Simple fit to the ground-state energy of the two-dimensional electron gas in the fractional quantum Hall regime

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We present a parametrization of the ground-state energy $E_g(\nu)$ of the two-dimensional electron gas as a function of the filling factor ν, in the fractional quantum Hall regime for the lowest Landau level. The cusp structure is included via an approximate model of the quasiparticle energies $\mu_\pm(\nu)$ at the cusps. The parametrization depends only on a *single* parameter fitted to the known E_g at $\nu = 1/3$. The energy functional is useful in density functional calculations for inhomogeneous systems and for approximate predictions of E_g, $\mu_\pm(\nu)$ at values of ν for which more fundamental calculations are not available.

I. INTRODUCTION

The fully spin polarized two-dimensional electron gas (2DEG) in the strong-field fractional quantum Hall (FQH) regime has been studied extensively.[1] Its rich structure, consisting of an interplay of a sequence of competing incompressible ground states,[2,3] quasi-Fermi liquids,[4] and Wigner crystal phases,[5,6] is reflected in the complicated, highly structured functional form of the ground-state energy $E_g(\nu)$ as a function of the filling factor ν. The filling factor ν is the number of electrons per flux quantum, and varies from zero to unity (filled Landau level) for the strong field limit considered here. The behavior of E_g as a function of ν is fundamental to the study of quasi-2DEG systems found in layered semiconductor structures and in nanostructures. The objective of this paper is to present a functional form which describes in sufficient detail, and with sufficient accuracy, the form of $E_g(\nu)$, inclusive of its cusp structure and consistent with available microscopic calculations. Thus the present effort goes beyond the work[7] of Levesque, Weis, and MacDonald (LWM) who fitted the E_g of simple FQH fluids for which $\nu = 1/q$, and of Fano and Ortolani[8] (FO) who provided a fit to include nonsimple FQH fluids where $\nu = p/q$, with $p \neq 1$. However, both LWM and FO did not include cusp structure and ν values in between the FQH fractions. Nevertheless, many authors have simply used these *smooth* functional forms for general ν and even assumed that these forms can be used for computing *derivatives*. Any attempt to treat the in-between regions need some modeling. Here, we include the cusp structure using the electron and hole quasiparticle energies at the FQH cusps. Incompressible-fluid ground states are experimentally observed if the fluid at the filling factor $\nu = p/q$ is more stable than the Wigner crystal (WC) which is the preferred phase for small ν. A reasonable choice for the set of *observed* FQH ground states is

$$\{\nu_i^{\rm obs}\} = \{i\} + \{*i\}, \tag{1}$$

$$\{i\} = 2/11, 1/5, 2/7, 1/3, 2/5, 3/7, 4/9, 5/11, 6/13,$$

$$\{*i\} = *1/9, *1/7, *3/7, *2/9, *3/13, *3/11, *4/13, *4/11.$$

The (*) fractions have been observed only as structure in the diagonal resistance, or in optical experiments. The FQH state at each cusp persists for a "window" $\Delta\nu$ due to sample-specific effects. Competition of the WC with FQH states leads to reentrant behavior and WC states have been claimed on both sides of $\nu = 1/5$. However, very little is known about E_g for ν in between two "adjacent" $\nu^{\rm obs}$. Ideally, given any pair $\nu_i^{\rm obs}$ and $\nu_{i+1}^{\rm obs}$, one can define other fractions p/q which fall between them. Then we have a densely discontinuous function $E_g(\nu)$ which is mostly unknown since exact diagonalizations, or calculations based on Laughlin-type wave functions exist only for a *limited number* of filling fractions $\nu^{\rm cal}$. It is also believed that $\nu = 1/4, 1/2, 3/4$, etc., probably[4] form a special type of Fermi liquid and their E_g are so far unknown. Any attempt to model such a complicated system seems hopeless. However, we have found a simple scheme which depends only on a *single* fitted parameter and yet reproduces the ground-state energies $E_g(\nu^{\rm cal})$ with high accuracy. The somewhat speculative procedure which enabled us to guess this functional form for the energy also yielded functional forms for the quasiparticle excitation energies $\mu_\pm(\nu)$. Unlike the ground state energies, microscopic calculations for the quasiparticle energies (QPE) are available only for a few ν as they are more difficult to calculate. Different QPE calculations may differ by a factor of 2 or more. The QPE calculated from our functional forms were found to be within the spread of reported $\mu_\pm(\nu^{\rm cal})$ obtained from microscopic calculations. Hence, the model presented here may be cautiously used for arbitrary values of ν within the first Landau level, when more fundamental calculations are unavailable.

II. THE GROUND-STATE ENERGY FUNCTION

A. Wigner crystal regime

Lam and Girvin[5] calculated the energy of the Wigner crystal in a magnetic field using a variational wave function which includes electron correlation effects. If the

electron number density per unit area is n, and the external magnetic field appled perpendicular to the 2DEG is B, the filling factor $\nu = n/(2\pi\lambda_0^2)$. Here, λ_0 is the magnetic length given by $\lambda_0^2 = \hbar c/eB$. Measuring energies in units of $e^2/\epsilon\lambda_0$, the Lam and Girvin expression is

$$E_g(\nu) = E_{\rm clas} + 0.2410\nu^{3/2} + 0.160\nu^{5/2} \quad (2)$$

where $E_{\rm clas} = -0.782133\nu^{1/2}$. Lam and Girvin[5] estimate that the crossover from the liquid to the WC occur around $1/\nu_c = 6.5 \pm 0.5$ using an estimate of the liquid phase obtained from the *smooth* functional form of LWM which has been fitted to the Laughlin liquids at the simple fractions $1/q$.

B. Liquid regime–LWM energy function

A many-body ground state wave function of FQH liquids at filling factors $\nu = 1/q$, where q is an odd integer, was given by Laughlin[2] where he also presented a mapping to a classical plasma. The E_g of the classical plasma can be obtained by hypernetted chain (HNC) integral equation methods or by molecular dynamics (MD) simulations. The E_g from HNC and MD approaches agree very well and LWM (Ref. 7) presented a smooth fit which included an approximate Wigner crystal energy for the small ν regime. The LWM energy per particle is

$$E_g(\nu) = E_{\rm clas} + 0.2110\nu^{1.24} - 0.012\nu^{2.2}. \quad (3)$$

This form does not explicitly satisfy particle-hole conjugation and should not be applied directly if $\nu > 0.5$. If $\nu > 0.5$ the ground-state energy can be calculated via particle-hole conjugation. Denoting $\nu^* = 1 - \nu$,

$$\nu^* E_g(\nu^*) = \nu E_g(\nu) + C_{\rm HF}(\nu^* - \nu). \quad (4)$$

Here, $C_{\rm HF} = -(\pi/8)^{1/2} = -0.626\,657\,1$ is the Hartree-Fock energy of a full Landau level. The LWM form yields the E_g of fluid states if and only if $\nu = 1/q$, and *does not* give the E_g of $\nu = p/q$ where $p \neq 1$. Note that Eq. (3) contains four fitted parameters, two of which are nonlinear. $E_g(\nu)$ calculated using LWM is given in the first row of Table I.

C. Liquid regime–FO energy function

Fano and Ortolani also presented a *continuous* fit valid for the lowest Landau level, using the available ground state energies for $\nu = p/q$, where $p \leq q$. Their first form[8] was replaced by a manifestly particle-hole conjugate form given by

$$E_g(\nu) = C_{\rm HF}\nu + E_{\rm clas} + 0.683\nu(\nu^*)^2 - 0.806\nu^{3/2}(\nu^*)^{5/2}. \quad (5)$$

The FO form has no cusp structure typical of the FQH regime. As in Lam *et al.*, and in LWM, the $E_{\rm clas}$ term implies a smooth crossover to the classical solid (WC) limit. The derivatives of $E_g(\nu)$ calculated from the FO-fit need not have any meaning, just as in the case of LWM. $E_g(\nu)$ calculated using the FO form is given in the third row of Table I.

D. Liquid regime–energy function with cusp structure

In the following, we present a functional form which reproduces the fluid state energies and the known cusp structure of the FQH regime. The fluid state energy need not smoothly reduce to the crystal energy at small ν and hence this limit is not forced into our functional form. At first we consider a *hypothetical model* plasma which is Hartree-Fock-like and has some additional Coulomb correlations but *does not have* FQH-like strong correlations. Since Coulomb correlations in 2D go as the electron spac-

TABLE I. The ground-state energy, quasiparticle energies $\mu_\pm$ and energy gaps $E_{\rm gap}$ in units of $e^2/\epsilon\lambda_0$, at some filling factors for which microscopic calculations are available. The parametrization of LWM (Ref. 7) for $1/q$ fractions, row 1, is to be compared with our $E_g^{(1)}$, given in row 2 where the single adjustable parameter is fitted to the energy at 1/3. Row 3 gives the FO (Ref. 8) for $\nu = p/q$. Row 4 gives the ground state energy E_{mb} from many-body calculations. The result from our fit is in row 5.

ν	1/3	1/5	2/5	2/7	3/7	2/9	4/9
LWM: Eq. (3)	-0.410	-0.328	-0.443	-0.384	-0.456	-0.343	-0.463
E_g^1: Eq. (8)	fitted	-0.329	-0.443	-0.384	-0.457	-0.344	-0.464
FO : Eq. (5)	-0.410	-0.329	-0.439	-0.385	-0.450	-0.345	-0.456
E_{mb} (Ref. 3)	-0.410	-0.328	-0.433	-0.381	-0.441	-0.339	-0.448
E_g, our Eq. (12)	-0.410	-0.328	-0.434	-0.379	-0.445	-0.340	-0.451
μ_-, our Eq. (14)	-0.155	-0.059	-0.109	-0.058	-0.084	-0.037	-0.066
μ_-, other works	-0.132[a]	-0.076[b]					
μ_+, our Eq. (15)	0.254	0.123	0.163	0.102	0.119	0.071	0.094
μ_+, other works	0.231[a]	0.107[b]					
$E_{\rm gap}$, our value	0.100	0.063	0.054	0.044	0.035	0.034	0.028
$E_{\rm gap}$, other works	0.099	0.031	0.061,[c] 0.135[d]	0.024,[e] 0.063[b]	0.049[e]		

[a]Reference 11.
[b]Reference 16.
[c]Reference 19.
[d]Reference 20.
[e]Reference 21.

ing parameter $r_s \propto n^{1/2}$, this additional correlation beyond Hartree-Fock would be expected to be of the form $\nu^{1/2}$ since $2\pi\lambda_0^2\nu = n$. Hence, we take the form

$$E^0 = C_{\rm HF}\nu - c\nu^{1/2}(1-\nu)^{1/2}, \tag{6}$$

where c is an unknown constant and the superfix 0 on E^0 indicates the *model* plasma. This and the following equations do not manifestly satisfy particle-hole conjugation, just as in the case of LWM. However, this is no limitation as they will be used for $\nu < 1/2$ or suitably corrected for $\nu > 1/2$. In order to go beyond the model plasma with its continuous $E_q(\nu)$, we introduce the *scaling ansatz* which asserts that FQH like correlations for $\nu = 1/q$ can be introduced into Eq. (6) by simply replacing ν by $\nu^{1/2}$. Thus

$$\nu \rightarrow \nu^{1/2}. \tag{7}$$

A rationalization of this ansatz will be attempted in the Appendix. Using this ansatz we have, for the simple fractions $\nu_q = 1/q$,

$$E_g^{(1)}(\nu_q) = C_{\rm HF}\nu^{1/2} - c\nu^{1/4}(1-\nu^{1/2})^{1/2}. \tag{8}$$

This equation applies *if and only if* $\nu_q = 1/q$. This is indicated by the superscript (1) on $E_g^{(1)}$. We fix the value of c to reproduce the known energy at $\nu = 1/3$, giving $c = 0.098$. Using this value of c it is found that Eq. (8) closely reproduces the LWM energies at arbitrary values of $\nu \geq 1/7$. This is seen by comparing rows 1 and 2 of Table I. These are FQH–ground-state energies *only* for $\nu = 1/q$ fractions.

To calculate $E_g(\nu)$ for nonsimple fractions $\nu = p/q$ where $p \neq 1$, without introducing additional fitting parameters, we proceed as follows. If $p/q < 1/2$ we can calculate an energy $E(p/q)$ from the model plasma and apply the ansatz. This gives us an energy

$$E(p/q) = C_{\rm HF}(p/q)^{1/2} - c(p/q)^{1/4}[1-(p/q)^{1/2}]^{1/2}, \tag{9}$$

which is simply $E_g^{(1)}(p/q)$, i.e, the energy estimate from the LWM form which is incorrect for $p \neq 1$. We can approximately correct for this inaccuracy by noting that we know the exact energy for $p = q-1$. That is, from Eq. (8), knowing the energy $E_g^{(1)}(\nu_q)$ we know, from Eq. (4) the energy $E_g^{p-h}(\nu_q) = E_g(\nu_q^*)$ at $\nu_q^* = (q-1)/q$. Hence, we define the quantities

$$\Delta(q) = [E_g^{p-h}(\nu_q) - E_g^{(1)}(\nu_q^*)]/(q-1), \tag{10}$$

$$\Delta^0(q) = [E^{0,p-h}(1/q) - E^0(\nu_q^*)]/(q-1). \tag{11}$$

The latter equation is for the hypothetical model plasma and will be useful when we consider the quasiparticle excitation energies. Using Eq. (10) our expression for the FQH–fluid energy at $\nu = p/q$, $\nu < 0.5$ is taken to be

$$E_g(p/q) = E_g^{(1)}(p/q) + \Delta(q)(p-1). \tag{12}$$

This completes our expression for the ground-state energy at FQH–filling factors $\nu = p/q$. The predictions of Eq. (12) are given in row 5 of Table I and should be compared with row 4 containing results of microscopic calculations,[3] and with the FO fit given in row 3. The agreement of our one parameter fit is excellent.

E. Liquid regime–energy function between FQHE cusps

Using Eq. (12), we can calculate E_g at any fraction p/q, q odd, for a 2DEG in the lowest Landau level. The ground-state energy E_g has downward cusps at these special filling factors. If we are slightly away from a given ν_i, say at $\nu_i \pm \delta\nu$, then the energy of the system can be calculated if the quasiparticle energies are known, since

$$\mu_\pm = d(\nu E_g(\nu))/d\nu|_\pm. \tag{13}$$

Hence, the two gradients on the two sides of the cusp can be evaluated and the energy at $\nu_i \pm \delta\nu$ can be estimated. Such an energy estimate is good only if $\delta\nu$ is sufficiently small. Using the spacing Δ_i found in the set of observed FQH fractions $\{\nu_i^{\rm obs}\}$ listed in the introduction, we have decided on a suitable set $\delta\nu_i^\pm$ applicable at each cusp. Thus, in the range of two adjacent cusps i and $i+1$, we have the energies at the four points ν_i, $\nu_i + \delta\nu_i^+$, $\nu_{i+1} - \delta\nu_{i+1}^-$, ν_{i+1}. We use these four known energies to interpolate (cubic splines) for other ν values between ν_i and ν_{i+1}. This is merely a prescription and need not pick up the correct physics of these largely unknown regions. However, the resulting functional form respects the cusp structure at the observed filling factors, and is guided by the correct gradients on the two sides of each cusp.

In treating the energies near $\nu = 1/2$, we interpolate between $\nu = 6/13$ and its conjugate fraction 7/13. Such a procedure need not produce the correct physics at $\nu = 1/2$. However, given more information for such unknown regions, the functional form of E_g can be modified to incorporate the new information.

F. Model for the quasiparticle energies of FQH fluids

The ground-state energy $E_g(\nu)$ of a FQH fluid can be determined either by basis-set dependent "exact" diagonalizations for a finite number N of electrons,[1,9] or using methods based on a form for the many-body wave function.[2,3,10] The different methods are in satisfactory agreement for the ground-state energy. However, the calculations of quasiparticle energies[11] turn out to be more difficult and less concordant than those for E_g. The hypernetted chain method now requires calculations of distribution functions in the presence of a quasiparticle or quasihole—i.e., three-body distributions. Although the quasihole energies calculated using integral equation methods tend to be in reasonable agreement with finite-N diagonalizations, this is not the case for electron quasiparticle energies. We have argued[12] that the treatment of inhomogeneous hypernetted chain equations[13,14] found in the FQH literature uses incorrect limiting procedures in going to limitingly low quasiparticle concentrations. It is also found that the quasiparticle ener-

gies and energy gaps reported by different microscopic calculations may differ by a factor of 2 or more, even for well established fractions like 2/5. Given these difficulties, it is not surprising that no microscopic results exist for most of the observed fractions. Hence, we seek a scheme which produce quasiparticle energies in *grosso modo* agreement with published values, and calculable from the energy parametrization scheme presented above. As explained in the Appendix, our parametrization method for the ground-state energy emerged from our belief that the lowest Landau level of a $1/q$ FQH fluid acquires a sub-Landau level structure containing q-sublevels.[16] The FQH gaps correspond to gaps in the sub-Landau levels. Creation of electron-quasiparticles corresponds to introducing particles above the gap. Creation of holes involves modifications in the already occupied sub-Landau levels below the gap. Using the *hypothetical* model plasma energy $E^0(\nu)$ which is differentiable, and then using the *scaling ansatz* etc., as discussed in the Appendix, we adopt the following expressions for the quasiparticle energies:

$$f(\nu) = C_{\rm HF}\nu - 0.5c(\nu^* - \nu)\nu^{1/2}/(\nu^*)^{1/2},$$

$$\mu_-(p/q) = (1/q)[E^0(\nu) + f(\nu) + \Delta^0(q)], \qquad (14)$$

$$\mu_+(p/q) = (1/q)[E_g^1(\nu) + f(\nu^{1/2}) + \Delta(q)(p-1)]. \qquad (15)$$

No new fit parameters have been introduced into the model in constructing the quasiparticle energies. That is, the model still contains only the *single* fit parameter $c = 0.098$ obtained by fitting the ground-state energy at $\nu = 1/3$. The quasiparticle energies and gaps calculated using these expressions are given in Table I. The agreement with microscopic calculations for $\nu = 1/3$ is reasonable and the energy gap $E_{\rm gap} = \mu_+ + \mu_-$ is in good agreement with microscopic calculations. The quasiparticle energies for 1/5 are similarly acceptable, while the energy gap is overestimated. Quasiparticle energies from microscopic calculations for other fractions are not available in the literature. However, some gap energies are available and the results from our model fall within the reported ranges of values. We have used our values of $\mu_\pm$ to determine the derivatives on the two sides of the cusps in the ground-state energy and constructed the functional forms shown in Figs. 1 and 2.

III. DISCUSSION

We see from Table I that the cusp values of the ground state energy at the FQH-filling factors $\nu = p/q$ are well reproduced by our one parameter fit. The ground state energy, with interpolation for in between regions, is displayed in Figs. 1 and 2, together with the LWM, FO and the WC energy (Lam and Girvin form). The reentrant behavior of the Wigner crystal phase for in between filling factors for ν less than about 0.3 is clearly evident. In fact the first reentrant region is between $\nu = 4/13$ and 1/3. The fraction 4/13 has not been clearly identified as a Hall plateau and hence the first effective reentrant region is between 2/7 and 1/3, as observed experimentally. The exact value of the intersection of the Wigner-crystal energy curve and fluid energy curve depends somewhat on the interpolation and the values of the gradients on the two sides of each cusp. Nevertheless, we believe that the results presented in Figs. 1 and 2 provide at least a semiquantitative picture of the behavior of the ground-state energy of a 2DEG in a strong magnetic field.

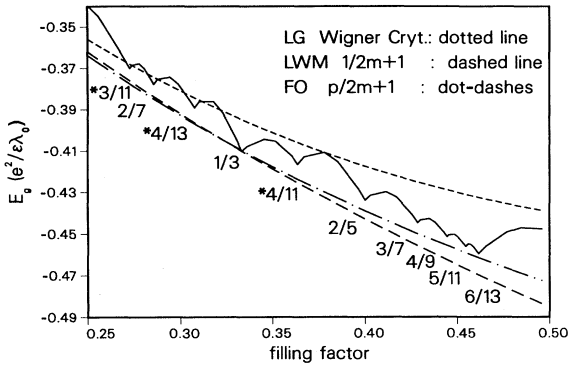

FIG. 1. Solid line: our ground-state energy $E_g(\nu)$ inclusive of the cusp structure. The LWM (Ref. 7) continuous fit to the $1/q$ energies as well as Lam and Girvin (Ref. 5) Wigner crystal energies are shown. The FO (Ref. 8) fit to the p/q energies is not shown and follows LWM closely in this range ($0.1 < \nu < 0.25$). The Wigner solid shows reentrant behavior in regions where the solid line goes above the dotted line.

The LWM and FO forms of E_g did not include the cusp structure of the 2DEG ground-state energy. Such functional forms are not able to treat the reentrant behavior of the Wigner-FQH phases.

In constructing our energy function we were guided by certain physical ideas which are of course not necessary for *using* the functional forms. That is, Eq. (12) etc., could be regarded as mere functional forms with an adjustable parameter which successfully reproduce the energies, etc. Some remarks about the physical ideas that point in the direction of the present parametrization are given in the Appendix.

The energy fits discussed here apply only to the strong

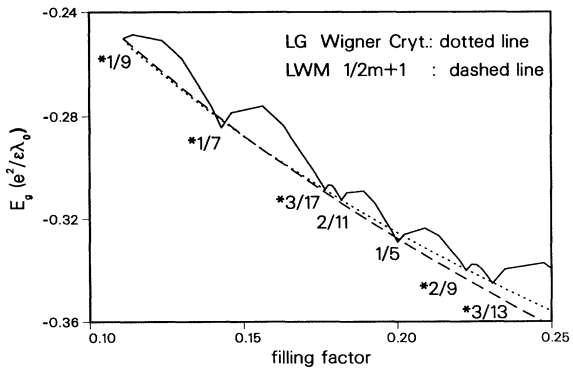

FIG. 2. Same as Fig. 1, but for the range $0.25 < \nu < 0.5$. Here, we have included the FO (Ref. 8) fit as a dash-dot-dash ($\cdot - \cdot -$) line.

field limit. Since the Hartree energy for a uniform system is zero, the $E_g(\nu)$ fits are in fact the exchange-correlation energy functionals needed in density functional theory. When we go to weaker fields the higher Landau levels cannot be neglected and the assumption of full spin polarization breaks down. Thus, the energy has to have the form $E_g(\nu^{\uparrow}, \nu^{\downarrow})$ and should reduce to the zero field ground state energy[15] of the 2DEG as B goes to zero. However, more details regarding the behavior of the two-component 2DEG is needed before a parametrization of the exchange correlation energy could be attempted.

APPENDIX

1. Scaling ansatz

In Eq. (7), we scaled the filling factor ν at the FQH–filling factors to get $\nu_{\rm scl} = \nu^{1/2}$ and used them in Eq. (6) to pick up the special correlations typical of FQH fluids. The physical reasoning that lead us to try this scheme is based[16] on the following picture. At the special FQH–filling factors $\nu_q = 1/q$, the lowest Landau level is replaced by q sub-Landau levels (sLL) with gaps. The sub-Landau levels themselves give rise to sub-sLL of the hierarchy scheme, as in a fractal structure. The exchange-correlation effects manifest themselves as a Chern-Simons type field[17] (arising from an exchange-correlation vector potential) which replaces the external field B by an *effective* field $B^{\rm eff} = B/q$. Hence the effective energy scale in the gaped plasma depends on $e^2/\epsilon\lambda_0^{\rm eff}$ with $\lambda_0^{\rm eff} = \lambda_0 q^{1/2}$. Equivalently, the new energy scale is $e^2\nu_q^{1/2}/\epsilon\lambda_0$ where a $\nu^{1/2}$ appears.

Another way of looking at the scaling ansatz is to consider an N-electron droplet. The N-electrons in the lowest Landau level occupy the angular momentum states $l = 0, 1, \ldots, N-1$ with a total angular momentum $L_{\rm min} = N(N-1)/2$, and a filling factor $\nu = 1$. When an arbitrary number of states with $L > L_{\rm min}$ is available, the filling factor is $\nu = L/L_{\rm min}$. For a given number N of electrons, with N large, $\nu \propto N^2$. However, for certain filling factors the competing requirements of electron exchange and Coulomb repulsion can be optimized by having compact droplets where the ground state is formed by $L_{\rm sml} \approx NN_{\rm sml}/2$ where the electrons occupy the $N_{\rm sml}$ smallest-l orbitals. This effectively constrains the $\nu \propto N^2$ relation to a $\nu \propto N$, or, ν is scaled to $\nu^{1/2}$. This is our scaling ansatz.

2. Quasiparticle energies

In writing down the quasiparicle energies we begin with the model plasma energy of Eq. (6) and assume that when an electronlike quasiparticle is created (a flux quantum is removed), the quasiparticle is placed in the *empty* sub-Landau level above the gap and hence it has no special correlation effects associated with the compact droplet. Thus the first guess for the electron-quasiparticle energy is to use E^0 in Eq. (13). This gives us

$$\mu^-(1/q) = (1/q)[E^0(\nu_q) + f(\nu_q)]. \tag{A1}$$

To approximately treat the case $\nu = p/q$ and make some corrections for particle-hole conjugation effects we add the term $\Delta^0(q)$ and obtain the form given in Eq. (14). The quasiparticle energy for adding a flux quantum arises from creating a hole in the correlated fluid, i.e, in the *filled* sub-Landau level below the gap. Thus our first guess for μ_+ is to use the scaling ansatz and replace ν_q by $\nu_q^{1/2}$ in Eq. (A1). The generalization to $\nu = p/q$ now involves the approximate correction $\Delta(q)(p-1)$ as indicated in Eq. (15).

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