Matrix model approach to the flux-lattice melting in two-dimensional superconductors

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(Received 6 January 1995)

We investigate a gauged matrix model in the large-N limit that is closely related to the superconductor fluctuation and the flux-lattice melting in two dimensions. With the use of the saddle-point method, the free energy is expanded up to eighth order for the coupling constant g. In the case that the coefficient of the quadratic term of the Ginzburg-Landau matrix model is negative, a critical point $g = g_c$ is obtained in the large-N limit and the relation between this phase transition and the two-dimensional flux-lattice melting transition is discussed.

I. INTRODUCTION

In the study of the fluctuation of superconductors and the melting transition of the Abrikosov flux-lattice in a strong magnetic field, the Ginzburg-Landau (GL) model is especially useful with the lowest Landau level approximation. This approximation is valid in the region very near to H_{c2} and neglects all the contributions from upper Landau levels except the lowest one, and reduces the effective dimension of the system by two due to the Landau quantization perpendicular to the magnetic field axis.

The perturbational studies[1–5] for GL free energy up to high orders are unable to find a flux-lattice melting transition in two dimensions. However, the comparison of the low temperature expansion to the high temperature expansion for the free energy gives the estimation of the melting transition point.[3] On the other hand, the numerical studies such as Monte Carlo simulations show that there exists the first-order transition in a finite temperature well below the mean field superconducting transition point H_{c2}.[6–9] Although both analytical and numerical calculations agree quite well in derivation of the statistical amounts, the nature of the melting transition of flux lattice is not yet fully understood.[10,11] The first-order transition has been suggested by the renormalization group analysis,[12] but two dimensions are far from the valid region around six dimensions.

In this study we generalize the GL Hamiltonian to a gauged matrix model in which the order parameter is expressed by a complex $N \times N$ matrix.[13] The matrix model in the large-N limit recently attracted much theoretical interest since it has a close relation to the string field theory[14] and has been studied in many fields such as two-dimensional (2D) quantum gravity coupled to matter field,[15] mesoscopic fluctuations,[16] and electron correlations.[17] In several matrix models the exact solutions have been obtained[18] and these solutions give the clues to the perturbative analysis of other unsolved matrix models. It is known that the matrix model has a phase transition in the large-N limit when the coefficient of the quadratic term is negative or in the lattice gauge theory.[19–21] This may be true for our gauged matrix model and we are interested in how this phase transition is related to the superconducting flux-lattice melting. The usual Ginzburg-Landau model corresponds to the $N = 1$ case, but we generalize the order parameter to an $N \times N$ complex matrix and take the large-N limit. This large-N limit should be distinguished from the large-N case of the N-vector model,[22,23] which becomes equivalent to the Hartree-Fock approximation and has no phase transition in two dimensions.

In the usual GL model, the perturbation series becomes asymptotic expansions and one needs a Borel summation for such divergent series. In the large-N limit of the GL matrix model, the perturbation series about g is convergent, and the precise analysis of the free energy becomes possible. Thus it is natural to consider the solution of the GL matrix model in the large-N limit as a first approximation of the 2D superconductor phase transition. We find indeed a phase transition point $g_c/\alpha_B^2 = 0.07$ for the large-N limit of the GL matrix model and this point corresponds to $y = -2.7$ in terms of the reduced relative temperature. In a previous paper,[3] we evaluated the flux-lattice melting temperature for 2D superconductors ($N = 1$) as $y = -10$ by the analysis of the perturbation series of the usual GL free energy. Although the phase transition for $N = \infty$ occurs at a considerably higher temperature than the usual GL model of $N = 1$, we consider that this difference may result from the first approximation for 2D superconductors. The improvement of the evaluation of this phase transition point in the gauged matrix model is suggested by taking the higher-order terms in the $1/N^2$ expansion.

This paper is organized as follows: In Sec. II, a gauged matrix model with the interaction of a Ginzburg-Landau type $\frac{g}{N}\mathrm{Tr}(M^*M)^2$ is introduced in the lowest Landau level approximation and the free energy is obtained as the power series of coupling constant g using the saddle-point method. In Sec. III, the critical point of the free energy is obtained for the case where the coefficient of the $\mathrm{Tr}M^*M$ term is negative. This phase transition is investigated in nature and in Sec. IV the relation to the melting transition of flux lattice is discussed.

II. GAUGED MATRIX MODEL IN TWO DIMENSIONS

In the Ginzburg-Landau model, the order parameter ϕ is a complex field and can be expanded with the Landau levels. If the field is near the critical point H_{c2}, the lowest Landau level approximation, which neglects all contributions to the order parameter from the upper Landau levels and takes into account only the lowest one, is justified for the strong magnetic field due to the absence of mixing between Landau levels. We consider this strong magnetic field case and generalize this complex order parameter ϕ to a complex matrix $\tilde{\phi}_{ij}$.

The Hamiltonian is given by

$$H(\tilde{\phi}) = \frac{1}{2m}\mathrm{Tr}|(-i\nabla_\mu - eA_\mu)\tilde{\phi}|^2 + \alpha \mathrm{Tr}|\tilde{\phi}|^2 + \frac{\beta}{2N}\mathrm{Tr}|\tilde{\phi}|^4 , \tag{2.1}$$

where $\tilde{\phi}$ is a rank N complex matrix and $\mu = x, y$. We denote the charge of superconductor by e, and this charge e is of course twice the charge of a single electron. Here the Abelian gauge field A_μ is a vector potential of a magnetic field. α and β are the usual GL parameters. We choose the Landau gauge $\mathbf{A} = (0, Bx)$. The complex matrix $\tilde{\phi}$ is written by the projection to the lowest Landau level as

$$\tilde{\phi}_{ij} = \sum_q M_{ij}(q)(L_y)^{-1/2}\left(\frac{eB}{\pi}\right)^{1/4} e^{iqy} \times \exp\left[-\frac{eB}{2}\left(x - \frac{q}{eB}\right)^2\right] , \tag{2.2}$$

where M_{ij} is an $N \times N$ complex matrix and we put $m = 1$ and $\hbar = 1$. Then the Hamiltonian is rewritten as

$$H(M) = \alpha_B \sum_q \mathrm{Tr}|M(q)|^2 + \sum_{q_i} \frac{\beta}{2L_y N}\left(\frac{eB}{2\pi}\right)^{1/2} \times \exp\left\{-\frac{1}{2eB}\left[\sum q_i^2 - \frac{1}{4}\left(\sum q_i\right)^2\right]\right\} \times \delta_{q_1+q_2,q_3+q_4} \mathrm{Tr}\left[M^*(q_1)M(q_3)M^*(q_2)M(q_4)\right] , \tag{2.3}$$

where $\alpha_B = \alpha + \frac{eB}{m}$ is related to the reduced temperature $[T - T_c(B)]/T_c(B)$. The free energy F is obtained

$$F = -\frac{1}{N^2}\ln Z , \tag{2.4}$$

$$Z = \int dM e^{-H}. \tag{2.5}$$

The perturbation series for the free energy is expanded by a new variable defined by

$$g = \frac{\beta e B}{4\pi\alpha_B^2}. \tag{2.6}$$

This is the same as the expansion parameter in the $N = 1$ case.[1,2] Since we have generalized the GL order parameter to a matrix variable, the perturbation series has an N dependence. In the diagrammatic language, the planar diagrams are obtained in the large-N limit. We have a nonlocal interaction in Eq. (2.3) and have to perform a Gaussian integration, which is equivalent to count the number of Euler paths T of each diagram.[2] Thus there appears an additional factor $1/T$ to the combinatorial factor. In the large-N limit, we have to select only planar diagrams among various terms which have been worked before by considering an N dependence.[1]

Here we follow the new calculational method instead of selecting the planar diagrams as proposed in Ref. 13. Except the factor T of the number of the Euler path, the combinatorial factor of each diagram in the planar limit becomes the same as the one-matrix model. There is no difference between a complex matrix model and a Hermitian matrix model in the leading order except a trivial factor 2. It has been recognized that the renormalized expansion simplifies remarkably the diagrammatic expansion for matrix models: one needs to consider only the irreducible diagrams to obtain the perturbation series of the free energy. This renormalized expansion can be applied successfully to our case, since the extra factor $1/T$ due to the Euler path is factorized. This factorization is easily understood for any diagram by the definition of the Euler path. Therefore, we follow the procedure of the renormalized expansion method,[13] which gives a remarkably simplified method for the perturbation expansion.

We introduce the equivalent $2N^2$ real vector model which is expressed by the $2N^2$ dimensional real vector field $\mathbf{r}$ and the effective Hamiltonian for the large-N limit is written as

$$H_{\mathrm{eff}} = x + \frac{1}{N^{4k-2}}\sum_{k=1}^{\infty} f_k g^k x^{2k} , \tag{2.7}$$

where

$$x = \mathbf{r}^2 = \langle \mathrm{Tr} M^* M\rangle . \tag{2.8}$$

Note that $\mathrm{Tr}M^*M$ is the sum of the square of the absolute value for the matrix element. We integrate out the angular variables of this $2N^2$-dimensional coordinate by keeping the radial part $|\mathbf{r}|$. We choose the appropriate coefficients f_k so as to make the free energy in the large-N limit become the same as that of the original gauged matrix model defined by Eq. (2.3). This coefficient, f_k, turns out to be determined by the irreducible diagrams in the original matrix model.

By the saddle-point method, the free energy for the Hamiltonian, Eq. (2.3), becomes, in the large-N limit,

$$\frac{F}{N^2} = x + \sum_{k=1}^{\infty} f_k g^k x^{2k} - \ln x , \tag{2.9}$$

where we have replaced $x \to N^2 x$. The saddle-point equation is derived as

$$\frac{x}{N^2}\frac{\partial F}{\partial x} = x + \sum_{k=1}^{\infty} f_k (2K) g^k x^{2k} - 1 = 0. \tag{2.10}$$

This relation is also expressed simply by

TABLE I. Number of planar irreducible diagrams relevant in the derivation of Eq. (2.13).

	$O(g)$	$O(g^2)$	$O(g^3)$	$O(g^4)$	$O(g^5)$	$O(g^6)$	$O(g^7)$	$O(g^8)$
No. of planar diagrams	1	1	1	2	3	9	22	61

$$x = 1 - 2gy \,, \tag{2.11}$$

where we define

$$y = \frac{1}{N^2}\frac{\partial F}{\partial g}. \tag{2.12}$$

The relevant irreducible diagrams in each order of g is reduced a great deal in number and in Table I we give these numbers of diagrams. The self-energy part is completely renormalized into the quantity $x = \langle \mathrm{Tr} M^* M \rangle$ and only irreducible diagrams should be taken into account. In other words, our irreducible diagrams do not contain the self-energy diagrams. We briefly explain how we obtain the irreducible planar diagrams: we generate nth order planar irreducible diagrams from $(n-1)$th order ones by cutting the two lines and add one vertex. The choice of the two lines to be cut is restricted by the condition that generated next-order diagrams should also be planar, and the directions of the lines are uniquely determined also by this condition. First we make the generated graphs undirected and drop all the isomorphic graphs obtained by this process. Then for the remaining graphs, we choose properly directed graphs and calculate the combinatorial factor by counting the number of iso-

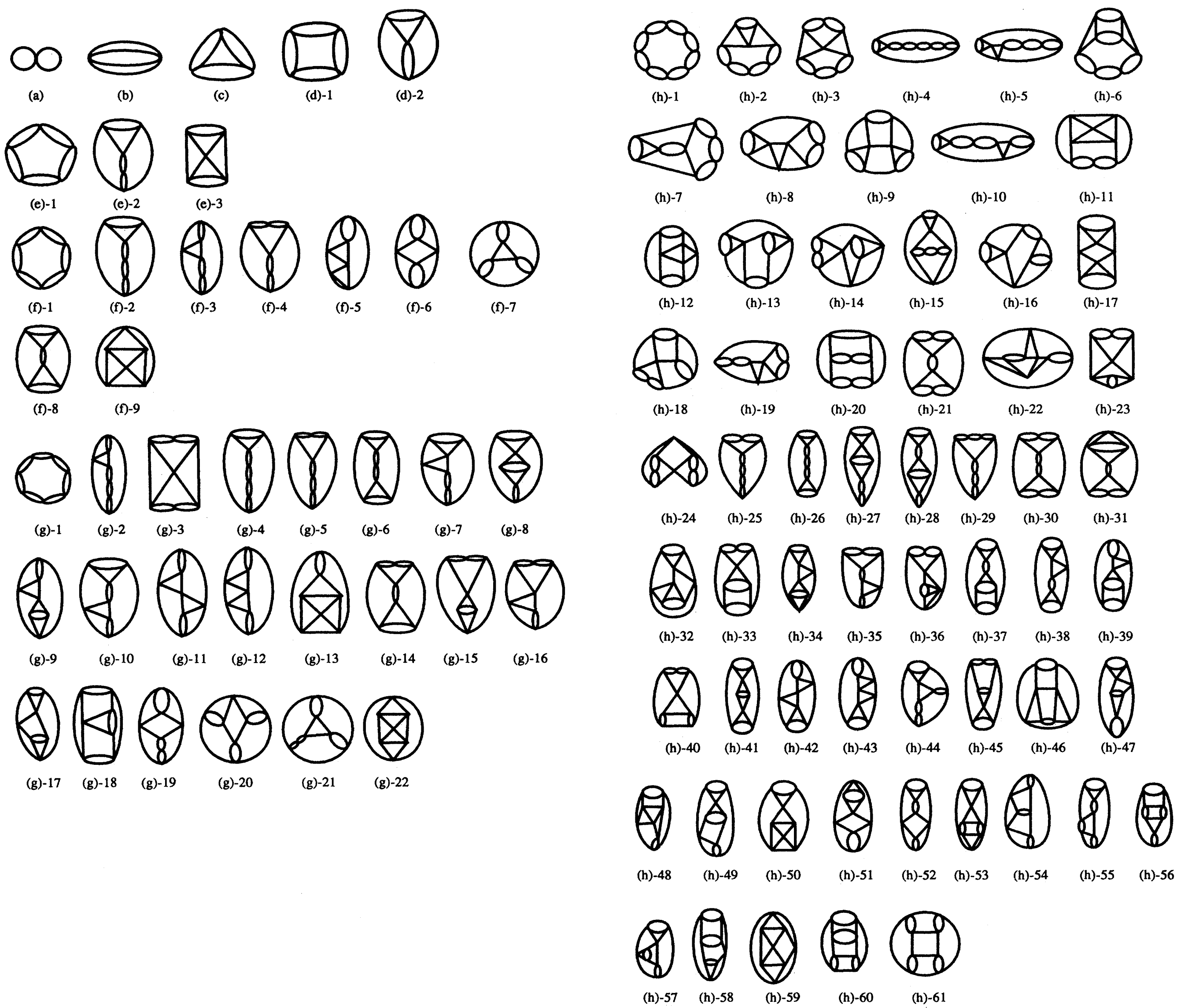

FIG. 1. Planar irreducible diagrams up to $O(g^8)$.

morphisms of the graphs. The number of the Euler path of a given graph is determined by evaluating the determinant of the adjacency matrix.[2] In Fig. 1 we show planar irreducible diagrams to eighth order, where the direction of the line (arrow) is not shown and the line is denoted simply by a single line instead of double lines.

Up to the eighth order we obtain the perturbation series as

$$\frac{F}{N^2} = -\ln x + x + 2gx^2 - \frac{1}{2}g^2x^4 + \frac{8}{9}g^3x^6 - 2\frac{3}{5}g^4x^8 + 9\frac{149}{175}g^5x^{10} - 43\frac{488}{715}g^6x^{12} + 212\frac{48\,357\,908}{101\,846\,745}g^7x^{14} -1144\frac{21\,940\,452\,333\,362}{33\,393\,321\,606\,645}g^8x^{16} , \tag{2.13}$$

where the coefficients are evaluated solely from the irreducible diagrams.

It may be instructive to see how this renormalized expansion method gives an efficient result for the case of one matrix model where the exact solution is known: We now consider the case without a magnetic field in $d = 0$ dimensions. In the one-matrix model, the Hamiltonian becomes, with Hermitian matrix M,

$$H = \frac{1}{2}\text{Tr}M^2 + \frac{g}{N}\text{Tr}M^4 \tag{2.14}$$

from which we obtain the perturbation series for the free energy in the large-N limit with the same irreducible diagrams. (Here we omit the factor $1/T$ for the Euler path.) Since M is Hermitian in Eq. (2.14), there appears a difference of a factor 2 compared to the complex matrix case except the number T of the Euler path,

$$\frac{F}{N^2} = -\frac{1}{2}\ln x + \frac{1}{2}x + 2gx^2 - 2g^2x^4 + \frac{32}{3}g^3x^6 -96g^4x^8 + \cdots . \tag{2.15}$$

The saddle-point equation is given by

$$\frac{x}{N^2}\left(\frac{\partial F}{\partial x}\right) = -\frac{1}{2} + \frac{1}{2}x + 4gx^2 - 8g^2x^4 + 64g^3x^6 - \cdots = 0 . \tag{2.16}$$

From Eq. (2.15) we obtain, for $y = \partial(F/N^2)/\partial g$,

$$y = 2x^2 - 4gx^4 + 32g^2x^6 - 384g^3x^8 + O(g^4). \tag{2.17}$$

In the iteration for small g with Eq. (2.17), x^2 is reexpressed as

$$x^2 = \frac{1}{2}y + \frac{1}{2}gy^2 - g^2y^3 + \frac{9}{2}g^3y^4 - \cdots . \tag{2.18}$$

By the investigation of the ratio of the coefficients in the series, Eq. (2.18), we find that the ratio $R_k = c_k/c_{k-1}$ is given by $R_k = -12 + 30/k$ ($k \geq 3$). Then we have the following closed equation:

$$gx^2 = \frac{1}{108}\left[(1+12gy)^{3/2} - 1\right] + \frac{gy}{3}. \tag{2.19}$$

From this result, we can determine the coefficients of the free energy in Eq. (2.15) up to any order of g for the one-matrix model.

We now come back to the case with a strong magnetic field. Including the factor $1/T$ of the Euler path, we are able to check the result of Eq. (2.13) comparing with Eq. (2.15). We repeat the same procedure as the one-matrix model. From Eq. (2.13) the saddle-point equation becomes

$$\frac{x}{N^2}\left(\frac{\partial F}{\partial x}\right) = -1 + x + 4gx^2 - 2g^2x^4 + \frac{16}{3}g^3x^6 - \frac{104}{5}g^4x^8 + O(g^5) = 0 . \tag{2.20}$$

We rewrite this equation by the iteration for small g as

$$x = 1 - 4g + 34g^2 - \frac{1120}{3}g^3 + \cdots . \tag{2.21}$$

Using the relation of Eq. (2.11) [note y is defined by $y = \partial(F/N^2)/\partial g$], the free energy is obtained in the large-N limit,

$$\frac{F(g)-F(0)}{N^2} = 2g - \frac{17}{2}g^2 + 62\frac{2}{9}g^3 - 585\frac{14}{15}g^4 +6396\frac{272}{525}g^5 - 77\,001\frac{3702}{5005}g^6 +993\,805\frac{7\,137\,857}{33\,948\,915}g^7 -1.351\,344\,723\times 10^7 g^8 . \tag{2.22}$$

We correct here the result of the previous calculation of order g^7 and g^8 in Ref. 13, where a small deviation was represented. It is remarkable that we easily obtain the series expansion by considering rather small diagrams.

III. THE CRITICAL POINT

Our main object is to apply the matrix model to the investigation of the flux-lattice melting transition of a superconductor in a strong magnetic field for two dimensions. Hereafter we consider the gauged matrix model with negative mass $\alpha_B < 0$ and positive coupling case $g > 0$.

From the study of the large-N limit of the two-dimensional $U(N)$ lattice gauge theory[19,20] and one-matrix model,[21] it is known that there is a critical point of g and the free energy shows different behaviors in the small coupling ($g < g_c$) and the strong coupling ($g > g_c$) regions. Also the third derivative of the free energy with respect to the temperature becomes discontinuous at this point and third-order phase transitions are observed in these models. This phase transition is equivalent to the freezing of the saddle-point value of x. In the equivalent

$2N^2$-vector model representation, x is frozen below this critical point as

$$x = \frac{s\alpha_B}{g} \,, \tag{3.1}$$

where s is a certain negative value to be determined.

We again go back to the relation between $x = \langle \mathrm{Tr} M^* M \rangle$ and $y = \langle \mathrm{Tr}(M^* M)^2 \rangle$. From Eq. (2.13) y is expanded by

$$y = \frac{1}{N^2}\frac{\partial F}{\partial g} = 2x^2 - gx^4 + \frac{8}{3}g^2x^6 - \frac{52}{5}g^3x^8 + \cdots \,. \tag{3.2}$$

This relation remains true even when we introduce the negative α_B. We reexpress this perturbation series as

$$gx^2 = \frac{1}{2}gy + \frac{1}{8}g^2y^2 - \frac{5}{48}g^3y^3 + \frac{299}{1920}g^4y^4 - \frac{8249}{26\,880}g^5y^5 + \frac{4\,624\,511}{6\,589\,440}g^6y^6 - \frac{83\,706\,754\,319}{49\,662\,412\,800}g^7y^7 + \frac{26\,766\,869\,658\,031\,714\,037}{5\,471\,161\,812\,032\,716\,800}g^8y^8 + O(g^9). \tag{3.3}$$

For the discussion of the region below the mean-field transition temperature, $\alpha_B < 0$, we write the free energy with an explicit α_B dependence in the term of order x,

$$F = -\ln x + \alpha_B x + 2gx^2 - \frac{1}{2}g^2x^4 + \cdots \,. \tag{3.4}$$

In this case, we change the definition of g from the previous one, Eq. (2.6), to

$$g = \frac{\beta e B}{4\pi} \,. \tag{3.5}$$

The saddle-point equation becomes

$$\alpha_B x = 1 - 2gy \,. \tag{3.6}$$

Thus we have for the temperature below the freezing point, with Eq. (3.1),

$$gy = \frac{g - s\alpha_B^2}{2g} \,. \tag{3.7}$$

As shown in Ref. 13, the critical point is obtained exactly for the one-matrix model from the exact equation of Eq. (2.19) by insertion of Eq. (3.1) and $x = 1 - 4gy$. For our present model, the exact equation is unknown. Thus we approximate Eq. (3.3), which corresponds to Eq. (2.19) for the one-matrix model by the Padé form, and repeat the same analysis which determines the critical point. The approximated Padé form of Eq. (3.3) becomes

$$gx^2 = \frac{1}{2}gy \cdot \frac{1 + b_1 gy + b_2(gy)^2 + \cdots + b_{p-1}(gy)^{p-1}}{1 + c_1 gy + c_2(gy)^2 + \cdots + c_q(gy)^q} \colon [p,q] \text{ Padé}. \tag{3.8}$$

We put Eq. (3.7) into this equation then we have

$$-sz = \frac{(1+z)(B_0 + B_1 z + \cdots + B_{p-1}z^{p-1})}{C_0 + C_1 z + \cdots + C_q z^q} \,, \tag{3.9}$$

where $z = -s\alpha_B^2/g$ and the coefficients $B_0, B_1, \ldots, C_0, C_1, \ldots$ are obtained by the Padé coefficients in Eq. (3.8). The degenerate solution of z is obtained when the line $-sz$ becomes tangent to the curve given by the rhs of Eq. (3.9) (Fig. 2). This degenerate solution gives the critical point z_c. We apply various Padé methods for the rhs of Eq. (3.9). In Table II we list the critical point z_c as a Padé table form. From this table we see that the convergency of the Padé methods which involve an even number of terms of the original series is good. In the case that $p \le q$, there appears a pole in the positive z and we consider that the critical point is effected by this pole. We find the transition point $z_c = 5.6$, $s_c = -0.4$, and $\alpha_B^2/g = 14$ by [5,3] Padé.

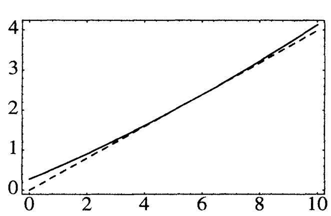

FIG. 2. The approximated [5,3] Padé form of the rhs of Eq. (3.9) is plotted against z by the solid line. The dotted tangent line determines the phase transition point $z_c = 5.64$ and $-s_c = 0.4$ which is the slope of this line.

In the one-matrix model, we have an explicit expression which corresponds to Eq. (3.9) as[13]

$$-sz = \frac{1}{108}[(4+3z)^{3/2} - 1] + \frac{1}{12} + \frac{z}{12} \tag{3.10}$$

and the exact transition point $z = 4$, $s = -1/4$, and $\alpha_B^2/g = 16$. The exponent 3/2 is of course related to the critical exponent of the free energy, $1 - \gamma_{\mathrm{st}}$, where γ_{st} is the exponent of the string susceptibility $\gamma_{\mathrm{st}} = -1/2$. In our case, this string susceptibility exponent is considered to be zero, since our system has a central charge $c = 1$ as shown in Ref. 13. Thus, we expect that the singularity of the rhs of Eq. (3.9) at the negative z_c becomes $(1 - z/z_c)\ln(1 - z/z_c)$. Then we must consider the logarithmic singularity and our simple Padé form loses the validity.

TABLE II. Phase transition point z_c obtained by various $[p,q]$ Padé methods in the $\alpha_B < 0$ case.

$q \backslash p$	3	4	5
2	19.6	5.64	4.62
3	5.18	4.12	5.64
4	4.43	5.19	

However, we are interested in the region of positive z_c, and the assumption of Eq. (3.9) may be valid.

In a previous paper,[3] we discussed the flux-lattice melting point from the direct calculation of the perturbation series for the GL free energy. In this study we used the convenient reduced temperature y_t which is defined as

$$y_t = \frac{\alpha_B}{\sqrt{eB\beta/2\pi}}. \qquad (3.11)$$

This reduced temperature y_t is expressed by the definition of z and s,

$$y_t = \sqrt{\frac{z}{-2s}}. \qquad (3.12)$$

Table III shows the obtained critical point in terms of reduced temperature y_t by the various Padé method.

For $g < g_c$, the saddle point x_c is frozen, and s becomes a constant s_c. The derivative of the free energy $y = \partial F/\partial g$ is given by Eq. (3.7) as

$$y = \frac{1}{2g} - \frac{s_c\alpha_B^2}{2g^2}. \qquad (3.13)$$

Then the free energy becomes, for the low temperature phase, as

$$\frac{F(g) - F(0)}{N^2} = \frac{1}{2}\ln g + \frac{s_c\alpha_B^2}{2g}. \qquad (3.14)$$

The phase transition occurs at $z = z_c = -s_c\alpha_B^2/g_c$.

In order to compare our model in the large-N limit with the usual $N = 1$ case, we write the free energy F and the specific heat C in modified parameters in Refs. 1 and 3. Instead of $g = \beta eB/4\pi\alpha_B^2$ in Eq. (2.6), we introduce a new variable $\tilde{g} = \beta eB/4\pi\tilde{\alpha}^2$, where $\tilde{\alpha}$ is a mass of Hartree-Fock approximation and related to α_B as

$$\alpha_B = \tilde{\alpha}(1 - 4\tilde{g}) , \qquad (3.15)$$

$$\tilde{g} = \frac{\beta eB}{4\pi\tilde{\alpha}^2} . \qquad (3.16)$$

Then the Gibbs free energy G in two dimensions is given by

$$G = \frac{eB}{2\pi}\left[\ln\left(\frac{\alpha_B}{\pi}\right) + f\left(\frac{\beta eB}{4\pi\alpha_B^2}\right)\right] = \frac{eB}{2\pi}\left[\ln\left(\frac{\tilde{\alpha}}{\pi}\right) + \tilde{f}(\tilde{g})\right] . \qquad (3.17)$$

The function $f(g)$ is equal to $[F(g) - F(0)]/N^2$ in Eq. (2.22). We have changed the variable g to $\tilde{g}$, since g is divergent at $\alpha_B = 0$. The new $\tilde{f}(\tilde{g})$ is readily obtained from Eq. (2.22) as

$$\tilde{f}(\tilde{g}) = -2\tilde{g} - \frac{1}{2}\tilde{g}^2 + \frac{8}{9}\tilde{g}^3 - 4\frac{3}{5}\tilde{g}^4 + 28\frac{272}{525}\tilde{g}^5 - 188\frac{5393}{6435}\tilde{g}^6 + 1309\frac{133\,391}{2\,263\,261}\tilde{g}^7 - 9471\frac{128\,589\,189\,285\,247}{166\,966\,608\,033\,225}\tilde{g}^8 . \qquad (3.18)$$

This expression should be compared to the result for the $N = 1$ case; here we represent the previous result,[1]

$$\tilde{f}(\tilde{g})|_{N=1} = -2\tilde{g} - \tilde{g}^2 + 4\frac{2}{9}\tilde{g}^3 - 39\frac{29}{30}\tilde{g}^4 + 471.396\,594\,517\tilde{g}^5 - 6471.562\,574\,96\tilde{g}^6 + \cdots . \qquad (3.19)$$

It is easily seen that in the $N = \infty$ case, the series is convergent, while the series for $N = 1$ in Eq. (3.19) is an asymptotic expansion.

For the low temperature region, we have from Eq. (3.14)

$$\frac{G}{\left(\frac{eB}{2\pi}\right)} = \frac{1}{2}\ln\left[\frac{\tilde{g}}{(1-4\tilde{g})^2}\right] + \frac{s_c(1-4\tilde{g})^2}{2\tilde{g}} \simeq -\frac{1}{2}\ln\tilde{g} + 8\tilde{g}s_c \qquad (3.20)$$

for $\tilde{g} \to \infty$. Since we have found $s_c \simeq -0.4$, this behavior coincides with the low temperature free energy[3,4] of $N = 1$ case,

$$\frac{G}{\left(\frac{eB}{2\pi}\right)} = -\frac{1}{2}\ln\tilde{g} - \frac{4\tilde{g}}{1.16} , \qquad (3.21)$$

where a factor 1.16 is the Abrikosov ratio $\beta_A = \langle|\psi|^4\rangle/\langle|\psi|^2\rangle^2$ for the triangular lattice.

The specific heat C, normalized by $\Delta C = v/\beta$ (v is volume), is obtained by the derivative of entropy S,

$$\frac{C}{\Delta C} = \frac{1}{\Delta C}\frac{dS}{d\alpha_B} , \qquad (3.22)$$

and the entropy S is given by

$$S = -\langle|\psi|^2\rangle = -\frac{\partial G}{\partial\alpha_B} . \qquad (3.23)$$

It may be interesting to compare the $N = \infty$ result of the specific heat with the previous result of the $N = 1$ case. The reduced temperature y_t defined by Eq. (2.17) becomes

$$y_t = \frac{\alpha_B}{\sqrt{\frac{eB\beta}{2\pi}}} = \frac{1 - 4\tilde{g}}{\sqrt{2\tilde{g}}} . \qquad (3.24)$$

The specific heat in the high temperature region is analyzed by the Padé method, which has been studied before for $N = 1$ case. Remarkably the large-N result agrees with the previous result of the $N = 1$ case. For the low temperature region, the specific heat is simply given by

TABLE III. Reduced relative phase transition temperature which corresponds to the critical points z_c and s_c obtained by the $[p,q]$ Padé method.

$q \backslash p$	3	4	5
2	−5.16	−2.66	−2.38
3	−2.54	−2.23	−2.66
4	−2.32	−2.54	

$$\frac{C}{\Delta C} \simeq \frac{1}{y_t^2} - 2s_c \ . \tag{3.25}$$

In Fig. 3, we represent the curves of the specific heat in the low temperature and in the high temperature regions. The values in the high temperature region are obtained by the [5,3] Padé result of Eq. (3.9). In Fig. 3, we also represent the curve of the specific heat of the $N = 1$ case. There is a phase transition of third order at $y_t \simeq -2.7$ in the large-N limit.

It may also be interesting to consider the Abrikosov factor β_A. In our matrix model, it is given precisely as

$$\beta_A = \frac{y}{x^2}, \tag{3.26}$$

since $y = \mathrm{Tr}(M^*M)^2$ and $x = \mathrm{Tr}(M^*M)$. From Eqs. (3.1) and (3.7) we obtain for the phase below the transition point,

$$\beta_A = -\frac{1}{2s_c} + \frac{g}{2s_c^2\alpha_B^2} \ , \tag{3.27}$$

where s_c is a frozen constant and estimated as $s_c \simeq -0.4$. In the low temperature limit $\alpha_B \to -\infty$, β_A becomes $\beta_A = -1/2s_c$, and we have $\beta_A \simeq 1.2$ which is close to the Abrikosov value 1.16. Our estimation gives a slightly larger value and suggests that the low temperature phase is different from the Abrikosov phase.

IV. DISCUSSION

In this paper, we have developed a new series expansion for the matrix Ginzburg-Landau model and in the large-N limit, we have obtained a phase transition which corresponds to the superconductor transition in two dimensions. We have found remarkable agreement about the specific heat between the large-N limit and the usual GL model ($N = 1$) in the high temperature region and in the low temperature region. The phase transition point for the $N = \infty$ case is higher than the value of $N = 1$. We have obtained the transition point $y_t = -2.7$, while we have $y_t = -10$ for the usual Ginzburg-Landau model of $N = 1$.

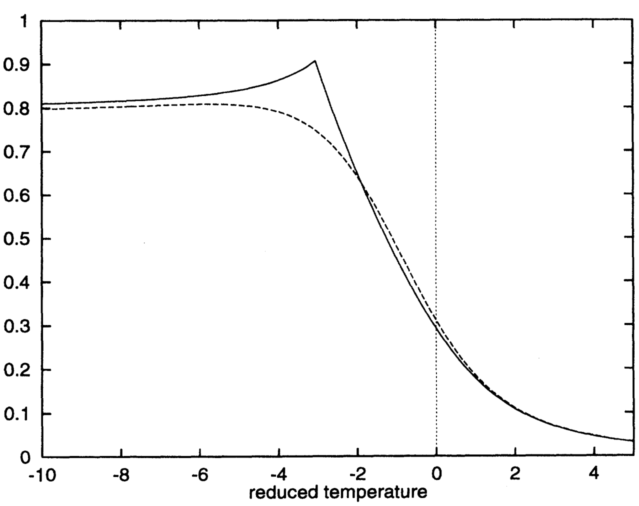

FIG. 3. The specific heat scaled by ΔC against the reduced temperature y_t. For the derivation of this line $\tilde{f}(\tilde{g})$ is approximated by the [5,3] Padé form. The low temperature side is obtained from reexpressing Eq. (3.14) as the function of y_t and evaluating its second derivative with respect to y_t. The dotted line is the specific heat for the $N = 1$ case obtained in Ref. 3.

It is easy to evaluate the next order $1/N^2$, which corresponds with the diagrams of the genus one. Also it is interesting to perform the numerical simulation by the Langevin method or Monte Carlo method for the matrix Ginzburg-Landau model and to find the melting transition point which depends upon N. We will represent this simulation result elsewhere.

As a theoretical interest, our matrix model may be interesting for several reasons. (i) In the one-matrix model, the phase transition corresponds to the singularity of the density of state, where the density of the eigenvalue of the Hermitian matrix has a gap at the band center.[21] Our gauged model is considered to be similar, although the eigenvalue representation is difficult due to the complex matrix. (ii) The matrix model in the large-N limit represents the string behavior as a random surface in the double scaling limit at negative g. We have discussed the phase transition at a positive g, and also the double scaling limit is expected at the phase transition point. In the superconductor, the vortex is a string, and it is interesting to think the phase transition, which we found in the large-N limit, is related to the string theory. We note that in a different context, the analogy of the phase transition in a strong magnetic field to string theory has been discussed.[24] As the same as the other matrix models,[19,20] the transition may correspond to the condensation, like the ideal Bose-Einstein gas. It is interesting to consider further the tachyon condensation for our phase transition. Our result of the new renormalized expansion becomes useful for the phase transition on a random surface in the case of the central charge $c > 1$. (iii) In the presence of the impurity, the melting phase transition is considered to be second order,[25,26] and a vortex glass or a gauge glass phase appears instead of the Abrikosov vortex lattice phase. This gauge glass phase has no long range order. It is interesting to note that for the matrix model in the large-N limit, below the phase transition point, there is no symmetry breaking, as seen in the density of state of the eigenvalue. The density of state has a gap, but is still symmetric for the positive and negative eigenvalue. This is a common behavior seen in freezing phase transitions.[27] Thus a matrix model in the large-N limit may become a model of a gauge glass state in the two-dimensional superconductor in a magnetic field, in which the freezing transition is essential.

ACKNOWLEDGMENTS

This work was supported in part by the cooperative research project between CNRS and JSPS, and by a Grant-in-Aid for Scientific Research by the Ministry of Education, Science and Culture.

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