Magnetoelectric effect in fibrous composites with piezoelectric and piezomagnetic phases

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Fibrous composite systems characterized by a cylindrical microgeometry and consisting of a piezoelectric and a piezomagnetic phase with thermal effects are considered. These composites exhibit a magnetoelectric effect that is not present in the constituents. Exact connections are derived between the effective moduli of such systems. These relations are independent of the details of the microgeometry and of the particular choice of the averaging model. In the case of overall transverse isotropy the composite is characterized by twenty one constants. We show that there are sixteen connections for a subset of twenty effective moduli. Simple expressions are also derived for the effective constants of the composite cylinder assemblage microgeometry in such systems.

I. Introduction. Composite aggregates which consist of a piezoelectric and a piezomagnetic phase exhibit a magnetoelectric effect which is not present in the constituents. Such systems have been recently studied by Harshé *et al.*,[1,2] Avellaneda and Harshé,[3] and Nan.[4] Multilayer media[1,3] and particulate composites[2,4] have been considered in these works which include a list of further references as well as a discussion of possible applications of these composites. A special category of particulate media is one in which the aspect ratio of the aligned inclusions is large. An idealization of these aggregates is achieved by letting the fibers to be cylindrical so that the transverse microgeometry is invariant. The present work is concerned with such composites.

This Brief Report is a generalization of the procedures used in the author's recent studies in piezoelectric composites[5–7] to the present systems. Section II deals with the concept of exact connections between the effective moduli of the composite. The idea here is to look for connections which need to be satisfied irrespective of the details of the transverse microgeometry and of the choice of the adopted averaging scheme. Section III gives a very simple derivation of the effective moduli of the composite cylinder assemblage[8] model for the present systems. Since the analysis in both sections is along the same lines of the previous works by the author on piezoelectric composites, we have maintained it here at a minimum, emphasizing only the results which concern the piezomagnetic and magnetoelectric effects.

II. Constitutive Laws and Exact Connections Between the Effective Moduli. Let us consider a two-phase composite medium with a fibrous structure characterized by the fact that the phase boundaries are surfaces which can be generated by straight lines parallel to the x_3 axis. The system of primary interest is that in which one phase is piezoelectric and the other is piezomagnetic. Yet, in order to have a unified analysis, both phases will be assumed to be piezomagnetoelectric. The phases, as well as the composite, will be assumed to be transversely isotropic. The constitutive laws are given by

$$\begin{aligned}\boldsymbol{\sigma}^{(r)} &= \mathbf{L}^{(r)}\,\boldsymbol{\epsilon}^{(r)} - \mathbf{e}^{T(r)}\mathbf{E}^{(r)} - \mathbf{q}^{T(r)}\mathbf{H}^{(r)} + \boldsymbol{\beta}^{(r)}\,\theta_0\,,\\ \mathbf{D}^{(r)} &= \mathbf{E}^{(r)}\,\boldsymbol{\epsilon}^{(r)} + \boldsymbol{\kappa}^{(r)}\mathbf{E}^{(r)} + \boldsymbol{\alpha}^{(r)}\mathbf{H}^{(r)} + \mathbf{p}^{(r)}\,\theta_0\,,\\ \mathbf{B}^{(r)} &= \mathbf{q}^{(r)}\,\boldsymbol{\epsilon}^{(r)} + \boldsymbol{\alpha}^{(r)}\mathbf{E}^{(r)} + \boldsymbol{\mu}^{(r)}\mathbf{H}^{(r)} + \mathbf{m}^{(r)}\,\theta_0\,,\end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}^{(r)}$, $\mathbf{D}^{(r)}$, $\mathbf{B}^{(r)}$, $\boldsymbol{\epsilon}^{(r)}$, $\mathbf{E}^{(r)}$, $\mathbf{H}^{(r)}$, θ_0 denote, respectively ,the stresses, electric displacements, magnetic fluxes, strains, electric field intensities, magnetic field intensities, and the temperature. The elastic properties are given by the fourth-order tensor $\mathbf{L}^{(r)}$, whereas the piezoelectric and piezomagnetic properties are denoted by third-order tensors $\mathbf{e}^{(r)}$ and $\mathbf{q}^{(r)}$. The second-order tensors $\boldsymbol{\kappa}^{(r)}$ and $\boldsymbol{\mu}^{(r)}$ are the dielectric and magnetic permeabilities. Finally, $\boldsymbol{\beta}^{(r)}$ denotes the thermal stress tensor, and $\mathbf{p}^{(r)}$ and $\mathbf{m}^{(r)}$ are the pyroelectric and pyromagnetic vectors. The explicit expressions for $\mathbf{L}^{(r)}$, $\mathbf{e}^{(r)}$, $\boldsymbol{\kappa}^{(r)}$, $\boldsymbol{\beta}^{(r)}$, and $\mathbf{p}^{(r)}$ can be found in the previous works of the author.[5–7] In the adopted matrix notation the properties $\mathbf{q}^{(r)}$, $\boldsymbol{\mu}^{(r)}$, $\boldsymbol{\alpha}^{(r)}$, and $\mathbf{m}^{(r)}$ are given by

$$\mathbf{q}^{(r)} = \begin{bmatrix} 0 & 0 & 0 & 0 & q_{15} & 0\\ 0 & 0 & 0 & q_{15} & 0 & 0\\ q_{31} & q_{31} & q_{33} & 0 & 0 & 0\end{bmatrix}^{(r)},$$

$$\boldsymbol{\mu}^{(r)} = \begin{bmatrix} \mu_{11} & 0 & 0\\ 0 & \mu_{11} & 0\\ 0 & 0 & \mu_{33}\end{bmatrix}^{(r)},$$

$$\boldsymbol{\alpha}^{(r)} = \begin{bmatrix} \alpha_{11} & 0 & 0\\ 0 & \alpha_{11} & 0\\ 0 & 0 & \alpha_{33}\end{bmatrix}^{(r)}, \quad \mathbf{m}^{(r)} = \begin{Bmatrix} 0\\ 0\\ m_3\end{Bmatrix}^{(r)}. \tag{2}$$

Consider now a loading of the composite aggregate in the form

$$u_i(S) = \epsilon^0_{ij}\,x_j\ ,\quad \phi_e(S) = -E^0_i\,x_i\ ,$$
$$\phi_m(S) = -H^0_i\,x_i\ ,\quad \theta(S) = \theta_0\,, \tag{3}$$

where u_i, ϕ_e, and ϕ_m denote, respectively, the mechanical displacements and electric and magnetic potentials, and ϵ^0_{ij}, E^0_i, H^0_i, and θ_0 are constant strains, electric fields, magnetic fields, and temperature.

The loading in (3) can be decomposed into two parts as follows:

$$\begin{Bmatrix} u_1(S) \\ u_2(S) \\ u_3(S) \end{Bmatrix} = \begin{bmatrix} \epsilon^0_{11} & \epsilon^0_{12} & 0 \\ \epsilon^0_{12} & \epsilon^0_{22} & 0 \\ 0 & 0 & \epsilon^0_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} , \tag{4}$$

$$\phi_e(S) = -E^0_3\, x_3 \;,\; \phi_m(S) = -H^0_3\, x_3 \;,\; \theta(S) = \theta_0 \,,$$

and

$$\begin{Bmatrix} u_1(S) \\ u_2(S) \\ u_3(S) \end{Bmatrix} = \begin{bmatrix} 0 & 0 & \epsilon^0_{13} \\ 0 & 0 & \epsilon^0_{23} \\ \epsilon^0_{13} & \epsilon^0_{23} & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} , \tag{5}$$

$$\phi_e(S) = -E^0_1\, x_1 - E^0_2\, x_2 \;,\; \phi_m(S) = -H^0_1\, x_1 - H^0_2\, x_2 \,.$$

Making use of the described fibrous microgeometry, the transversely isotropic structure of the constituents, and the steady state equilibrium conditions,

$$\sigma_{ij,j} = 0 \;,\; D_{i,i} = 0 \;,\; B_{i,i} = 0 \,, \tag{6}$$

it can be readily proved that the fields induced under (4) are decoupled from those induced by (5).

Next, we turn again to (3) and ask whether there exists a specific choice of (ϵ^0, $\mathbf{E^0}$, $\mathbf{H^0}$, θ_0), denoted by ($\hat{\epsilon}$, $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}$, θ_0) such that the strains and electric and magnetic fields be uniform throughout:

$$\epsilon_{ij}(\mathbf{x}) = \hat{\epsilon}_{ij} \;,\; E_i(\mathbf{x}) = \hat{E}_i \;,\; H_i(\mathbf{x}) = \hat{H}_i \;,\; \theta = \theta_0 \,. \tag{7}$$

Following the procedure in Benveniste,[5] it can be shown that the only possible sets of ($\hat{\epsilon}$, $\hat{\mathbf{E}}$, $\hat{\mathbf{H}}, \theta_0$) resulting in uniform fields are given by

$$\begin{Bmatrix} \hat{\epsilon}_1 \\ \hat{\epsilon}_2 \\ \hat{\epsilon}_3 \\ \hat{E}_3 \\ \hat{H}_3 \end{Bmatrix} = \eta_1 \begin{Bmatrix} 1 \\ 1 \\ r_3 \\ 0 \\ 0 \end{Bmatrix} + \eta_2 \begin{Bmatrix} 0 \\ 0 \\ 1 \\ S_3 \\ 0 \end{Bmatrix} + \eta_3 \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ T_3 \end{Bmatrix} + \eta_4 \begin{Bmatrix} 0 \\ 0 \\ \tilde{r}_3 \\ 0 \\ 0 \end{Bmatrix} \theta_0 \,, \tag{8}$$

$$\hat{\epsilon}_4 = \hat{\epsilon}_5 = \hat{\epsilon}_6 = \hat{E}_1 = \hat{E}_2 = \hat{H}_1 = \hat{H}_2 = 0,$$

where η_i, $i = 1, 2, 3, 4$ are arbitrary constants and

$$\begin{aligned} r_3 &= -2(k_1 - k_2)/(\ell_1 - \ell_2), \\ S_3 &= (\ell_1 - \ell_2)/(e^{(1)}_{31} - e^{(2)}_{31}) \,, \\ T_3 &= (\ell_1 - \ell_2)/(q^{(1)}_{31} - q^{(2)}_{31}), \\ \tilde{r}_3 &= -(\beta^{(1)}_1 - \beta^{(2)}_1)/(\ell_1 - \ell_2) \,, \end{aligned} \tag{9}$$

with k being the plane strain bulk modulus for lateral dilatation without axial extension and ℓ the cross modulus in longitudinal uniaxial straining.[5,7] It is thus noted that uniform fields can be generated under the loading of the type (4), but not in the case of (5).

These uniform fields allow the derivation of exact connections between some of the effective properties. The effective law of the piezomagnetoelectric composite aggregate described by the effective tensors $\mathbf{L}$, $\mathbf{e}$, $\mathbf{q}$, $\boldsymbol{\kappa}$, $\boldsymbol{\alpha}$, $\boldsymbol{\mu}$, $\boldsymbol{\beta}, \mathbf{p}$, $\mathbf{m}$ can be represented by

$$\begin{aligned} \bar{\boldsymbol{\sigma}} &= \mathbf{L}\, \bar{\epsilon} - \mathbf{e}^T\, \bar{\mathbf{E}} - \mathbf{q}^T \bar{\mathbf{H}} + \boldsymbol{\beta}\theta_0 \\ &= \sum_{r=1}^{2} c_r [\mathbf{L}^{(r)} \bar{\epsilon}^{(r)} - (\mathbf{e}^T)^{(r)} \bar{\mathbf{E}}^{(r)} - (\mathbf{q}^T)^{(r)} \bar{\mathbf{H}}^{(r)} + \boldsymbol{\beta}^{(r)}\theta_0] \,, \\ \bar{\mathbf{D}} &= \mathbf{e}\, \bar{\epsilon} + \boldsymbol{\kappa} \bar{\mathbf{E}} + \boldsymbol{\alpha} \bar{\mathbf{H}} + \mathbf{p}\theta_0 \\ &= \sum_{r=1}^{2} c_r (\mathbf{e}^{(r)} \bar{\epsilon}^{(r)} + \boldsymbol{\kappa}^{(r)}\, \bar{\mathbf{E}}^{(r)} + \boldsymbol{\alpha}^{(r)} \bar{\mathbf{H}}^{(r)} + \mathbf{p}^{(r)}\theta_0) \,, \\ \bar{\mathbf{B}} &= \mathbf{q}\, \bar{\epsilon} + \boldsymbol{\alpha} \bar{\mathbf{E}} + \boldsymbol{\mu} \bar{\mathbf{H}} + \mathbf{m}\theta_0 \\ &= \sum_{r=1}^{2} c_r (\mathbf{q}^{(r)} \bar{\epsilon}^{(r)} + \boldsymbol{\kappa}^{(r)}\, \bar{\mathbf{E}}^{(r)} + \boldsymbol{\alpha}^{(r)} \bar{\mathbf{H}}^{(r)} + \mathbf{m}^{(r)}\theta_0) \,, \end{aligned} \tag{10}$$

where an overbar denotes a representative volume average, whereas $(\cdot\bar{\cdot}\cdot)^{(r)}$ stands for an average over the phase r in that representative volume.

Subject now the composite to the boundary conditions (4) with the specific choice of (ϵ^0, $\mathbf{E^0}$, $\mathbf{H^0}$, θ_0), given by (8) and (9), so that uniform fields are generated throughout,

$$\bar{\boldsymbol{\epsilon}} = \bar{\boldsymbol{\epsilon}}^{(r)} = \hat{\boldsymbol{\epsilon}} \;,\; \bar{\mathbf{E}} = \bar{\mathbf{E}}^{(r)} = \hat{\mathbf{E}} \;,\; \bar{\mathbf{H}} = \bar{\mathbf{H}}^{(r)} = \hat{\mathbf{H}} \,. \tag{11}$$

Next, substitute (11) and (8) into (10) and equate the coefficients of η_i, $i = 1, 2, 3, 4$. This provides a set of exact connections between the effective properties. For the sake of brevity, we state here only the connections which are in addition to those given before[5] for the piezoelectric case:

$$\frac{q_{31} - \sum_{r=1}^{2} c_r\, q^{(r)}_{31}}{\ell - \sum_{r=1}^{2} c_r\, \ell_r} = \frac{q_{33} - \sum_{r=1}^{2} c_r\, q^{(r)}_{33}}{n - \sum_{r=1}^{2} c_r\, n_r} = \frac{\sum_{r=1}^{2} c_r\, \mu^{(r)}_{33} - \mu_{33}}{q_{33} - \sum_{r=1}^{2} c_r\, q^{(r)}_{33}} = \frac{q^{(1)}_{31} - q^{(2)}_{31}}{\ell_1 - \ell_2} \,, \tag{12}$$

$$\frac{e_{33} - \sum_{r=1}^{2} c_r\, e^{(r)}_{33}}{\sum_{r=1}^{2} c_r\, \alpha^{(r)}_{33} - \alpha_{33}} = \frac{\ell_1 - \ell_2}{q^{(1)}_{31} - q^{(2)}_{31}} \,, \tag{13}$$

$$m_3 = \left(\sum_{r=1}^{2} c_r\, q^{(r)}_{33} - q_{33} \right) \tilde{r}_3 + \sum_{r=1}^{2} c_r\, m^{(r)}_3 \,. \tag{14}$$

In Eq. (12), the parameter n denotes the modulus for longitudinal uniaxial stretching.[5] These relations are supplemented by equations (59), (60), (62), (64), and (65) of the previous paper by the author,[5] as applied to the case of overall transverse isotropy. Alternative, but equivalent forms for m_3 can be written on the basis of (12) above and (59), (60). A variant of (13) is also obtained by manipulating (60) in the previous paper[5] together with (12) and (13) and is given by

$$\frac{q_{33} - \sum_{r=1}^{2} c_r\, q^{(r)}_{33}}{\sum_{r=1}^{2} c_r\, \alpha^{(r)}_{33} - \alpha_{33}} = \frac{\ell_1 - \ell_2}{e^{(1)}_{31} - e^{(2)}_{31}} \,. \tag{15}$$

On the whole, there are nine exact connections between the ten effective parameters k, ℓ, n, e_{33}, e_{31}, κ_{33}, q_{33}, q_{31}, μ_{33}, α_{33}, so that knowledge of one of them allows the determination of the rest. Furthermore, effective thermal terms can be determined from the constituent properties and the knowledge of the effective constants ℓ, n, e_{33}, and q_{33}.

Of particular interest are Eqs. (13) and (15), which show that the magnetoelectric coefficient α_{33} of such fibrous composites can be expressed in terms of the constituent properties and either the effective piezoelectric constant e_{33} or the piezomagnetic one q_{33}. All of these relations are universal in the sense that the details of the transverse microgeometry and averaging assumptions have not been invoked in their derivation. It may be noted that all of the effective constants which appear in these connections are those which would enter in the effective description of the composite under a loading of the type (4). Another effective constant, the transverse shear modulus G_T, is also induced under this loading. This property, however, does not appear in any of the exact connections.

Under a loading of the type (5), the effective behavior of the composite involves the remaining six constants G_L, e_{15}, q_{15}, κ_{11}, α_{11}, μ_{11}. Exact connections between these constants can be derived by using a procedure described by Milgrom and Shtrikman.[11] It is first noted that the solution for the mechanical displacements u_i and electric and magnetic potentials induced under (5) can be represented by

$$\begin{aligned} &u_1^{(r)} = \epsilon_{13}^0\, x_3, \quad u_2^{(r)} = \epsilon_{23}^0\, x_3\,, \\ &u_3^{(r)} = \psi^{(r)}\,(x_1,\, x_2) - \epsilon_{13}^0\, x_1 - \epsilon_{23}^0\, x_2\,, \\ &\phi_e^{(r)} = \phi_e^{(r)}(x_1,\, x_2), \quad \phi_m^{(r)} = \phi_m^{(r)}(x_1,\, x_2)\,, \end{aligned} \tag{16}$$

where we have introduced a new function $\psi^{(r)}(x_1,\, x_2)$. With this notation, the constitutive laws of the constituents and of the composite may conveniently be casted in the form

$$\left\{\begin{matrix} \boldsymbol{\sigma} \\ \mathbf{D} \\ \mathbf{B} \end{matrix}\right\}^{(r)} = \begin{bmatrix} G_L & e_{15} & q_{15} \\ e_{15} & -\kappa_{11} & \alpha_{11} \\ q_{15} & \alpha_{11} & -\mu_{11} \end{bmatrix}^{(r)} \left\{\begin{matrix} \boldsymbol{\nabla}\psi \\ \boldsymbol{\nabla}\phi_e \\ \boldsymbol{\nabla}\phi_m \end{matrix}\right\}^{(r)}, \qquad r = 1, 2, c, \tag{17}$$

where the subscript c refers to the effective law, and the following definitions have been made:

$$\boldsymbol{\sigma} = (\sigma_{13},\, \sigma_{23}), \quad \mathbf{D} = (D_1,\, D_2), \quad \mathbf{B} = (B_1,\, B_2)\,,$$

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial\psi}{\partial x_1}\,,\, \frac{\partial\psi}{\partial x_2}\right), \quad \boldsymbol{\nabla}\phi_e = \left(\frac{\partial\phi_e}{\partial x_1}\,,\, \frac{\partial\phi_e}{\partial x_2}\right),$$

$$\boldsymbol{\nabla}\phi_m = \left(\frac{\partial\phi_m}{\partial x_1}\,,\, \frac{\partial\phi_m}{\partial x_2}\right)\,. \tag{18}$$

The method of Milgrom and Shtrikman[11] concerns coupled field phenomena like magnetoelectricity in which there are divergenceless fluxes together with driving fields derivable from scalar potentials. Basically, it establishes a correspondence between the uncoupled and coupled problems. Although piezomagnetoelectricity does not fall in general in the category of coupled field phenomena treated in that work, the procedure is applicable to two-phase fibrous composites with transversely isotropic constituents and the mode of deformation described in (16). As this procedure has been illustrated and implemented in the context of piezoelectricity by the present author,[6,7] we limit ourselves here to stating the main result only. Let us first define the 3×3 matrices $\mathbf{L}_r$ representing the constitutive law (17):

$$\mathbf{L}_r = \begin{bmatrix} G_L & e_{15} & q_{15} \\ e_{15} & -\kappa_{11} & \alpha_{11} \\ q_{15} & \alpha_{11} & -\mu_{11} \end{bmatrix}^{(r)}, \quad r = 1, 2, c\,. \tag{19}$$

The main finding of Milgrom and Shtrikman[11] consists in establishing the correspondence

$$\mathbf{L}_r^* = \mathbf{W}\mathbf{L}_r\mathbf{W}^T, \quad r = 1, 2, c\,, \tag{20}$$

where $\mathbf{L}_r^*$ is a diagonal matrix and $\mathbf{W}$ is the 3×3 matrix which diagonalizes not only the constituent matrices $\mathbf{L}_1$ and $\mathbf{L}_2$ but also the effective matrix $\mathbf{L}_c$. The implication of this result is the existence of the following constraint relation between the components of the effective matrix $\mathbf{L}_c$:

$$\mathbf{L}_c\, \mathbf{L}_1^{-1}\, \mathbf{L}_2 - \mathbf{L}_2\, \mathbf{L}_1^{-1}\, \mathbf{L}_c = \mathbf{0}\,. \tag{21}$$

It can be readily verified that the resulting matrix on the left-hand side of (21) is antisymmetric so that this equation provides three connections between the components G_L, e_{15}, q_{15}, κ_{11}, α_{11}, μ_{11} of the effective $\mathbf{L}_c$.

To summarize, therefore, the analysis of this section establishes 16 exact connections [Eqs. (59), (60), (62), (64),(65) in the previous paper[5] plus (12), (13), (14), (21) herein] for a subset of 20 effective constants entering in the characterization of the fibrous composite.

III. Composite Cylinder Assemblage Results for the Fibrous Composite. The composite cylinder assemblage model has been introduced by Hashin and Rosen[8] to model the uncoupled mechanical behavior of fibrous composites. It has also counterparts in particulate composites with spherical particles as well as conductivity problems; see Hashin[12] and Hashin and Shtrikman.[13] We now follow the approach used by the author[7] for piezocomposites and provide a very brief derivation of the composite cylinder assemblage moduli for the present composite aggregate.

It is first observed that the transverse bulk modulus k of the piezomagnetoelectric composite is the same as that of the purely elastic composite. This is proved by subjecting the composite aggregate to the boundary conditions necessary to determine the effective modulus k,

$$\begin{aligned} &u_1(S) = \epsilon^0\, x_1, \quad u_2(S) = \epsilon^0\, x_2, \quad u_3(S) = 0, \\ &\phi_e(S) = 0, \quad \phi_m(S) = 0, \quad \theta(S) = 0, \end{aligned} \tag{22}$$

and looking for the transverse dilatational strain in the composite cylinder element. Using the constitutive equations (1) in the equilibrium equations (6) shows that this is a purely mechanical problem. The effective transverse bulk modulus k of the composite cylinder assemblage in the present case is therefore given by its exact expression in the uncoupled mechanical context:

$$k = k_1 + \cfrac{c_2}{\cfrac{1}{k_2 - k_1} + \cfrac{c_1}{k_1 + (G_T)^{(1)}}}, \tag{23}$$

where c_1 and c_2 denote the volume fractions of the matrix and inclusions, respectively. Having obtained an expression for k, the moduli ℓ, n, e_{33}, e_{31}, κ_{33}, q_{33}, q_{31}, μ_{33}, α_{33}, β_1, β_3, p_3, q_3, and m_3 follow from Eqs. (59), (60), (62), (64), (65) of the previous paper[5] and (12), (13), (14) herein. For example, the magnetoelectric coefficient α_{33} is given by

$$\alpha_{33} = c_1\alpha_{33}^{(1)} + c_2\alpha_{33}^{(2)} + \frac{(q_{31}^{(1)} - q_{31}^{(2)})(e_{31}^{(1)} - e_{31}^{(2)})c_1c_2}{[(k_1 + G_T^{(1)}) + c_1(k_2 - k_1)]}. \tag{24}$$

Clearly, this equation shows in an explicit manner the presence of an effective magnetoelectric effect even in situations when it is not present in the phases.

The six moduli G_L, e_{15}, q_{15}, κ_{11}, α_{11}, μ_{11} are easily derived by using the approach of Milgrom and Shtrikman.[11] Consider first the uncoupled elastic, electric, and magnetic behavior of the composite and denote the longitudinal shear modulus and the transverse dielectric and magnetic permeability coefficients by G_L^*, κ_{11}^*, and μ_{11}^*. In the framework of the composite cylinder assemblage model these are given by[14]

$$\begin{aligned} G_L^* &= (G_L^*)_1 \, \frac{(G_L^*)_1 \, c_1 + (G_L^*)_2(1+c_2)}{(G_L)_1(1+c_2) + (G_L)_2 \, c_1} \, , \\ \kappa_{11}^* &= (\kappa_{11}^*)_1 \, \frac{(\kappa_{11}^*)_1 \, c_1 + (\kappa_{11}^*)_2(1+c_2)}{(\kappa_{11}^*)_1(1+c_2) + (\kappa_{11}^*)_2 \, c_1} \, , \\ \mu_{11}^* &= (\mu_{11}^*)_1 \, \frac{(\mu_{11}^*)_1 \, c_1 + (\mu_{11}^*)_2(1+c_2)}{(\mu_{11}^*)_1(1+c_2) + (\mu_{11}^*)_2 \, c_1} \, . \end{aligned} \tag{25}$$

We now define the matrices $\mathbf{L}_r^*$,

$$\mathbf{L}_r^* = \begin{bmatrix} (G_L^*)_r & 0 & 0 \\ 0 & (-\kappa_{11}^*)_r & 0 \\ 0 & 0 & (-\mu_{11}^*)_r \end{bmatrix} \quad , \quad r = 1, 2, c \, , \tag{26}$$

and cast (25) in the following form:

$$\begin{aligned} \mathbf{L}_c^* = \mathbf{L}_1^* \, &[(1+c_2)\mathbf{L}_1^* + (1-c_2)\mathbf{L}_2^*]^{-1} \\ &\times[(1-c_2)\mathbf{L}_1^* + (1+c_2)\mathbf{L}_2^*] \, . \end{aligned} \tag{27}$$

Making now use of the correspondences in (20) shows readily that the effective moduli of the piezomagnetoelectric composite is given by a formula similar to (27):

$$\begin{aligned} \mathbf{L}_c = \mathbf{L}_1 \, &[(1+c_2)\mathbf{L}_1 + (1-c_2)\mathbf{L}_2]^{-1} \\ &\times[(1-c_2)\mathbf{L}_1 + (1+c_2)\mathbf{L}_2] \, , \end{aligned} \tag{28}$$

where $\mathbf{L}_1$, $\mathbf{L}_2$, and $\mathbf{L}_c$ have been defined in (19).

This concludes the determination of all the effective constants of the composite cylinder assemblage model except for G_T. This last constant needs to be determined by a different micromechanical model like, for example, the generalized self-consistent scheme of Christensen and Lo.[15] Yet, as in the case of the transverse bulk modulus k, it is readily recognized that its determination does not involve any electric or magnetic effects and it is, therefore, equal to the corresponding constant of the uncoupled mechanical problem.

IV. Conclusions. The exact connections derived in this paper have several applications. They have allowed, for example, a very brief derivation of the composite cylinder assemblage moduli in Sec. III.

Since the transverse bulk modulus k given in Eq. (23) is known to be exact for this particular microgeometry,[8,14] it has not been necessary to provide a separate derivation of the constants ℓ, n, e_{33}, e_{31}, κ_{33}, q_{33}, q_{31}, μ_{33}, α_{33}, β_1, β_3, p_3, m_3. They have simply followed from the existence of the exact connections. The derivation of the effective moduli G_L, e_{15}, κ_{11}, q_{15}, μ_{11}, α_{11}, on the other hand, has been slightly different. First, in Sec. II we have noticed that the work of Milgrom and Shtrikman[11] has allowed the derivation of three exact scalar connections (21) among these constants. Yet, the explicit expression (28) for these constants have been directly derived from the knowledge of the effective moduli (25) of the uncoupled problems. Since the formula (28) is exact, it is expected that it satisfies the connections (21). That this is indeed so can be readily verified by substituting (28) into (21).

Another application of the exact connections is in verifying the internal consistency of several averaging schemes. Namely, a given micromechanics model will possess such a consistency property if the approximate moduli it predicts satisfy the exact connections given in the previous paper by the author[5] and the present one. Consider, for example, the so-called non-self-consistency approximation (NSC) of Nan and Jin.[9,10,4] Although the majority of their moduli in the case of fibrous systems can be shown to satisfy the exact connections, the moduli G_L, e_{15}, κ_{11} in their piezoelectric paper[9] fail to do so. Thus, although explicit expressions have not been given for the corresponding moduli G_L, e_{15}, q_{15}, κ_{11}, α_{11}, μ_{11} in Nan's piezomagnetoelectric paper,[4] it is questionable that they would fulfill Eq. (21).

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