparable to the level spacing and, therefore, the conductance is provided by many levels. The interference of these contributions to the total amplitude of the transition through the dot can be described as a diffusion in the complex plane.[11] The constraint $g \leq 1$ leads then to the uniform distribution of the transition amplitude inside the unit circle, as we have in Eq. (30). According to Refs. 4–6 the distribution (30) realizes also for a dot coupled to ideal one-mode leads.

If we insert in Eq. (27) α in the form

$$\alpha = -i\pi(\omega + i0)/\Delta \ , \tag{31}$$

we get the distribution for the frequency-dependent impedance $g(\omega)$. With Eq. (27) the average impedance can be found to be ($t_1 = t_2 = t$)

$$\langle g(\omega) \rangle = 2t \int_0^\infty \frac{ds}{(s+1)^2} \exp\left(-\frac{2i\omega s}{\gamma}\right) \times \left[1 + \frac{1 - e^{2ix}}{2ix}\left(1 - \frac{t}{s+1} - \frac{1-t}{(s+1)^2}\right)\right] , \tag{32}$$

where $x = \pi\omega/\Delta$ and $\gamma = t\Delta/\pi$. The oscillating term in x in Eq. (32) comes from the interlevel transitions near the contacts. The rest of Eq. (32) describes the one-level contribution[16] or the Debye losses.[17] The integration over the variable s in Eq. (32) corresponds to the averaging over the distribution of relaxation times or level widths.

In conclusion, the statistics of mesoscopic fluctuation of the conductance in the quantum dot with point contacts has been studied within the supersymmetry formalism.

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