Failure of hydrodynamics within the vortex-liquid phase

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The recent discovery that some of the coefficients of the viscosity tensor are negative is shown to invalidate the hydrodynamic approach to the vortex-liquid phase of a type-II superconductor. A satisfactory theory requires retention of all the spatial gradients of the velocities or electric fields and not just the first derivatives, as assumed in a hydrodynamic theory. We illustrate such a procedure by using time-dependent Ginzburg-Landau theory to determine the electric-field distribution near a single "twin-plane boundary" due to a current passing through the boundary.

There exists an extensively developed phenomenological hydrodynamic theory[1-6] for superconductors in the vortex-liquid regime. In the recent work of Mou et al.,[7] the coefficients of the viscosity tensor were calculated from a more fundamental theory, in this case Ginzburg-Landau theory, and some were found to be negative. We show explicitly that this invalidates the hydrodynamic approach. We find that a satisfactory theory requires retention of all the spatial gradients of the velocities and not just the first derivatives, as assumed in a hydrodynamic theory. We illustrate our procedure by calculating the electric-field distribution near a single "twin-plane boundary" when a constant current is being passed through that boundary. The electric field decays to its bulk value; whereas, the hydrodynamic theory with negative viscosity coefficients (as calculated) has unphysical oscillatory behavior of the electric field as a function of distance from the boundary.

Because the motion of extended flux lines plays a vital role, the transport properties of type-II superconductors near and below the $H_{c2}(T)$ line are expected to be non-local. For instance, the current **j** produced by a small electric field **E**, is given by

$$j_\mu(\mathbf{r}) = \int \sigma_{\mu\nu}(\mathbf{r}, \mathbf{r}') \, E_\nu(\mathbf{r}') \, d\mathbf{r}', \qquad (1)$$

where the conductivity tensor $\sigma_{\mu\nu}$ is nonlocal, i.e., $\sigma_{\mu\nu}(\mathbf{r}, \mathbf{r}') \neq \sigma_{\mu\nu}\delta(\mathbf{r} - \mathbf{r}')$. Assuming translational invariance, Eq. (1) then becomes in momentum space: $j_\mu(\mathbf{k}) = \sigma_{\mu\nu}(\mathbf{k}) \, E_\nu(\mathbf{k})$, where nonlocality now implies $\sigma_{\mu\nu}(\mathbf{k}) \neq$ const. Recent experiments[8] found the nonlocal effects to be substantial.

The "hydrodynamic" approach[1-6] assumes that an adequate description of the long-time, large-distance behavior can be derived from a truncation of the small-wavevector, small-frequency expansion of the conductivity to the lowest nontrivial terms. For instance, the dc hydrodynamic conductivity tensor would be taken in the hydrodynamic approach to be

$$\sigma_{\mu\nu}(\mathbf{k}, \omega = 0) = \sigma_{\mu\nu}(0) + S_{\mu\alpha\beta\nu} \, k_\alpha k_\beta \qquad (2)$$

(in the notation of Mou et al.[7]). Moreover, hydrodynamics includes a tacit assumption that this truncation results in a stable theory; S_{zzzz}, e.g., must be positive. (We take the external field **B** to be in the $\hat{z}$ direction.)

However, recent calculations of Mou et al.[7] of the S's from Ginzburg-Landau theory found some of them to have signs opposite of those required for stable hydrodynamics. We extend their calculations and show that stability is regained if the full expansion in powers of the wave vector is considered. Then we proceed to calculate the zz component of the conductivity in the presence of a boundary and with it, we calculate the electric field for the case of a constant current j_z being passed through a "twin-plane boundary" (at $z = 0$). The results are compared with those from the hydrodynamic theory.[2] This example also illustrates the necessity for stability of including the normal component of the conductivity (assumed to be local).

Our starting point is the time-dependent Ginzburg-Landau equation

$$\left[\frac{1}{\Gamma} \frac{\partial}{\partial t} - \sum_{j=1}^{3} \frac{\hbar^2}{2m_j} \left(\frac{\partial}{\partial r_j} - \frac{ie^* A_j}{\hbar} \right)^2 + a \right] \psi(\mathbf{r}, t)$$

$$+ \frac{b}{2} |\psi(\mathbf{r}, t)|^2 \, \psi(\mathbf{r}, t) = \eta(\mathbf{r}, t), \quad (3)$$

with noise correlations

$$\langle \eta^*(\mathbf{r}, t) \, \eta(\mathbf{r}', t') \rangle = \frac{2k_B T}{\Gamma} \, \delta(\mathbf{r} - \mathbf{r}') \, \delta(t - t') \qquad (4)$$

and with $m_{1,2} = m_{ab}$ and $m_3 = m_c$. We use the symmetric gauge to describe the uniform field: $A_x = -By/2$ and $A_y = Bx/2$. As the parameter Γ sets the time scale, time will here after be measured in units of Γ; $\hbar$ and k_B will be set to 1. The following calculations are based on the linearized version of Eq. (3), and so strictly speaking above the $H_{c2}(T)$ line; however, making the Hartree approximation $a_{\text{eff}} = a + b \langle |\psi|^2 \rangle$ can take one below the $H_{c2}(T)$ line.[9] It is conceivable that as one goes to lower temperatures still, to the regime where the Hartree approximation breaks down, that the viscosity coefficients become positive (see Ref. 7).

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First, let us sketch out the derivation of the bulk conductivity, returning to the case with a boundary afterward. The linear response to the field can be calculated via the Kubo formula as follows:

$$\sigma_{\mu\nu}^{(s)}(\mathbf{k},\omega) = \frac{1}{2T}\int d(\mathbf{r}-\mathbf{r}')\int d\tau\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega\tau}$$
$$\times \langle J_\mu(\mathbf{r},t+\tau)\, J_\nu(\mathbf{r}',t)\rangle_c\,. \tag{5}$$

Here, we will be interested in the nonlocal nature of the dc conductivity: $\sigma_{\mu\nu}^{(s)}(\mathbf{k},\omega=0)$. [The frequency dependence of the uniform conductivity $\sigma_{\mu\nu}^{(s)}(\mathbf{k}=\mathbf{0},\omega)$ was studied in Ref. 10.]

In order to calculate the current fluctuations, we will need the Green's function and correlation function. We can take the result for the Green's function from the path-integral approach:[11]

$$G(\mathbf{r},t+\tau;\mathbf{r}',t) = \left(\frac{m_c}{32\pi^3\ell^4\tau}\right)^{1/2}\frac{1}{\sinh(\omega_0\tau/2)}$$
$$\times\exp\left\{-a\tau - \frac{m_c}{2\tau}(z-z')^2\right\}$$
$$\times\exp\left\{-\frac{\coth(\omega_0\tau/2)}{4\ell^2}[(x-x')^2\right.$$
$$\left. +(y-y')^2] + \frac{i}{2\ell^2}(xy'-yx')\right\},\tag{6}$$

where $\omega_0 = e^*B/m_{ab}$ is the cyclotron frequency, $\ell = (e^*B)^{-1/2}$ is the magnetic length, and $\tau > 0$.

From the definition of a Green's function

$$\psi(\mathbf{r},t) = \int dt'\int d\mathbf{r}'\, G(\mathbf{r},t;\mathbf{r}',t')\,\eta(\mathbf{r}',t'),\tag{7}$$

which one can then use to calculate the correlation function $\langle\psi^*(\mathbf{r}',t')\,\psi(\mathbf{r},t)\rangle$. Using Eq. (4) for the noise correlations, it follows after some algebra that

$$\langle\psi^*(\mathbf{r}',t')\psi(\mathbf{r},t)\rangle = 2T\int_0^\infty ds\, G(\mathbf{r},\mathbf{r}';|t-t'|+2s).\tag{8}$$

The Kubo formula for the conductivity involves a product of correlation functions with some (covariant) derivatives acting on it, those derivatives coming from the usual definition of the current density

$$J_\mu(\mathbf{r},t) = \frac{e^*}{2m_\mu}\left[\psi^*\left(-i\frac{\partial}{\partial r_\mu} - e^*A_\mu\right)\psi + \text{c.c.}\right].\tag{9}$$

The next steps are relatively straightforward: (i) insert Eqs. (6), (8), and (9) into the Kubo formula (5); (ii) carry out the required derivatives; and (iii) perform the spatial integrations in Eq. (5), which are all Gaussian integrals. Three integrations over time variables remain. After some rearrangement of these time integrations, we find

$$\sigma_{\mu\nu}^{(s)}(\mathbf{k},0) = \frac{e^{*2}Tm_c^{1/2}}{8\pi^{3/2}m_\mu m_\nu\ell^2}\int_0^\infty d\tau\,\frac{\tau^{3/2}\,e^{-2a\tau}}{\sinh(\omega_0\tau)}\int_0^1 dv\, v\int_0^1 du\, C_{\mu\nu}(\tau,u,v,\mathbf{k})$$
$$\times\exp\left\{-\frac{\ell^2[\cosh(\omega_0\tau)-\cosh(uv\omega_0\tau)]}{2\sinh(\omega_0\tau)}(k_x^2+k_y^2) - \frac{\tau(1-u^2v^2)}{4m_c}k_z^2\right\},\tag{10}$$

where

$$C_{xx} = \frac{2\cosh(uv\omega_0\tau)}{\ell^2\sinh(\omega_0\tau)} + \frac{\sinh^2(uv\omega_0\tau)}{\sinh^2(\omega_0\tau)}k_x^2$$
$$+\frac{[\cosh(\omega_0\tau)-\cosh(uv\omega_0\tau)]^2}{\sinh^2(\omega_0\tau)}k_y^2,$$
$$C_{xy} = \frac{\sinh^2(uv\omega_0\tau)-[\cosh(\omega_0\tau)-\cosh(uv\omega_0\tau)]^2}{\sinh^2(\omega_0\tau)}k_x k_y,$$
$$C_{zz} = 2m_c/\tau + u^2v^2\,k_z^2,$$
$$C_{xz} = [uv\sinh(uv\omega_0\tau)/\sinh(\omega_0\tau)]\,k_x k_z.\tag{11}$$

The $\{x\leftrightarrow y\}$ symmetry gives one the other $C_{\mu\nu}$'s.

Expanding Eq. (10) to quadratic order in $\mathbf{k}$ and performing the u and v integrations reproduces the results of Mou et al.,[7] including results such as $S_{xxxx} < 0$ and $S_{zzzz} < 0$. However, one can see from Eqs. (10) and (11) that the full expression for $\sigma_{ii}^{(s)}(\mathbf{k})$ is greater than zero as required for stability.

In some special instances, we can resum the expansion in powers of $\mathbf{k}$; a case we will be using below is $\sigma_{zz}^{(s)}(k_x = 0, k_y = 0, k_z)$:

$$\sigma_{zz}^{(s)}(k_z) = \frac{e^{*2}m_c^{1/2}T}{\pi\ell^2}\frac{1}{k_z^2}\sum_{n=0}^\infty\left[(\alpha+2n\omega_0)^{-1/2}\right.$$
$$\left. -\left(\alpha+2n\omega_0+k_z^2/4m_c\right)^{-1/2}\right],\tag{12}$$

where $\alpha = 2a+\omega_0$ measures the distance from the $H_{c2}(T)$ line.

The above results are, however, for the bulk conductivity. In the case with boundaries, the calculation should be repeated with the Green's function $\mathcal{G}$ which satisfies the boundary conditions imposed on the Ginzburg-Landau equation (3). Let us consider an example with a boundary condition at $z = 0$:

$$\psi(x,y,z=0;t) = 0.\tag{13}$$

One could imagine that such a boundary condition might arise from having a defect like a twin plane at $z = 0$. We admit that this is not a realistic model in any sense for such a defect. It is just the simplest example we could devise which enables us to make some analytical progress. The response function $\mathcal{G}$ is then given by

$$\mathcal{G}(\mathbf{r}; \mathbf{r}',; t) = G(\mathbf{r}; \mathbf{r}',; t) - G(\mathbf{r} - 2\mathbf{z}; \mathbf{r}'; t), \qquad (14)$$

which is just the difference between the bulk Green's function and its image.

Suppose one passes a constant current in the $\hat{z}$ direction through this boundary, and that one would like to calculate the corresponding electric field. Since in this case the current and electric field both point in the $\hat{z}$ direction, only $\Sigma_{zz}^{(s)}$, the zz component of the conductivity in the presence of the boundary, is required. Also the electric field varies only in the $\hat{z}$ direction, so the x and y variables can be summed over (or in momentum space $k_x = 0$, $k_y = 0$). Thus we need to evaluate $\Sigma_{zz}^{(s)}(z, z')$. As from now on we shall only be considering z components, we shall drop the z indices; for example, $E_z \rightarrow E$, $\sigma_{zz}^{(s)}(\mathbf{k}) \rightarrow \sigma^{(s)}(\mathbf{k})$, $S_{zzzz} \rightarrow S$, etc. Furthermore, we will denote the uniform conductivity $\sigma^{(s)}(0)$ simply as $\sigma^{(s)}$.

The calculation of the conductivity involves a product of Green's functions. When repeating the steps outlined earlier with $\mathcal{G}$ instead of G, one can identify four pieces:

$$\Sigma^{(s)}(z, z') = \sigma^{(s)}(z - z') - \sigma^{(s)}(z + z')$$
$$+ \varsigma^{(s)}(z, z') - \varsigma^{(s)}(-z, z'). \qquad (15)$$

The first term comes from the product $G(z - z')G(z - z')$ and is the bulk conductivity $\sigma^{(s)}(z - z')$, the Fourier transform of which was given in Eq. (12). The second term arises from $G(z+z')G(z+z')$ and is the image of the first $-\sigma^{(s)}(z + z')$; the negative sign resulting from the derivative with respect to z associated with the current operator. The third term originates from the cross term $G(z - z')G(z + z')$ and takes on the form

$$\varsigma(z, z') = A \int_0^\infty d\tau\, \tau^2 \int_0^1 dv\, v \int_{-1}^1 du\, \frac{(\kappa_+\kappa_-)^{1/2}e^{-2a\tau}}{\sinh(\omega_0\tau)}$$
$$\times \left[2\tau\kappa_+\kappa_-/m_c - \kappa_-^2(z - z')^2 + \kappa_+^2(z + z')^2\right]$$
$$\times \exp\left\{-\kappa_-(z - z')^2/2 - \kappa_+(z + z')^2/2\right\}, \qquad (16)$$

where $A = e^{*2}T/32\pi^2 m_c^2\ell^2$ and $\kappa_\pm = m_c/\tau(1\mp uv)$. And finally, the fourth term is the image of the third, again with a negative sign.

The first piece, the bulk conductivity $\sigma^{(s)}(z - z')$, decays away from the source at z'; the second piece, its image, decays away from the image of the source at $-z'$. The cross term $\varsigma(z, z')$, on the other hand, decays away from both source and image. Hence, it only produces a significant contribution near the boundary where it is close to both source and image. However, at the boundary, its contribution is killed off by its image $-\varsigma(-z, z')$. Thus, one expects the cross terms to play a lesser role. For this reason, though mainly for sake of calculational convenience, we will drop the cross terms. (Of course, the cross terms while cumbersome could be handled in numerical work if quantitative results were required.)

Thus far the calculation has concerned only the superconducting contribution to the conductivity. Before proceeding to determine the electric field, the normal component must be included. Here we will simply add a normal contribution $\sigma^{(n)}$, neglecting any interference be-

tween the two. On the scales of interest, the normal conductivity is local. Hence our final expression for the zz component of combined conductivity is

$$\Sigma(z, z') = \frac{2}{\pi} \int_0^\infty dk_z\, \sin(k_z z')\, \sin(k_z z)$$
$$\times \left[\sigma^{(n)} + \sigma^{(s)}(k_z)\right]. \qquad (17)$$

Since in the present scenario, we know the current and want the electric field, we need the inverse of $\Sigma(z, z')$, the resistivity $\rho(z, z')$. For an operator of the form (17), the inversion is readily performed:

$$\rho(z, z') = \frac{2}{\pi} \int_0^\infty dk_z\, \frac{\sin(k_z z')\, \sin(k_z z)}{\left[\sigma^{(n)} + \sigma^{(s)}(k_z)\right]} \qquad (18)$$

[this ease of inversion is the reason why the "cross terms" in Eq. (15) were dropped]. Notice that without the inclusion of $\sigma^{(n)}$, the normal component, this integral would have been divergent. The electric field is then obtained by integrating over the source $j(z')$:

$$E(z) = \int_0^\infty dz'\, \rho(z, z')\, j(z'). \qquad (19)$$

For a constant current $j(z') = j_0$, one finds

$$E(z) = \frac{2\, j_0}{\pi} \int_0^\infty dk_z\, \frac{\sin(k_z z)}{k_z\left[\sigma^{(n)} + \sigma^{(s)}(k_z)\right]}. \qquad (20)$$

Inserting the form of $\sigma^{(s)}(k_z)$ provided in Eq. (12) provides the electric-field distribution shown in Fig. 1. At the boundary the electric field E_z takes on the value $j_0/\sigma^{(n)}$. Since $\psi(z = 0) = 0$, the current there must be purely normal in composition. The electric field then decays toward its bulk value $j_0/[\sigma^{(n)} + \sigma^{(s)}]$. Within the bulk, the superconducting channel becomes available and thus a smaller electric field is required to produce the same current. In expression (20), the length scale associated with the decay of the field to its bulk value depends not only on static quantities such as α [which is related to the c-axis correlation length $\xi \sim (m_c\alpha)^{-1/2}/2$] but also on dynamic quantites such as the normal conductivity $\sigma^{(n)}$.

What would have resulted from the hydrodynamic approach? With a negative coefficient S (as calculated), Eq. (20) would yield unphysical oscillatory behavior, in particular

$$E(z) = j_0\left[1 - \cos(z/\delta)\right]/(\sigma^{(n)} + \sigma^{(s)}) \qquad (21)$$

with $\delta^{-1} = [(\sigma^{(n)} + \sigma^{(s)})/|S|]^{1/2}$ (see Fig. 1).

The attraction of the hydrodynamic approach is its simplicity. The calculation considered above is cumbersome even for a single boundary and with simplifying assumptions. The complications will only multiply as one considers geometries more appropriate for modeling multiterminal transport measurements. We suggest the following ad hoc procedure which has the virtue of simplicity, gives reasonable agreement with the full theory, and which could be generalized to other geometries. We approximate the conductivity by
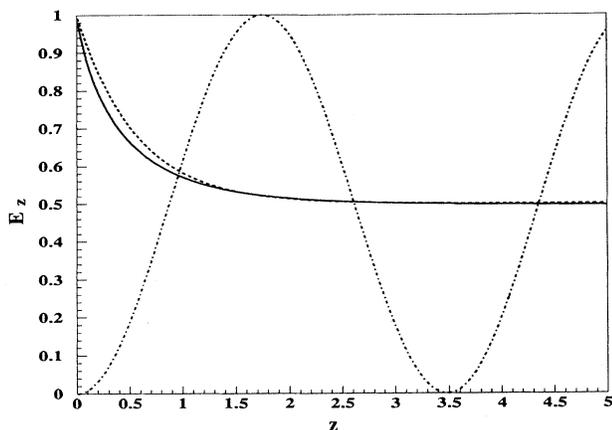
FIG. 1. The solid line shows the electric field (normalized by $j_0/\sigma^{(n)}$) as a function of distance from the boundary [measured in units of $(m_c\alpha)^{-1/2}/2$, the mean-field c-axis correlation length] as calculated from Eq. (20) with $\sigma^{(s)}(k_z)$ given by Eq. (12) with $\sigma^{(n)}/\sigma^{(s)} = 1$ and $\omega_0/\alpha = 2$. The dashed line shows $E(z)$ resulting from Eq. (23). The third line shows the oscillatory "hydrodynamic" result using the same parameters with $S < 0$ (as calculated).

$$\sigma(k_z) = \frac{\sigma^{(n)}\,\sigma^{(s)} + [\sigma^{(s)}]^2 + \sigma^{(n)}\,|S|\,k_z^2}{\sigma^{(s)} + |S|\,k_z^2}, \qquad (22)$$

which has the correct small-k behavior $(\sigma^{(n)} + \sigma^{(s)} +$

$S\,k_z^2$, as derived from Landau-Ginzburg theory), has the correct large-k behavior $(\sigma^{(n)})$, and is positive definite (i.e., stable). With this form for $\sigma(k_z)$, Eq. (20) yields:

$$E(z) = \frac{j_0}{\sigma^{(n)} + \sigma^{(s)}}\left[1 + \frac{\sigma^{(s)}}{\sigma^{(n)}}\exp\left(-z/\tilde{\delta}\right)\right], \qquad (23)$$

where $\tilde{\delta}^{-1} = [\sigma^{(s)}/\sigma^{(n)}]^{1/2}\delta^{-1}$. Note it has the correct limits: $E(0) = j_0/\sigma^{(n)}$ and $E(\infty) = j_0/(\sigma^{(n)} + \sigma^{(s)})$. Furthermore, it provides a simple expression for the dependence of the length scale associated with the decay of E to its bulk value. Its similarity to the full expression can be judged from Fig. 1.

As multiterminal transport measurements represent one of the key experimental probes of superconducting materials, calculations within such geometries are important for relating theory and experiment. Steps toward this end were initiated within the phenomenological hydrodynamic approach;[4] however, the subsequent calculation of negative viscosity coefficients[7] within Ginzburg-Landau theory invalidates the hydrodynamic approach. Here, we have presented a computation which while more cumbersome than hydrodynamics has the advantage that it begins with a more fundamental theory and leads to sensible results.

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