

Generalized circular ensemble of scattering matrices for a chaotic cavity with nonideal leads

P. W. Brouwer

Instituut-Lorentz, University of Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

(Received 6 January 1995)

We consider the problem of the statistics of the scattering matrix S of a chaotic cavity (quantum dot), which is coupled to the outside world by nonideal leads containing N scattering channels. The Hamiltonian H of the quantum dot is assumed to be an $M \times M$ Hermitian matrix with probability distribution $P(H) \propto \det[\lambda^2 + (H - \varepsilon)^2]^{-(\beta M + 2 - \beta)/2}$, where λ and ε are arbitrary coefficients and $\beta = 1, 2, 4$ depending on the presence or absence of time-reversal and spin-rotation symmetry. We show that this ‘‘Lorentzian ensemble’’ agrees with microscopic theory for an ensemble of disordered metal particles in the limit $M \rightarrow \infty$, and that for any $M \geq N$ it implies $P(S) \propto |\det(1 - \bar{S}^\dagger S)|^{-(\beta M + 2 - \beta)}$, where \bar{S} is the ensemble average of S . This ‘‘Poisson kernel’’ generalizes Dyson’s circular ensemble to the case $\bar{S} \neq 0$ and was previously obtained from a maximum entropy approach. The present work gives a microscopic justification for the case that the chaotic motion in the quantum dot is due to impurity scattering.

I. INTRODUCTION

Theoretical work^{1–9} on phase-coherent conduction through cavities in which the classical electron motion can be regarded as chaotic has stimulated recent experiments^{10–13} on conductance fluctuations and weak-localization effects in quantum dots. If the capacitance of the quantum dot is large enough, a description in terms of noninteracting electrons is appropriate (otherwise the Coulomb blockade becomes important^{2,4}).

For an isolated chaotic cavity, it has been conjectured and confirmed by many examples that the statistics of the Hamiltonian H agrees with that of the Gaussian ensemble of random-matrix theory.^{14,15} If the chaotic behavior is caused by impurity scattering, the agreement has been established by microscopic theory: Both the Gaussian ensemble and the ensemble of Hamiltonians with randomly placed impurities are equivalent to a certain nonlinear σ model.^{16,17} Transport properties can be computed by coupling M eigenstates of H to N scattering channels.^{17–20} Since $N \ll M$ this construction introduces a great number of coupling parameters, whereas only a few independent parameters determine the statistics of the scattering matrix S of the system.¹⁷

For transport properties at zero temperature and infinitesimal applied voltage, one only needs to know S at the Fermi energy E_F , and an approach which starts directly from the ensemble of scattering matrices at a given energy is favorable. Following up on earlier work on chaotic scattering in billiards,²¹ two recent papers^{5,6} have studied the transport properties of a quantum dot under the assumption that S is distributed according to Dyson’s circular ensemble.^{22,23} In Refs. 5 and 6 the coupling of the quantum dot to the external reservoirs was assumed to occur via ballistic point contacts (or ‘‘ideal leads’’). The extension to coupling via tunnel barriers (nonideal leads) was considered in Ref. 8. In all cases complete agreement was obtained with results which were obtained from the

Hamiltonian approach.^{4,18,19} This agreement calls for a general demonstration of the equivalence of the scattering matrix and the Hamiltonian approach, for arbitrary coupling of the quantum dot to the external reservoirs. It is the purpose of this paper to provide such a demonstration. A proof of the equivalence of the Gaussian and circular ensembles has been published by Lewenkopf and Weidenmüller,²⁰ for the special case of ideal leads. The present proof applies to nonideal leads as well, and repairs a weak spot in the proof of Ref. 20 for the ideal case.

The circular ensemble of scattering matrices is characterized by a probability distribution $P(S)$ that is constant, that is to say, each unitary matrix S is equally probable. As a consequence, the ensemble average \bar{S} is zero. This is appropriate for ideal leads. A generalization of the circular ensemble which allows for nonzero \bar{S} (and can, therefore, be applied to nonideal leads) has been derived by Mello, Pereyra, and Seligman,^{24,25} using a maximum entropy principle. The distribution function in this generalized circular ensemble is known in the mathematical literature²⁶ as the Poisson kernel,

$$P(S) \propto |\det(1 - \bar{S}^\dagger S)|^{-(\beta N + 2 - \beta)}. \quad (1.1)$$

Here, $\beta \in \{1, 2, 4\}$ is the symmetry index of the ensemble of scattering matrices: $\beta = 1$ or 2 in the absence or presence of a time-reversal-symmetry breaking magnetic field; $\beta = 4$ in zero magnetic field with strong spin-orbit scattering. (In Refs. 24 and 25 only the case $\beta = 1$ was considered.) One verifies that $P(S) = \text{constant}$ for $\bar{S} = 0$. Equation (1.1) was first recognized as a possible generalization of the circular ensemble by Krieger,²⁷ for the special case that \bar{S} is proportional to the unit matrix.

In this paper, we present a microscopic justification of the Poisson kernel, by deriving it from an ensemble of random Hamiltonians, which is equivalent to an ensemble of disordered metal grains. For the Hamiltonian en-

semble, we can use the Gaussian ensemble, or any other ensemble to which it is equivalent in the limit $M \rightarrow \infty$.²⁸ (The microscopic justification of the Gaussian ensemble only holds for $M \rightarrow \infty$.) For technical reasons, we use a Lorentzian distribution for the Hamiltonian ensemble, which in the limit $M \rightarrow \infty$ can be shown to be equivalent to the usual Gaussian distribution. The technical advantage of the Lorentzian ensemble over the Gaussian ensemble is that the equivalence to the Poisson kernel holds for arbitrary $M \geq N$, and does not require taking the limit $M \rightarrow \infty$.

The outline of this paper is as follows: In Sec. II, the usual Hamiltonian approach is summarized, following Ref. 17. In Sec. III, the Lorentzian ensemble is introduced. The eigenvalue and eigenvector statistics of the Lorentzian ensemble are shown to agree with the Gaussian ensemble in the limit $M \rightarrow \infty$. In Sec. IV, we then compute the entire distribution function $P(S)$ of the scattering matrix from the Lorentzian ensemble of Hamiltonians, and show that it agrees with the Poisson kernel (1.1) for arbitrary $M \geq N$. In Sec. V, the Poisson kernel is shown to describe a quantum dot, which is coupled to the reservoirs by means of tunnel barriers. We conclude in Sec. VI.

II. HAMILTONIAN APPROACH

The Hamiltonian approach^{17,20,29} starts with a formal division of the system into two parts, the leads and the cavity [see Fig. 1(a)]. The Hamiltonian of the total system is represented in the following way: Let the set $\{|a\rangle\}$ represent a basis of scattering states in the lead at the Fermi energy E_F ($a = 1, \dots, N$), with N the number of propagating modes at E_F . The set of bound states in the cavity is denoted by $\{|\mu\rangle\}$ ($\mu = 1, \dots, M$). We assume $M \geq N$. The Hamiltonian \mathcal{H} is then given by¹⁷

$$\mathcal{H} = \sum_a |a\rangle E_F \langle a| + \sum_{\mu,\nu} |\mu\rangle H_{\mu\nu} \langle \nu| + \sum_{\mu,a} (|\mu\rangle W_{\mu a} \langle a| + |a\rangle W_{\mu a}^* \langle \mu|). \quad (2.1)$$

The matrix elements $H_{\mu\nu}$ form a Hermitian $M \times M$ matrix H , with real ($\beta = 1$), complex ($\beta = 2$), or real quaternion ($\beta = 4$) elements. The coupling constants $W_{\mu a}$ form a real (complex, real quaternion) $M \times N$ matrix W . The $N \times N$ scattering matrix $S(E_F)$ associated

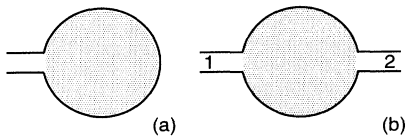


FIG. 1. Schematic drawing of a disordered cavity (gray) attached to a lead. There are N scattering channels in the lead, which are coupled to M bound levels in the cavity. In (a) only one lead is drawn. A system with more leads (b) is described by combining them formally into one lead.

with this Hamiltonian is given by

$$S(E_F) = 1 - 2\pi i W^\dagger (E_F - H + i\pi W W^\dagger)^{-1} W. \quad (2.2)$$

For $\beta = 1, 2, 4$, the matrix S is, respectively, unitary symmetric, unitary, and unitary self-dual.

Usually one assumes that H is distributed according to the Gaussian ensemble,

$$P(H) = \frac{1}{V} \exp\left(-\frac{1}{4}\beta M \lambda^{-2} \text{tr} H^2\right), \quad (2.3)$$

with V a normalization constant and λ an arbitrary coefficient that determines the density of states at E_F . The coupling matrix W is fixed. Notice that $P(H)$ is invariant under transformations $H \rightarrow U H U^\dagger$, where U is orthogonal ($\beta = 1$), unitary ($\beta = 2$), or symplectic ($\beta = 4$). This implies that $P(S)$ is invariant under transformations $W \rightarrow U W$, so that it can only depend on the invariant $W^\dagger W$. The ensemble-averaged scattering matrix \bar{S} can be calculated analytically in the limit $M \rightarrow \infty$, at fixed N , E_F , and fixed mean level spacing Δ . The result is¹⁷

$$\bar{S} = \frac{1 - \pi W^\dagger W / \lambda}{1 + \pi W^\dagger W / \lambda}. \quad (2.4)$$

It is possible to extend the Hamiltonian (2.1) to include a “background” scattering matrix S_0 , which does not couple to the cavity.³⁰ The matrix S_0 is symmetric for $\beta = 1$ and can be decomposed as $S_0 = O e^{2i\Phi} O^T$, where the matrix O is orthogonal and Φ is real and diagonal. In the limit $M \rightarrow \infty$, the average scattering matrix \bar{S} is now given by³⁰

$$\bar{S} = O e^{i\Phi} \frac{1 - \pi W^\dagger W / \lambda}{1 + \pi W^\dagger W / \lambda} e^{i\Phi} O^T. \quad (2.5)$$

Lewenkopf and Weidenmüller²⁰ used this extended version of the theory to relate the Gaussian and circular ensembles, for $\beta = 1$ and $\bar{S} = 0$. Their argument is based on the assumption that Eq. (2.5) can be inverted, to yield $W^\dagger W$ and S_0 as a function of \bar{S} . Then $P(S) = P_{\bar{S}}(S)$ is fully determined by \bar{S} (and does not require separate knowledge of $W^\dagger W$ and S_0). Under the transformation $S \rightarrow U S U^T$ (with U an arbitrary unitary matrix), \bar{S} is mapped to $U \bar{S} U^T$, which implies

$$P_{\bar{S}}(S) = P_{U \bar{S} U^T}(U S U^T). \quad (2.6)$$

For $\bar{S} = 0$ one finds that $P(S)$ is invariant under transformations $S \rightarrow U S U^T$, so that $P(S)$ must be constant (circular ensemble). There is, however, a weak spot in this argument: Equation (2.5) cannot be inverted for the crucial case $\bar{S} = 0$. It is only possible to determine $W^\dagger W$, not S_0 . This is a serious objection, since S_0 is not invariant under the transformation $S \rightarrow U S U^T$, and one cannot conclude that $P(S) = \text{const}$ for $\bar{S} = 0$. We have not succeeded in repairing the proof of Ref. 20 for $\bar{S} = 0$, and instead present in the following sections a different proof (which, moreover, can be extended to nonzero \bar{S}).

A situation in which the cavity is coupled to n reservoirs by n leads, having N_j scattering channels ($j = 1, \dots, n$) each, can be described in the framework pre-

sented above by combining the n leads formally into a single lead with $N = \sum_{j=1}^n N_j$ scattering channels. Scattering matrix elements between channels in the same lead correspond to reflection from the cavity, elements between channels in different leads correspond to transmission. In this notation, the Landauer formula for the conductance G of a cavity with two leads [Fig. 1(b)] takes the form

$$G = \frac{2e^2}{h} \sum_{i=1}^{N_1} \sum_{j=N_1+1}^{N_1+N_2} |S_{ij}|^2. \quad (2.7)$$

III. LORENTZIAN ENSEMBLE

For technical reasons, we wish to replace the Gaussian distribution (2.3) of the Hamiltonians by a Lorentzian distribution,

$$P(H) = \frac{1}{V} \lambda^{M(\beta M+2-\beta)/2} \times \det [\lambda^2 + (H - \varepsilon)^2]^{-(\beta M+2-\beta)/2}, \quad (3.1)$$

where λ and ε are parameters describing the width and center of the distribution, and V is a normalization constant independent of λ and ε . The symmetry parameter $\beta \in \{1, 2, 4\}$ indicates whether the matrix elements of H are real [$\beta = 1$, Lorentzian orthogonal ensemble (LOE)], complex [$\beta = 2$, Lorentzian unitary ensemble (LUE)], or real quaternion [$\beta = 4$, Lorentzian symplectic ensemble (LSE)]. (We abbreviate ‘‘Lorentzian’’ by a capital lambda, because the letter L is commonly used to denote the Laguerre ensemble.)

The replacement of (2.3) by (3.1) is allowed because the eigenvector and eigenvalue distributions of the Gaussian and the Lorentzian ensemble are equal on a fixed energy scale, in the limit $M \rightarrow \infty$ at a fixed mean level spacing Δ . The equivalence of the eigenvector distributions is obvious: The distribution of H depends solely on the eigenvalues for both the Lorentzian and the Gaussian ensemble, so that the eigenvector distribution is uniform for both ensembles. In order to prove the equivalence of the distribution of the eigenvalues E_1, E_2, \dots, E_M (energy levels), we compare the n -level cluster functions $T_n(E_1, E_2, \dots, E_n)$ for both ensembles. The general definition of the T_n 's is given in Ref. 23. The first two T_n 's are defined by

$$T_1(E) = \left\langle \sum_{i=1}^M \delta(E - E_i) \right\rangle, \quad (3.2a)$$

$$T_2(E_1, E_2) = \left\langle \sum_{i,j=1}^M \delta(E_1 - E_i) \delta(E_2 - E_j) \right\rangle \quad (3.2b)$$

$$- \left\langle \sum_{i=1}^M \delta(E_1 - E_i) \right\rangle \left\langle \sum_{j=1}^M \delta(E_2 - E_j) \right\rangle. \quad (3.2c)$$

The brackets $\langle \rangle$ denote an average over the ensemble.

The cluster functions in the Gaussian ensemble are known for arbitrary n ,²³ for the Lorentzian ensemble, we compute them below.

From Eq. (3.1), one obtains the joint probability distribution function of the eigenvalues,

$$P(\{E_j\}) = \frac{1}{V} \lambda^{M(\beta M+2-\beta)/2} \prod_{i<j} |E_i - E_j|^\beta \times \prod_i [\lambda^2 + (E_i - \varepsilon)^2]^{-(\beta M+2-\beta)/2}. \quad (3.3)$$

We first consider the case $\lambda = 1$, $\varepsilon = 0$. We make the transformation

$$S = \frac{1 + iH}{1 - iH}. \quad (3.4)$$

The eigenvalues $e^{i\phi_j}$ of the unitary matrix S are related to the energy levels E_j by

$$e^{i\phi_j} = \frac{1 + iE_j}{1 - iE_j} \iff \phi_j = 2 \arctan E_j. \quad (3.5)$$

The probability distribution of the eigenphases follows from Eqs. (3.3) and (3.5),

$$P(\{\phi_j\}) = \frac{1}{V} 2^{-M(\beta M+2-\beta)/2} \prod_{i<j} |e^{i\phi_i} - e^{i\phi_j}|^\beta. \quad (3.6)$$

This is precisely the distribution of the eigenphases in the circular ensemble. The cluster functions in the circular ensemble are known.^{22,23} The n -level cluster functions T_n^Λ in the Lorentzian ensemble are thus related to the n -level cluster functions T_n^C in the circular ensemble by

$$T_n^\Lambda(E_1, \dots, E_n) = T_n^C(2 \arctan E_1, \dots, 2 \arctan E_n) \prod_{j=1}^n \frac{2}{1 + E_j^2}. \quad (3.7)$$

For $n = 1$, one finds the level density

$$\rho(E) = \frac{M}{\pi(1 + E^2)}, \quad (3.8)$$

independent of β . For $n = 2$, one finds the pair-correlation function,

$$T_2^\Lambda(E_1, E_2) = \frac{4 \sin^2(M \arctan E_1 - M \arctan E_2)}{(1 + E_1^2)(1 + E_2^2) \sin^2(\arctan E_1 - \arctan E_2)}. \quad (3.9)$$

Equation (3.9) holds for $\beta = 2$. The expressions for $\beta = 1, 4$ are more complicated.

The n -level cluster functions for arbitrary λ and ε can be found after a proper rescaling of the energies. Equation (3.7) generalizes to

$$T_n^\Lambda(E_1, \dots, E_n) = T_n^C \left(2 \arctan \frac{E_1 - \varepsilon}{\lambda}, \dots, 2 \arctan \frac{E_n - \varepsilon}{\lambda} \right) \prod_{j=1}^n \frac{2\lambda}{\lambda^2 + (E_j - \varepsilon)^2}. \quad (3.10)$$

The large- M limit of the T_n 's is defined as

$$Y_n(\xi_1, \dots, \xi_n) = \lim_{M \rightarrow \infty} \Delta^n T_n(\xi_1 \Delta, \dots, \xi_n \Delta). \quad (3.11)$$

For both the Gaussian and the Lorentzian ensembles, the mean level spacing Δ at the center of the spectrum in the limit $M \rightarrow \infty$ is given by $\Delta = \lambda\pi/M$. Therefore, the relevant limit $M \rightarrow \infty$ at fixed level spacing is given by $M \rightarrow \infty$, $\lambda \rightarrow \infty$, $\Delta = \lambda\pi/M$ fixed for both ensembles. Equation (3.10) allows us to relate the Y_n 's in the Lorentzian and circular ensembles,

$$\begin{aligned} Y_n^\Lambda(\xi_1, \dots, \xi_n) &= \lim_{M \rightarrow \infty} (2\pi/M)^n T_n^C(2 \arctan(\pi\xi_1/M), \dots, 2 \arctan(\pi\xi_n/M)) \\ &= \lim_{M \rightarrow \infty} (2\pi/M)^n T_n^C(2\pi\xi_1/M, \dots, 2\pi\xi_n/M) \\ &= Y_n^C(\xi_1, \dots, \xi_n). \end{aligned} \quad (3.12)$$

It is known that the cluster functions Y_n^C in the circular ensemble are equal to the cluster functions Y_n^G in the Gaussian ensemble.²³ Equation (3.12), therefore, shows that the Lorentzian and the Gaussian ensembles have the same cluster functions in the large- M limit.

The technical reason for working with the Lorentzian ensemble instead of with the Gaussian ensemble is that the Lorentzian ensemble has two properties, which make it particularly easy to compute the distribution of the scattering matrix. The two properties are as follows.

Property 1. If H is distributed according to a Lorentzian ensemble with width λ and center ε , then H^{-1} is again distributed according to a Lorentzian ensemble, with width $\tilde{\lambda} = \lambda/(\lambda^2 + \varepsilon^2)$ and center $\tilde{\varepsilon} = \varepsilon/(\lambda^2 + \varepsilon^2)$.

Property 2. If the $M \times M$ matrix H is distributed according to a Lorentzian ensemble, then every $N \times N$ submatrix of H obtained by omitting $M-N$ rows and the corresponding columns is again distributed according to a Lorentzian ensemble, with the same width and center.

The proofs of both properties are essentially contained in Ref. 26. In order to make this paper self-contained, we briefly give the proofs in the Appendix.

IV. SCATTERING MATRIX DISTRIBUTION FOR THE LORENTZIAN ENSEMBLE

The general relation between the Hamiltonian H and the scattering matrix S is given by Eq. (2.2). After some matrix manipulations, it can be written as

$$S = [1 + i\pi W^\dagger (H - E_F)^{-1} W] \times [1 - i\pi W^\dagger (H - E_F)^{-1} W]^{-1}. \quad (4.1)$$

We can write the coupling matrix W as

$$W = UQ\tilde{W}, \quad (4.2)$$

where U is an $M \times M$ orthogonal ($\beta = 1$), unitary ($\beta = 2$), or symplectic ($\beta = 4$) matrix, \tilde{W} is an $N \times N$ matrix, and Q is an $M \times N$ matrix with all elements zero, except

$Q_{nn} = 1$, $1 \leq n \leq N$. Substitution into Eq. (2.2) gives

$$S = \left(1 + i\pi \tilde{W}^\dagger \tilde{H} \tilde{W} \right) \left(1 - i\pi \tilde{W}^\dagger \tilde{H} \tilde{W} \right)^{-1}, \quad (4.3)$$

where we have defined $\tilde{H} \equiv Q^T U^\dagger (H - E_F)^{-1} U Q$.

We assume that H is a member of the Lorentzian ensemble, with width λ and center 0. Then the matrix $H - E_F$ is also a member of the Lorentzian ensemble, with width λ and center E_F . Property 1 implies that $(H - E_F)^{-1}$ is distributed according to a Lorentzian ensemble with width $\tilde{\lambda} = \lambda/(\lambda^2 + E_F^2)$ and center $\tilde{\varepsilon} = E_F/(\lambda^2 + E_F^2)$. Orthogonal (unitary, symplectic) invariance of the Lorentzian ensemble implies that $U^\dagger (H - E_F)^{-1} U$ has the same distribution as $(H - E_F)^{-1}$. Using property 2 we then find that \tilde{H} [being an $N \times N$ submatrix of $U^\dagger (H - E_F)^{-1} U$] is distributed according to the same Lorentzian ensemble (width $\tilde{\lambda}$ and center $\tilde{\varepsilon}$). We now compute the distribution of the scattering matrix, first for a special coupling, then for the general case.

A. Special coupling matrix

First, we will consider the special case that

$$\tilde{W} = \pi^{-1/2} \delta_{nm} \quad (4.4)$$

is proportional to the unit matrix. The relation (4.3) between the S and \tilde{H} is then

$$S = \frac{1 + i\tilde{H}}{1 - i\tilde{H}}. \quad (4.5)$$

Thus, the eigenvalues \tilde{E}_j of \tilde{H} and $e^{i\phi_j}$ of S are related via

$$e^{i\phi_j} = \frac{1 + i\tilde{E}_j}{1 - i\tilde{E}_j} \iff \phi_j = 2 \arctan \tilde{E}_j. \quad (4.6)$$

Since transformations $\tilde{H} \rightarrow U\tilde{H}U^\dagger$ (with arbitrary orthogonal, unitary, or symplectic $N \times N$ matrix U) leave

$P(\tilde{H})$ invariant, $P(S)$ is also invariant under $S \rightarrow USU^\dagger$. So $P(S)$ can only depend on the eigenvalues $e^{i\phi_j}$ of S . The distribution of the \tilde{E} 's is [cf. Eq. (3.3)]

$$P(\{\tilde{E}_j\}) = \frac{1}{V} \tilde{\lambda}^{N(\beta N+2-\beta)/2} \prod_{j < k} |\tilde{E}_j - \tilde{E}_k|^\beta \times \prod_j [|\tilde{\lambda}^2 + (\tilde{E}_j - \tilde{\varepsilon})^2|^{-(\beta N+2-\beta)/2}]. \quad (4.7)$$

From Eqs. (4.6) and (4.7), we obtain the probability distribution of the ϕ 's,

$$P(\{\phi_j\}) = \frac{1}{V} \left(\frac{1 - \sigma\sigma^*}{2} \right)^{N(\beta N+2-\beta)/2} \times \prod_{j < k} |e^{i\phi_j} - e^{i\phi_k}|^\beta \times \prod_j |1 - \sigma^* e^{i\phi_j}|^{-(\beta N+2-\beta)}, \quad (4.8a)$$

$$\sigma = \frac{1 - \tilde{\lambda} - i\tilde{\varepsilon}}{1 + \tilde{\lambda} + i\tilde{\varepsilon}} = \frac{\lambda^2 + E_F^2 - \lambda - iE_F}{\lambda^2 + E_F^2 + \lambda + iE_F}. \quad (4.8b)$$

Equation (4.8) implies that $P(S)$ has the form of a Poisson kernel,

$$P(S) = \frac{\det(1 - \tilde{S}\tilde{S}^\dagger)^{(\beta N+2-\beta)/2}}{2^{N(\beta N+2-\beta)/2} V} \times |\det(1 - \tilde{S}^\dagger S)|^{-(\beta N+2-\beta)}, \quad (4.9)$$

the average scattering matrix \tilde{S} being given by

$$\tilde{S}_{nm} = \sigma \delta_{nm}. \quad (4.10)$$

B. Arbitrary coupling matrix

Now we turn to the case of arbitrary coupling matrix \tilde{W} . We denote the scattering matrix at coupling \tilde{W} by S , and denote the scattering matrix at the special coupling (4.4) by S_0 . The relation between S and S_0 is

$$S = r + t'S_0(1 - r'S_0)^{-1}t \iff S_0 = (t')^{-1}(S - r)(1 - r^\dagger S)^{-1}t^\dagger, \quad (4.11)$$

where we abbreviated

$$r = (1 - \pi\tilde{W}^\dagger\tilde{W})(1 + \pi\tilde{W}^\dagger\tilde{W})^{-1}, \quad (4.12a)$$

$$r' = -\tilde{W}(1 - \pi\tilde{W}^\dagger\tilde{W})(1 + \pi\tilde{W}^\dagger\tilde{W})^{-1}\tilde{W}^{-1}, \quad (4.12b)$$

$$t = 2\pi^{1/2}\tilde{W}(1 + \pi\tilde{W}^\dagger\tilde{W})^{-1}, \quad (4.12c)$$

$$t' = 2\pi^{1/2}(1 + \pi\tilde{W}^\dagger\tilde{W})^{-1}\tilde{W}^\dagger. \quad (4.12d)$$

The symmetry of the coupling matrix \tilde{W} is reflected in the symmetry of the $2N \times 2N$ matrix

$$S_1 = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}, \quad (4.13)$$

which is unitary symmetric ($\beta = 1$), unitary ($\beta = 2$), or unitary self-dual ($\beta = 4$).

The probability distribution P_0 of S_0 is given by Eq. (4.9). The distribution P of S follows from

$$P(S) = P_0(S_0) \frac{dS_0}{dS}, \quad (4.14a)$$

where the Jacobian dS_0/dS is the ratio of infinitesimal volume elements around S_0 and S . This Jacobian is known,^{26,31}

$$\frac{dS_0}{dS} = \left(\frac{\det(1 - r^\dagger r)}{|\det(1 - r^\dagger S)|^2} \right)^{(\beta N+2-\beta)/2}. \quad (4.14b)$$

After expressing S_0 in terms of S by means of Eq. (4.11), we find that $P(S)$ is given by the same Poisson kernel as Eq. (4.9), but with a different \tilde{S} ,

$$\tilde{S} = \frac{1 - \pi(\tilde{\lambda} + i\tilde{\varepsilon})W^\dagger W}{1 + \pi(\tilde{\lambda} + i\tilde{\varepsilon})W^\dagger W}. \quad (4.15)$$

In the limit $M \rightarrow \infty$, at fixed level spacing $\Delta = \lambda\pi/M$, Eq. (4.15) simplifies to

$$\tilde{S} = \frac{\Delta M - \pi^2 W^\dagger W}{\Delta M + \pi^2 W^\dagger W}. \quad (4.16)$$

The extended version of the Hamiltonian approach, which includes a background scattering matrix S_0 , can be mapped to the case without background scattering matrix by a transformation $S \rightarrow S' = USU^T$ ($\beta = 1$), $S \rightarrow S' = USV$ ($\beta = 2$), or $S \rightarrow S' = USU^R$ ($\beta = 4$), where U and V are unitary matrices,³⁰ (U^T is the transposed of U , U^R is the dual of U .) The Poisson kernel is covariant under such transformations,²⁴ i.e., it maps to a Poisson kernel with $\tilde{S}' = U\tilde{S}U^T$ ($\beta = 1$), $\tilde{S}' = U\tilde{S}V$ ($\beta = 2$), or $\tilde{S}' = U\tilde{S}U^R$ ($\beta = 4$). As a consequence, the distribution of S is given by the Poisson kernel for arbitrary coupling matrix W and background scattering matrix S_0 . This proves the general equivalence of the Poisson kernel and the Lorentzian ensemble of Hamiltonians.

V. IDEAL VERSUS NONIDEAL LEADS

The circular ensemble of scattering matrices is appropriate for a chaotic cavity that is coupled to the leads by means of ballistic point contacts ("ideal" leads). In this section we will demonstrate that the generalized circular ensemble described by the Poisson kernel is the appropriate ensemble for a chaotic cavity which is coupled to the leads by means of tunnel barriers ("nonideal" leads).

The system considered is shown schematically in Fig. 2. We assume that the segment of the lead between the tunnel barrier and the cavity is long enough, so that both

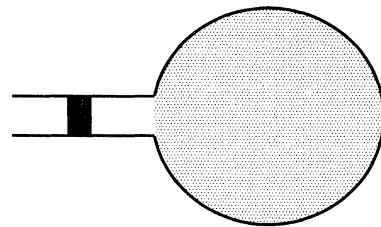


FIG. 2. Schematic drawing of the chaotic cavity and the nonideal lead containing a tunnel barrier.

the $N \times N$ scattering matrix S_0 of the cavity and the $2N \times 2N$ scattering matrix S_1 of the tunnel barrier are well defined. The scattering matrix S_0 has the probability distribution $P_0 = \text{const}$ of the circular ensemble, whereas the scattering matrix S_1 is kept fixed.

We decompose S_1 in terms of $N \times N$ reflection and transmission matrices,

$$S_1 = \begin{pmatrix} r_1 & t'_1 \\ t_1 & r'_1 \end{pmatrix}. \quad (5.1)$$

The $N \times N$ scattering matrix S of the total system is related to S_0 and S_1 by

$$S = r_1 + t'_1(1 - S_0 r'_1)^{-1} S_0 t_1. \quad (5.2)$$

This relation has the same form as Eq. (4.11). We can, therefore, directly apply Eq. (4.14), which yields

$$P(S) \propto |\det(1 - r_1^\dagger S)|^{-(\beta N + 2 - \beta)}. \quad (5.3)$$

Hence, S is distributed according to a Poisson kernel, with $\tilde{S} = r_1$.

VI. CONCLUSION

In conclusion, we have established by explicit computation the equivalence for $M \geq N$ of a generalized circular ensemble of scattering matrices (described by a Poisson kernel) and an ensemble of $M \times M$ Hamiltonians with a Lorentzian distribution. The Lorentzian and Gaussian distributions are equivalent in the large- M limit. Moreover, the Gaussian Hamiltonian ensemble and the microscopic theory of a metal particle with randomly placed impurities give rise to the same nonlinear σ model.^{16,17} Altogether, this provides a microscopic justification of

the Poisson kernel in the case that the chaotic motion in the cavity is caused by impurity scattering. For the case of a ballistic chaotic cavity, a microscopic justification is still lacking.

The equivalence of the Poisson kernel and an arbitrary Hamiltonian ensemble can be reformulated in terms of a central limit theorem: The distribution of a submatrix of H^{-1} of fixed size N tends to a Lorentzian distribution when $M \rightarrow \infty$, independent of the details of the distribution of H . A central limit theorem of this kind for $N = 1$ has previously been formulated and proved by Mello.²⁵

ACKNOWLEDGMENTS

This work was motivated by a series of lectures by P. A. Mello at the summer school in Les Houches on "Mesoscopic Quantum Physics." Discussions with C. W. J. Beenakker, K. Frahm, P. A. Mello, and H. A. Weidenmüller are gratefully acknowledged. This research was supported by the "Stichting voor Fundamenteel Onderzoek der Materie" (FOM) and by the "Nederlandse organisatie voor Wetenschappelijk Onderzoek" (NWO).

APPENDIX: PROOF OF PROPERTIES 1 AND 2 OF SEC. III

The two proofs given below are adapted from Ref. 26. The matrix H and its inverse H^{-1} have the same eigenvectors, but reciprocal eigenvalues. Therefore, property 1 of the Lorentzian ensemble is proved by showing that the distribution of the eigenvalues of H^{-1} is given by Eq. (3.3), with the substitutions $\lambda \rightarrow \tilde{\lambda}$ and $\varepsilon \rightarrow \tilde{\varepsilon}$. This is easily done,

$$\begin{aligned} P(\{E_j^{-1}\}) &= \frac{1}{V} \lambda^{M(\beta M + 2 - \beta)/2} \prod_{i < j} |E_i - E_j|^\beta \prod_i \left[[\lambda^2 + (E_i - \varepsilon)^2]^{-(\beta M + 2 - \beta)/2} \left| \frac{dE_i}{d(E_i^{-1})} \right| \right] \\ &= \frac{1}{V} \lambda^{M(\beta M + 2 - \beta)/2} \prod_{i < j} |E_i E_j (E_i^{-1} - E_j^{-1})|^\beta \prod_i \left\{ [\lambda^2 + (E_i - \varepsilon)^2]^{-(\beta M + 2 - \beta)/2} E_i^2 \right\} \\ &= \frac{1}{V} \lambda^{M(\beta M + 2 - \beta)/2} \prod_{i < j} |E_i^{-1} - E_j^{-1}|^\beta \prod_i [\lambda^2 E_i^{-2} + (1 - \varepsilon E_i^{-1})^2]^{-(\beta M + 2 - \beta)/2} \\ &= \frac{1}{V} \tilde{\lambda}^{M(\beta M + 2 - \beta)/2} \prod_{i < j} |E_i^{-1} - E_j^{-1}|^\beta \prod_i [\tilde{\lambda}^2 + (E_i^{-1} - \tilde{\varepsilon})^2]^{-(\beta M + 2 - \beta)/2}. \end{aligned} \quad (A1)$$

In order to prove property 2, we may assume that after rescaling of H , we have $\lambda = 1$, $\varepsilon = 0$. First consider $M = N - 1$. In this case, one can write

$$H = \begin{pmatrix} G & Y \\ Y^\dagger & Z \end{pmatrix}, \quad (A2)$$

where G is the $N \times N$ submatrix of H , whose distribution we want to compute, Y is a vector, with real ($\beta = 1$), complex ($\beta = 2$), or real quaternion elements ($\beta = 4$), and Z is a real number. For the successive integrations over Z and Y , we need two auxiliary results. First, for

real numbers a, b, c , such that $a > 0$ and $4ac > b^2$, and for real $m > 2$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} dx (ax^2 + bx + c)^{-m} \\ &= (ac - \frac{1}{4}b^2)^{-m+1/2} a^{m-1} \pi^{1/2} \Gamma(m-1/2) / \Gamma(m). \end{aligned} \quad (A3)$$

Second, if x is a d -dimensional vector with real components, and if $m > (d+1)/2$, then

$$\int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_d (1+x^2)^{-m} \\ = \pi^{d/2} \Gamma(m-d/2) / \Gamma(m). \quad (\text{A4})$$

Since $\det(1 + \tilde{H})^2$ is a quadratic function of Z , the integral over Z can now be carried out using Eq. (A3). The result is

$$\int dZ P(\tilde{H}) = \frac{\pi^{1/2} \Gamma[\frac{1}{2}(\beta M + 1 - \beta)]}{V \Gamma[\frac{1}{2}(\beta M + 2 - \beta)]} \\ \times \det(1 + G^2)^{(-\beta M - 2 + \beta)/2} \\ \times [1 + Y^\dagger(1 + G^2)^{-1}Y]^{-\beta M - 1 + \beta}. \quad (\text{A5})$$

Next, we integrate over Y . We may choose the basis for

the Y vectors so that $1 + G^2$ is diagonal, with diagonal elements $1 + G_i^2$. After rescaling of the Y vectors to $Y'_i = Y_i(1 + G_i^2)$ one obtains an integral similar to Eq. (A4), with $d = \beta(M - 1)$. The final result is

$$P(G) = \frac{\pi^{(\beta M - \beta + 1)/2} \Gamma[\frac{1}{2}(\beta M + 1 - \beta)]}{V \Gamma(\beta M + 1 - \beta)} \\ \times \det(1 + G^2)^{[-\beta(M-1) - 2 + \beta]/2}. \quad (\text{A6})$$

Property 2 now follows by induction. Notice that Eq. (A6) allows us to determine the normalization constant V ,

$$V = \pi^{(\beta M - \beta + 2)M/4} \prod_{j=1}^M \frac{\Gamma[\frac{1}{2}(\beta j + 1 - \beta)]}{\Gamma(\beta j + 1 - \beta)}. \quad (\text{A7})$$

-
- ¹ R. A. Jalabert, H. U. Baranger, and A. D. Stone, *Phys. Rev. Lett.* **65**, 2442 (1990).
² R. A. Jalabert, A. D. Stone, and Y. Alhassid, *Phys. Rev. Lett.* **68**, 3468 (1992).
³ H. U. Baranger, R. A. Jalabert, and A. D. Stone, *Phys. Rev. Lett.* **70**, 3876 (1993); *Chaos* **3**, 665 (1993).
⁴ V. N. Prigodin, K. B. Efetov, and S. Iida, *Phys. Rev. Lett.* **71**, 1230 (1993).
⁵ H. U. Baranger and P. A. Mello, *Phys. Rev. Lett.* **73**, 142 (1994).
⁶ R. A. Jalabert, J.-L. Pichard, and C. W. J. Beenakker, *Europhys. Lett.* **27**, 255 (1994).
⁷ Z. Pluhár, H. A. Weidenmüller, J. A. Zuk, and C. H. Lewenkopf, *Phys. Rev. Lett.* **73**, 2115 (1994).
⁸ P. W. Brouwer and C. W. J. Beenakker, *Phys. Rev. B* **50**, 11 263 (1994).
⁹ E. R. Mucciolo, V. N. Prigodin, and B. L. Altshuler, *Phys. Rev. B* **51**, 1714 (1995).
¹⁰ C. M. Marcus, A. J. Rimberg, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, *Phys. Rev. Lett.* **69**, 506 (1992); C. M. Marcus, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, *Phys. Rev. B* **48**, 2460 (1993); *Chaos* **3**, 643 (1993).
¹¹ M. W. Keller, O. Millo, A. Mittal, D. E. Prober, and R. N. Sacks, *Surf. Sci.* **305**, 501 (1994).
¹² A. M. Chang, H. U. Baranger, L. N. Pfeiffer, and K. W. West, *Phys. Rev. Lett.* **73**, 2111 (1994).
¹³ J. P. Bird, K. Ishibashi, Y. Aoyagi, T. Sugano, and Y. Ochiai (unpublished).
¹⁴ O. Bohigas, M. J. Giannoni, and C. Schmit, *Phys. Rev. Lett.* **52**, 1 (1984); O. Bohigas, in *Chaos and Quantum Physics*, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland, Amsterdam, 1991).
¹⁵ M. V. Berry, *Proc. R. Soc. London Ser. A* **400**, 229 (1985).
¹⁶ K. B. Efetov, *Adv. Phys.* **32**, 53 (1983).
¹⁷ J. J. M. Verbaarschot, H. A. Weidenmüller, and M. R. Zirnbauer, *Phys. Rep.* **129**, 367 (1985).
¹⁸ S. Iida, H. A. Weidenmüller, and J. Zuk, *Phys. Rev. Lett.* **64**, 583 (1990); *Ann. Phys. (N.Y.)* **200**, 219 (1990).
¹⁹ A. Altland, *Z. Phys. B* **82**, 105 (1991).
²⁰ C. H. Lewenkopf and H. A. Weidenmüller, *Ann. Phys. (N.Y.)* **212**, 53 (1991).
²¹ R. Blümel and U. Smilansky, *Phys. Rev. Lett.* **60**, 477 (1988); **64**, 241 (1990); U. Smilansky, in *Chaos and Quantum Physics*, edited by M.-J. Giannoni, A. Voros, and J. Zinn-Justin (North-Holland Amsterdam, 1991).
²² F. J. Dyson, *J. Math. Phys.* **3**, 140 (1962).
²³ M. L. Mehta, *Random Matrices* (Academic, New York, 1991).
²⁴ P. A. Mello, P. Pereyra, and T. H. Seligman, *Ann. Phys. (N.Y.)* **161**, 254 (1985).
²⁵ P. A. Mello, in *Mesoscopic Quantum Physics*, edited by E. Akkermans, G. Montambaux, J.-L. Pichard, and J. Zinn-Justin (North-Holland, Amsterdam, in press).
²⁶ L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains* (American Mathematical Society, Providence, 1963).
²⁷ T. J. Krieger, *Ann. Phys.* **42**, 375 (1967).
²⁸ G. Hackenbroich and H. A. Weidenmüller (unpublished).
²⁹ C. Mahaux and H. A. Weidenmüller, *Shell-Model Approach to Nuclear Reactions* (North-Holland, Amsterdam, 1969).
³⁰ H. Nishioka and H. A. Weidenmüller, *Phys. Lett. B* **157**, 101 (1985).
³¹ W. A. Friedman and P. A. Mello, *Ann. Phys. (N.Y.)* **161**, 276 (1985).

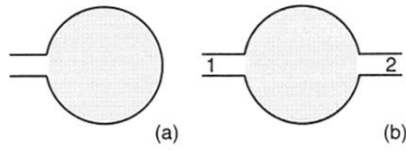


FIG. 1. Schematic drawing of a disordered cavity (gray) attached to a lead. There are N scattering channels in the lead, which are coupled to M bound levels in the cavity. In (a) only one lead is drawn. A system with more leads (b) is described by combining them formally into one lead.