

Crossover and scaling in a nearly antiferromagnetic Fermi liquid in two dimensions

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We consider two-dimensional Fermi liquids in the vicinity of a quantum transition to a phase with commensurate, antiferromagnetic long-range order. Depending upon the Fermi-surface topology, mean-field spin-density-wave theory predicts two different types of such transitions, with *mean-field* dynamic critical exponents $z = 1$ (when the Fermi surface does not cross the magnetic zone boundary, type *A*) and $z = 2$ (when the Fermi surface crosses the magnetic zone boundary, type *B*). The type-*A* system only displays $z = 1$ behavior at all energies and its scaling properties are similar (though not identical) to those of an insulating Heisenberg antiferromagnet. Under suitable conditions precisely stated in this paper, the type-*B* system displays a crossover from relaxational behavior at low energies to type-*A* behavior at high energies. A scaling hypothesis is proposed to describe this crossover: we postulate a universal scaling function which determines the entire, temperature-, wave-vector-, and frequency-dependent, dynamic, staggered spin susceptibility in terms of four experimentally measurable, $T = 0$ parameters. The scaling function contains the full scaling behavior in all regimes for both type-*A* and -*B* systems. The crossover behavior of the uniform susceptibility and the specific heat is somewhat more complicated and is also discussed. Explicit computation of the crossover functions is carried out in a large N expansion on a mean-field model. Some new results for the critical properties on the ordered side of the transition are also obtained in a spin-density-wave formalism. The possible relevance of our results to the doped cuprate compounds is briefly discussed.

I. INTRODUCTION

A number of recent works¹⁻⁸ have proposed a description of the low-temperature spin dynamics of strongly correlated electronic systems using their proximity to an actual or hypothetical zero-temperature, magnetic, quantum phase transition. Such an approach has the advantage of allowing development of a systematic expansion about a point with nontrivial spin correlations and with strongly interacting excitations. Some of these proposals have arisen in the context of two-dimensional interacting electron models of the doped cuprate superconductors, while others considered the heavy-fermion compounds. The ideas of this paper will be presented in the former context, although our approach may be of a more general utility.

We begin our discussion by reviewing two recent complementary approaches to magnetic quantum transitions in the cuprate compounds. A unified picture of spin fluctuations in the lightly doped cuprates obtained from these past approaches, and our present work, is provided towards the end of this paper in Sec. V. We wish to emphasize at the outset that all of the discussion in this paper refers to spin fluctuations associated with *commen-*

surate antiferromagnetic ordering; for the cuprates this corresponds to an ordering wave vector $Q = (\pi, \pi)$. We have little to say here about the case of incommensurate ordering.

The first approach begins from an insulating parent compound, like La_2CuO_4 , whose spin fluctuations can be modeled by a spin-1/2 Heisenberg antiferromagnet on a square lattice.¹ The low-energy excitations of the antiferromagnet are believed to be well described by a continuum $O(3)$ nonlinear σ model field theory. The nonlinear σ model is parametrized by a single coupling constant g , which measures the strength of quantum fluctuations in the system; for $g < g_c$, the system has Néel order, while for $g > g_c$, it is in the quantum-disordered phase. The transition at $g = g_c$ has been studied in some detail⁴—it has a dynamic critical exponent $z = 1$ and leads to a quantum-disordered phase in which at $T = 0$ the low-energy magnon excitations have a gap and an infinite lifetime. The current experimental and theoretical consensus is that the $S = 1/2$ Heisenberg antiferromagnet has macroscopic Néel order in the ground state, and therefore should map onto a σ model with $g < g_c$. Now consider doping this antiferromagnet with a small number of holes. At very small doping, the holes form small ellipti-

cal pockets at $(\pi/2, \pi/2)$ and symmetry-related points in the magnetic Brillouin zone.^{9–11} Quantum fluctuations associated with these holes decrease the spin stiffness of the antiferromagnet⁹ and should therefore drive the effective value of g closer to the quantum transition point at $g = g_c$. (We are assuming here that conditions are such that there is a direct transition from a commensurate long-range-ordered state to a commensurate quantum-disordered state with increasing doping;¹² we are neglecting the possibility of an intermediate incommensurate long-range-ordered state.^{13,14}) Suppose that quantum fluctuations are strong enough that the disordering transition occurs while the holes still occupy small pockets. The critical properties of such a transition were studied by one of us¹² in the framework of the Shraiman-Siggia model.¹³ It was found that the dynamic critical exponent remained at $z = 1$, and the mobile charge carriers only introduced a small damping of the magnon excitations at the critical point. The full structure of the quantum-disordered phase in this model and, in particular, the topology of its Fermi surface are not well understood: We will discuss these issues further in Sec. III C.

The second approach begins with the opposite limit of large doping, where the dilute system of electrons is presumably well described as a Fermi liquid. Using mean-field spin-density-wave ideas, a scenario for the onset of antiferromagnetic order in such a Fermi liquid was proposed many years ago by Hertz¹⁵ and extended recently by Millis.⁵ It turns out to be important to distinguish two cases⁵ depending upon the value the ordering wave vector Q [for our case $Q = (\pi, \pi)$], and the shape of the Fermi surface in the quantum-disordered phase.

(A) *Damping of spin excitations with momenta near Q , due to conversion into particle-hole pairs, is forbidden:* This is the case when Q cannot connect two points on the Fermi surface. For a circular Fermi surface this corresponds to $Q > 2k_F$ where k_F is the Fermi wave vector. The transition then has the mean-field exponent $z = 1$ and is rather similar to the discussion above on the first approach. The scaling results of Ref. 4 apply mostly unchanged: Some simple modifications are necessary for the uniform susceptibility, and are discussed in Sec. IV A.

(B) *Damping of spin excitations with momenta near Q , due to conversion into particle-hole pairs, is allowed:* This is the case when Q can connect two points on the Fermi surface, which for a circular Fermi surface corresponds to $Q < 2k_F$. This transition has $z = 2$ in mean-field theory. Accordingly, the magnon excitations in the quantum-disordered phase are overdamped and relaxational.

We will not discuss the special case $Q = 2k_F$ in this paper.

A phenomenological form for the magnetic susceptibility near a type- B transition was introduced in context of cuprate superconductors by Millis, Monien, and Pines¹⁶ to explain NMR data in $\text{YBa}_2\text{Cu}_3\text{O}_7$. The $z = 1$ to $z = 2$ crossover with doping was later proposed by Sokol and Pines⁶ and by Barzykin *et al.*⁷ to describe the evolution of the experimental NMR and neutron scattering data with oxygen concentration for $\text{YBa}_2\text{Cu}_3\text{O}_{6+x}$. Very recently, Liu and Su⁸ considered a two-component

“Kondo-lattice”-type model of the cuprates, with separate localized spin and itinerant electron degrees of freedom. They assumed conditions that were appropriate to have a type- B phase transition with $z = 2$, and found behavior characteristic of this value of z at the lowest energies or temperatures. At higher temperatures, however, they found a crossover to behavior characteristic of $z = 1$. Earlier, the temperature-induced crossover to the $z = 1$ regime was discussed in Ref. 17 in relation to $\text{YBa}_2\text{Cu}_3\text{O}_{6.63}$. It is not difficult to see that this crossover is in fact a rather general phenomenon for type- B systems — there will always be reactive terms $\sim \omega^2$ (ω is a measuring frequency) in spin response functions which will overwhelm dissipative terms at large enough ω . The crossover should be especially pronounced in systems where the damping constant γ is small.

The main purpose of this paper is to discuss the scaling properties of the quantum-disordered phase of systems near transitions of type B . We will also make a few comments about type- A systems to which the analysis in Ref. 4 will mostly apply. We will assume that conditions in the type- B system are such that the higher-energy crossover to type- A behavior occurs at energies which are significantly smaller than other high-energy cutoffs like the Fermi energy or the exchange constant. (For type- B systems which violate this condition, our results reproduce the correct asymptotic critical singularities, but make a particular choice for numerical scale factors which are in fact nonuniversal in this case; these statements will be made more precise later.) We also assume, of course, that no other unrelated low-energy scale appears. We will propose a scaling hypothesis, in which the wave vector (q), frequency (ω), and temperature (T) dependence of the staggered spin susceptibility can be expressed in terms of a scaling function which involves only four experimentally determinable $T = 0$ input parameters. Only these four parameters are dependent on the details of the microscopic interactions in the ground state; everything else is universal and can, in principle, be computed in a long-distance field theory. The general scaling arguments will be presented in the Sec. II. The crossover behavior of the uniform susceptibility and the specific heat is somewhat more complicated and requires additional microscopic parameters—this is discussed in Secs. IV A and IV B.

We will then illustrate these scaling ideas in two model calculations.

In Sec. III we consider magnetic phase transitions in a standard spin-density-wave (SDW) formalism. We will examine a simple model—a Hubbard model with first (t) and second neighbor hopping (t') for the fermions — which displays a transition of either type A or type B , depending on the ratio t/t' . Note that in both cases, the Fermi surface in the quantum-disordered phase is large (i.e., encloses a volume given by the total number of electrons). The SDW results for the magnetic susceptibility will illustrate the important differences in the nature of the spin excitations between type- A and - B transitions. SDW theory will also be used to obtain new mean-field results for critical properties on the magnetically ordered side of the transition. Finally, we

will briefly discuss the relation between the SDW results and those for the fluctuation-driven magnetic transition within the Shraiman-Siggia model.

In Sec. IV, we first use the results of Sec. III to motivate a model field theory S to describe the quantum-disordered phase in case- B and its crossover to type- A behavior. The model S , which relies on a mean-field approximation for the fermions, will turn out to be precisely the one proposed by Liu and Su;⁸ we caution, however, that the validity of S as a description of the underlying fermionic excitations has not been conclusively established asymptotically close to the critical point. We will use a large N expansion to compute explicit results for the crossover scaling functions of S .

Finally in Sec. V we will discuss the experimental relevance of our results and state our main conclusions.

II. SCALING HYPOTHESES

We now present some general scaling ideas to describe the crossover between type- A and type- B transitions. In the terminology of the well-developed theory of crossover phenomena between two critical points in classical phase transitions,^{18,19} we need to distinguish between the primary and secondary fixed points: The primary fixed point has two relevant directions, while the secondary point has only a single relevant direction associated with the “thermal” operator which drives one across the transition. In our case, it is clear that the primary fixed point is the type- A fixed point with $z = 1$. The mean-field value of $z = 1$ for type A is expected to be robust and not suffer any corrections from fluctuations, as the coupling between the spin fluctuations and fermionic quasiparticles is quite weak in this case. Adding any damping to the magnon excitations should be a relevant perturbation at this fixed point as it introduces additional low-energy excitations (it could be dangerously irrelevant, a possibility which we shall briefly refer to later, but not explore in any detail). The coupling, analogous to g , which tunes the system across the transition is the other relevant perturbation. The type- B fixed point must therefore be secondary. (We are also assuming here that the effect of the damping is associated with only a *single* relevant operator; there could be more than one. It should be easy to extend the following scaling analysis to this case, but we shall not do it in the interests of simplicity.) An important feature of the standard crossover theory^{18,19} is that the crossover scaling functions between the two fixed points are expressed in terms of eigenoperators and exponents of the primary fixed point, while the critical singularities of the secondary fixed point appear as non-analytic behavior in the crossover functions themselves. This result forms the basis of our analysis below.

We will restrict our scaling results below to the to the quantum-disordered side of the transition. We begin our discussion by introducing the four parameters which will characterize the quantum-disordered ground state; the finite-temperature properties of the staggered spin susceptibility χ_s will then be described by universal scaling

functions of these four parameters only. Imagine we have available, either through experiments or computer simulations, the $T = 0$ value of the imaginary part of the local, on-site, dynamic spin susceptibility $\chi_L''(\omega)$ of the system of interest; the local susceptibility is obtained by integrating the dynamic susceptibility over momenta in the vicinity of Q [where it equals $\chi_s(q, \omega)$]. In an insulating antiferromagnet, whose transition is described completely by the primary, $z = 1$ fixed point, χ_L'' would have the form shown in Fig. 1(a):⁴ There is gap below which the spectral density is strictly zero, a discontinuity at the gap, and for large ω we have $\chi_L'' \sim \omega^\eta$ where η is an anomalous dimension of the $z = 1$ fixed point.^{3,4} Here, by large ω , we mean frequencies which are large compared to the spin-fluctuation energy scale, but small compared to upper cutoffs like the an exchange constant or the Fermi energy; this and similar restrictions will be implicitly assumed in the remainder of the paper. Let us now examine the change in the spectrum due to a relevant perturbation which moves the system toward

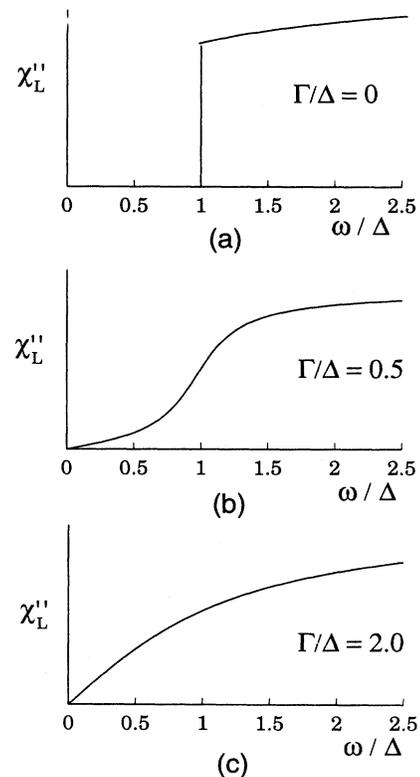


FIG. 1. Sketch of the imaginary part of the local dynamic susceptibility χ_L'' (obtained by integrating over momenta in the vicinity of the antiferromagnetic ordering wave vector) of a nearly antiferromagnetic Fermi liquid in two dimensions at zero temperature. (a) The $T = 0$ damping constant Γ is exactly 0, as is the case for an insulating antiferromagnet in a quantum-disordered phase; Δ is then the spin gap. (b) The consequences of a small value of Γ ; there is now no true gap, only a gaplike knee in the spectrum. (c) A large value of Γ makes the spectrum relatively featureless.

the secondary, type-*B*, fixed point. Under appropriate conditions, mobile fermionic carriers can act as such a perturbation, and the damping due to the particle-hole continuum will introduce some subgap absorption: χ_L'' will then look like Fig. 1(b). The gap has turned into a pseudogap, and $\chi_L'' \sim \omega$ for small ω . However, we will still have $\chi_L'' \sim \omega^\eta$ for large ω as the primary fixed point behavior is expected to continue to hold at large ω . We now extract three parameters from this form for χ_L'' .

(i) An overall amplitude \mathcal{Z} : This is defined by

$$\mathcal{Z} = 4 \lim_{\omega \rightarrow \infty} \frac{\chi_L''(\omega)}{(\hbar\omega)^\eta} \quad \text{at } T = 0. \quad (2.1)$$

The factor of 4 is for convenience in a later model calculation.

(ii) and (iii) Two energy scales Δ and Γ , which measure the pseudogap and the strength of the damping, respectively. These are obtained by solving the constraints

$$\chi_L''(\hbar\omega = \Delta) = \frac{\mathcal{Z}M^\eta}{8}, \quad \frac{\Gamma}{\Delta^2} = \frac{4\pi}{\mathcal{Z}M^\eta} \lim_{\omega \rightarrow 0} \frac{\chi_L''(\omega)}{\hbar\omega},$$

both at $T = 0$. (2.2)

The factor of 8 in the first equation is chosen to be exactly twice the factor 4 in (2.1); the factor of 4π in the second equation is arbitrary and is for future convenience. The parameter M is the energy below which the anomalous scaling of the primary fixed point $\chi_L'' \sim \omega^\eta$ stops, and a convenient choice is $M \equiv \alpha \max(\Gamma, \Delta)$, where α is of order, but larger than, unity. The factors of M^η above also ensure that Δ and Γ have units of energy. Because of the tiny value of $\eta \approx 0.03$, the M^η can be dropped while determining Γ and Δ from the data; this is, however, not true in any analytical computation of scaling functions, as they are essential in canceling cutoff dependences and obtaining a universal result. The pseudogap Δ has therefore been defined as the frequency at which χ_L'' falls to half its large frequency value (modulo factors of M^η), while Γ is determined from the subgap absorption. For $\Gamma \leq \Delta$, Δ is roughly the location of the knee in $\chi_L''(\omega)$, while Γ is its width [see Fig. 1(b)]; however, the definitions above hold for all Γ/Δ . For very small Γ , when the system is very close to the primary fixed point, Γ and Δ measure the strength of the two relevant perturbations away from it; this is no longer true for large Γ/Δ , but our scaling results below will continue to be valid.

(iv) The final parameter sets the normalization of length scales. A convenient choice is to use the $T = 0$ spin-wave velocity c , defined by the q dependence of the peak in the imaginary part of the staggered spin susceptibility $\chi_s(q, \omega)$ at large q (in all expressions for χ_s *only*, it is implicitly assumed that wave vectors q are deviations from the ordering wave vector Q).

Scaling functions of the primary $z = 1$ fixed point were studied in some detail in Ref. 4, and we can easily extend that analysis to write down the crossover scaling functions for the present case. The main difference is that we have a new relevant parameter Γ , which must

be included in the scaling functions. We consider only case of the scaling function of the staggered susceptibility χ_s here, leaving for later other observables which have a somewhat more complicated crossover behavior. Using the fact that Γ was defined above to have the dimensions of energy, we can write

$$\chi_s(q, \omega) = \frac{\mathcal{Z}}{(k_B T)^{-\eta}} \left(\frac{\hbar c}{k_B T} \right)^2 \Phi_s(\bar{q}, \bar{\omega}, \bar{\Delta}, \bar{\Gamma}), \quad (2.3)$$

where Φ_s is a fully universal, dimensionless crossover function, and the dimensionless arguments measure values of the parameters in units of $k_B T$: Thus

$$\bar{q} \equiv \frac{\hbar c q}{k_B T}, \quad \bar{\omega} \equiv \frac{\hbar \omega}{k_B T}, \quad \bar{\Delta} \equiv \frac{\Delta}{k_B T}, \quad \bar{\Gamma} \equiv \frac{\Gamma}{k_B T}. \quad (2.4)$$

All letter exponents η, ν (to be used later) refer to the primary fixed point. The main condition for the validity of (2.3) is that the five energy scales $\hbar c k$, $\hbar \omega$, Δ , Γ , and $k_B T$ are all significantly smaller than upper cutoffs like the Fermi energy or an exchange constant. The ratios of the five energy scales can, however, be arbitrary.

At $\bar{\Gamma} = 0$, Φ_s reduces to the result of Ref. 4 (there are some minor differences in conventions), and we obtain the scaling function for case *A*. The criticality of the secondary fixed point appears in the $\bar{\Gamma} \rightarrow \infty$ limit. Then Φ_s should collapse into a secondary scaling function Φ_s^S (Refs. 18, 19) with the $\bar{\Gamma}$ argument removed. The other arguments, and the overall scale, of Φ_s will be multiplied by powers of $\bar{\Gamma}$, so that Φ_s^S and its arguments have scaling dimensions appropriate to the secondary fixed point. To illustrate this point more clearly, let us assume that the structure of the type-*B* fixed point is identical to the Gaussian $z = 2$ fixed point obtained in spin-density-wave mean-field theory; then we will have

$$\lim_{\bar{\Gamma} \rightarrow \infty} \Phi_s(\bar{k}, \bar{\omega}, \bar{\Delta}, \bar{\Gamma}) = \frac{1}{\bar{\Gamma}^{1-\eta}} \Phi_s^S \left(\frac{\bar{q}}{\bar{\Gamma}^{1/2}}, \bar{\omega}, \frac{\bar{\Delta}}{\bar{\Gamma}^{1/2}} \right)$$

up to logarithmic corrections. (2.5)

The logarithmic corrections arise because the $z = 2$ critical point has marginal perturbations in $d = 2$.¹⁵ It is, however, important to note that these logarithmic corrections, and indeed the entire leading critical behavior of the secondary fixed point, are contained completely within the primary universal scaling function Φ_s . Everything about the logarithmic terms is universal and, in particular, there is no nonuniversal cutoff-dependent argument to the logarithms. This universality is a key point, and is a consequence of our earlier choice to define the scaling functions with respect to the primary fixed point. In systems with Γ so large that it is greater than a high-energy cutoff like the Fermi energy, this strong universality will not hold; however, the large Γ limit of Φ_s can still be used to describe the critical behavior, with the understanding that the various numerical scale factors are not correct as they are now nonuniversal.

III. MEAN-FIELD SDW CALCULATIONS FOR THE HUBBARD MODEL

The remainder of the paper will make the rather abstract arguments of Sec. I concrete by presenting explicit calculations in some simple models. We start with the mean-field SDW theory of the disordering transition in doped antiferromagnets. The simplest model which displays such transition is a one-band Hubbard model given by

$$\mathcal{H} = -t \sum_{\langle i,j \rangle} a_{i,\sigma}^\dagger a_{j,\sigma} - t' \sum_{\langle i,j' \rangle} a_{i,\sigma}^\dagger a_{j',\sigma} + U \sum_i n_{\uparrow} n_{\downarrow}. \quad (3.1)$$

Here j and j' label the nearest and the next-nearest neighbors, respectively, and $n = c^\dagger c$ is the particle density. Depending on the density of carriers and the ratio t'/t , the Fermi surface of free electrons can either be closed, in which case it is centered at $(0,0)$ and located entirely inside the magnetic Brillouin zone, or it can be an open Fermi surface which crosses the magnetic Brillouin zone boundary (Fig. 2). We will not specify the values of t and t' for which the Fermi surface is open or closed for a particular doping concentration, but rather consider the critical behavior of the dynamic spin susceptibility in both cases.

Let us assume that the model has commensurate antiferromagnetic long-range order down to a transition point (this assumption may not hold for a particular choice of t and t' , but it should always hold for related models with additional momentum dependence in U ;²⁰ this momentum dependence, however, does not lead to new physics near the transition, and we will not consider this complication here). The (π, π) ordering implies that, e.g., the z component of the spin-density operator

$\vec{S}(q) = (1/2) \sum_k a_{k+q,\alpha}^\dagger \vec{\sigma}_{\alpha,\beta} a_{k,\beta}$ has a nonzero expectation value at $q = Q \equiv (\pi, \pi)$. In the SDW approach,²¹ the relation $\langle \sum_k a_{k+Q,\uparrow}^\dagger a_{k,\uparrow} \rangle = -\langle \sum_k a_{k+Q,\downarrow}^\dagger a_{k,\downarrow} \rangle = \langle S_z \rangle$, is used to decouple the quartic term in (3.1). After decoupling and diagonalization of the quadratic form, one obtains²²

$$\mathcal{H}_{\text{SDW}} = \sum_k' E_k^c c_{k\sigma}^\dagger c_{k\sigma} + E_k^d d_{k\sigma}^\dagger d_{k\sigma}, \quad (3.2)$$

where the prime on the summation sign indicates that it is over the reduced magnetic Brillouin zone, and we introduced

$$E_k^c = E_k^- + \epsilon_k^+, \quad E_k^d = -E_k^- + \epsilon_k^+, \quad (3.3)$$

where $E_k^- = \sqrt{N_0^2 + (\epsilon_k^-)^2}$, and $N_0 = U \langle S_z \rangle$, $\epsilon_k^\pm = (\epsilon_k \pm \epsilon_{k+Q})/2$, and $\epsilon_k = -2t(\cos k_x + \cos k_y) - 4t' \cos k_x \cos k_y$. We will refer to the quasiparticles described by c and d operators as conduction and valence fermions, respectively. Finally, the self-consistency condition on $\langle S_z \rangle$ is

$$\frac{1}{U} = \sum_k' \frac{n_k^d - n_k^c}{E_k^-}. \quad (3.4)$$

Below we will assume that t' is negative. This is consistent with numerical calculations^{23,24} and fits to the measured²⁵ shape of the Fermi surface for $\text{YBa}_2\text{Cu}_3\text{O}_7$.

We now describe the evolution of the Fermi surface with doping. The shape of the Fermi surface is determined from $E^{c,d} = \mu$. At half-filling, the system is an insulator ($|\mu| < N_0$), all valence band states are occupied, and all conduction band states are empty. At finite doping, the chemical potential moves into the valence band. As the maximum of E_k^d is located at $(\pi/2, \pi/2)$ and symmetry-related points, the Fermi surface first opens in the form of hole pockets located around these points [Fig. 3(a)]. The subsequent evolution of the Fermi surface with doping in the SDW phase depends on the ratio t'/t . For sufficiently large $|t'|$, the bottom of the conduction band also becomes smaller than μ above some particular doping concentration, and electron pockets appear in the conduction band [Fig. 3(b)]. The energy gap between the hole and electron pockets shrinks gradually as N_0 decreases, and, at the transition point, the electron and hole pockets merge at $(x_0, \pi - x_0)$ and symmetry-related points, where $\cos^2 x_0 = |\mu/4t'|$, and the value of $|\mu|$ is determined from the solution of Eq. (3.4) with $N_0 \rightarrow 0$ [Fig. 3(c)]. It can be verified that this transition is of type *B*, and Q can span two points on the Fermi surface in the paramagnetic phase. For small enough $|t'|$, the situation is different: The size of hole pockets continues to increase with doping, and at some concentration x_1 , the hole pockets extend to $(\pi, 0)$ and related points while magnetic order is still present [Fig. 3(d)]. At concentrations smaller than x_1 , electron pockets either do not appear at all, or they first appear, their size passes through a maximum, and then they disappear at some doping concentration $x_2 < x_1$. At some concentration $x_3 > x_2$, the long-range order finally disappears,

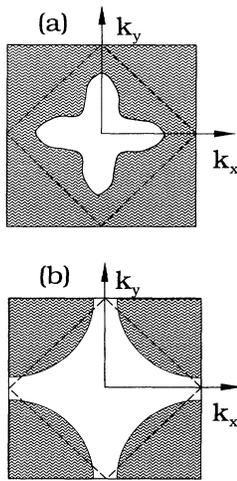


FIG. 2. (a) Fermi surface of type *A*: ordering wave vector $Q = (\pi, \pi)$ cannot connect two points on the Fermi surface. (b) Fermi surface of type *B*: Q can connect two points on the Fermi surface. Dark regions correspond to unoccupied electronic states in the first Brillouin zone.

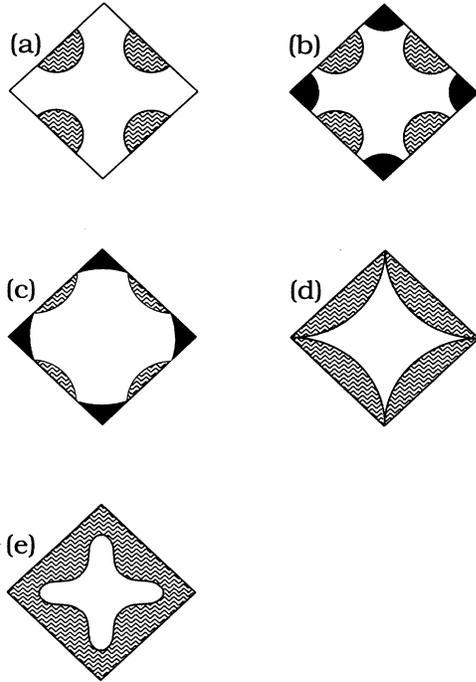


FIG. 3. The evolution of the Fermi surface with doping in the Hubbard model with first- and second-neighbor hopping. Shaded regions correspond to hole pockets and black regions correspond to electron pockets (i.e., doubly occupied states in the first Brillouin zone). The description of the figures is given in the text.

but the Fermi surface at the transition is located inside magnetic Brillouin zone boundary [Fig. 3(e)]. The disordering transition is therefore of type A.²⁶

We now present results from a mean-field SDW calculation of the uniform and staggered susceptibilities; details can be found in the Appendix. We shall mainly be interested in the small q and ω behavior of these responses when N_0 is small. We shall find that an important feature of type-*B* systems is that the limits $q, \omega \rightarrow 0$ and $N_0 \rightarrow 0$ do not commute. We consider the two cases separately.

A. Type A

The transverse, staggered, susceptibility is found to behave as

$$\chi_{s\perp}(q, \omega) = \frac{N_0^2}{\rho_s} \frac{1}{q^2 - \omega^2/c^2 - i\omega q N_0^2}, \quad (3.5)$$

where a, c are constants (i.e., finite as $N_0 \rightarrow 0$) and $\rho_s \sim N_0^2 \sim (g_c - g)$ is a spin stiffness. It is clear that the damping is negligible as $N_0 \rightarrow 0$, and that the limits $q, \omega \rightarrow 0$ and $N_0 \rightarrow 0$ do commute. The susceptibility is also that expected for a $z = 1$ system.

In the quantum-disordered phase we then have $\chi_s \sim (q^2 - \omega^2/c^2 + \Delta^2/c^2)^{-1}$, where the gap Δ vanishes at

the transition point. The pole in χ_s corresponds to a positive energy, spin-1 particle-hole bound state which has center-of-mass momentum near Q , and which lies below the bottom of the particle-hole continuum at these momenta.

The transverse, uniform susceptibility at $N_0 \rightarrow 0$ can be represented as a sum of the interband and intraband contributions

$$\chi_{u\perp} = \chi_{u\perp}^{\text{inter}} + \chi_{u\perp}^{\text{intra}}, \quad (3.6)$$

where we found that $\chi_{u\perp}^{\text{inter}} \sim O(N_0^2)$ and $\chi_{u\perp}^{\text{intra}} \sim \text{const} + O(N_0^2)$. So $\chi_{u\perp}$ is finite at the transition, as it should be when reaching a Fermi liquid. However, the assertion by Millis⁵ that the spin-wave velocity should be given by a hydrodynamic expression²⁷ $c_{\text{SW}}^2 = \rho_s/\chi_{u\perp}$ is seen to be incorrect: $c_{\text{SW}} = c$ is in fact finite at the transition. The correct relationship actually turns out to be $c_{\text{SW}}^2 = \rho_s/\chi_{u\perp}^{\text{inter}}$.

B. Type B

The noncommutativity of the limits $q, \omega \rightarrow 0$ and $N_0 \rightarrow 0$ now leads to much more complex behavior than for type A. Consider, first, the transverse, staggered susceptibility at $q = 0$; in this case a relatively complete evaluation is possible, and we find

$$\chi_{s\perp}(0, \omega) \sim -\frac{N_0^2}{\rho_s} \left[\frac{\omega^2}{\sqrt{4N_0^2 - \omega^2}} + O(\omega^2) \right]^{-1}, \quad (3.7)$$

where $\rho_s \sim N_0^2 \sim (g_c - g)$. The first, singular term arises from interband processes and is due to the integration over the region of momentum space where the Fermi surface crosses the magnetic Brillouin zone boundary; there is no such point for type-A systems. The second term is the regular contribution from the remainder of the momentum space; it is similar to that found in type-A systems. It is clear that the behavior of $\chi_{s\perp}$ is dramatically different for $\omega \ll N_0$ and $\omega \gg N_0$.

The q dependence of $\chi_{s\perp}$ only has a relatively simple form in the limiting regions. For $q, \omega \ll N_0$ we have

$$\chi_{s\perp}(q, \omega) = \frac{N_0^2}{\rho_s} \left[\left(q^2 - \frac{\omega^2}{c^2} \right) - \frac{\omega}{N_0} [b_1\omega - ib_2\psi(\vec{q}, \omega)] \right]^{-1}, \quad (3.8)$$

where c, b_1 , and b_2 are constants, and $\psi(\vec{q}, \omega) = \psi(|q_x + q_y|, |q_x - q_y|, \omega)$ is a symmetrical function of its first two arguments, which at $q \gg \omega/t$ is linear in q . The explicit form of ψ is presented in the Appendix. We see that the lowest-energy excitations are overdamped even in the magnetically ordered phase, although we indeed have a Goldstone mode at $q = 0$. Also, the renormalized spin-wave velocity (defined in the limit of small b_2) decreases as $c_{\text{SW}} \sim N_0^{1/2}$. In the opposite limit, $q, \omega \gg N_0$ (and for all q, ω at the transition point) both terms in the last

bracket in (3.8) become imaginary, and we find

$$\chi_{s\perp}(q, \omega) \sim (q^2 - \omega^2/c^2 - i\omega\gamma)^{-1}. \quad (3.9)$$

The magnon excitations are now clearly overdamped. In the SDW approximation we found that γ *does not* depend on the ratio ω/q . In the quantum-disordered phase we then have

$$\chi_s \sim (q^2 - \omega^2/c^2 + i\omega\gamma + \Delta^2/c^2)^{-1}, \quad (3.10)$$

where Δ is now a pseudogap and vanishes at the transition point. Notice also that c is *not* the velocity of the longest-wavelength spin-wave excitations in the ordered phase, though indeed it has the same order.

The same form of susceptibility was proposed earlier by Barzykin *et al.*⁷ on phenomenological grounds. We will use Eq. (3.10) below to motivate the field theory for the critical point and the paramagnetic phase.

As for type A, the transverse, uniform susceptibility is found to have two contributions from interband and intraband processes which now take the form

$$\chi_{u\perp} = \chi_{u\perp}^{\text{inter}} + \chi_{u\perp}^{\text{intra}}, \quad (3.11)$$

where now $\chi_{u\perp}^{\text{inter}} \sim O(N_0)$ and $\chi_{u\perp}^{\text{intra}} \sim \text{const} + O(N_0)$. Again, $\chi_{u\perp}$ is finite at the transition, and the relationship⁵ $c_{\text{SW}}^2 = \rho_s/\chi_{u\perp}$ does not work as it would predict $c_{\text{SW}} \sim N_0$. The correct relationship is $c_{\text{SW}}^2 = \rho_s/\chi_{u\perp}^{\text{inter}}$. In practice, however, it is difficult to find conditions when one can neglect the b_2 term in (3.8). Magnon excitations are then overdamped, and the issue of the behavior of the spin-wave velocity near the transition is not that relevant. Notice also that as shown in the Appendix, the $O(N_0)$ contributions in $\chi_{u\perp}^{\text{inter}}$ and $\chi_{u\perp}^{\text{intra}}$ cancel each other, so that $\chi_{u\perp} = \text{const} + O(N_0^2)$.

C. Relationship to results in the Shraiman-Siggia model

The main weakness of the above mean-field SDW computations is that they underestimate the contribution of magnetic fluctuations, which may destroy long-range order at a smaller doping. Such fluctuations may be more completely accounted for by studying the phase transition in the Shraiman-Siggia (SS) model,⁹ which, however, has other weaknesses to be described below. In this subsection we will review the results of such a study,¹² compare them with the above SDW results, and make some speculations on how they may be reconciled. Readers not interested in this issue can skip this subsection without any loss of continuity.

The SS model is a continuum theory of interactions between mobile holes and spin fluctuations in a t - J model, i.e., a model in which strong local repulsion between electrons allows one to project out states with more than a single electron on a site. In the language of the Hubbard model, there is a significant band gap between the lower and upper Hubbard bands, and the SS model focuses exclusively on physics within the lower band. At half fill-

ing, the lower and upper Hubbard bands can be identified with the valence and conduction bands, respectively, of the SDW theory discussed above. Therefore, in the SDW formalism the SS model describes the physics within the valence band. Furthermore, in the SDW mean-field theory, the minimum, direct band gap between the conduction and valence bands is proportional to the Néel order parameter N_0 , and therefore vanishes when long-range order disappears. This is, however, not the case in the SS model, in which N_0 and the band gap are independent parameters, and the band gap is assumed to remain finite at the point where N_0 vanishes. This is a key difference between the two approaches.

Let us now approach the magnetic transition from the ordered side in the SS model. A simple computation, similar to those in Ref. 12, shows that the transverse, staggered susceptibility has precisely the form (3.5) obtained above in the SDW theory of a type-A transition; in particular the result $\text{Im}(\chi_{s\perp})^{-1} \sim N_0^2\omega q$ is also obtained in the SS model [thus the proper interpretation of the factor of N_0^2 in (3.5) in SDW theory is that of the square of the magnetization order parameter, rather than the valence or conduction band gap]. At the point where N_0 vanishes, computations in the SS model show that¹² $\chi_{s\perp} \sim (q^2 - \omega^2/c^2 + ia'\omega^2q)^{-1}$. In the mean-field SDW theory above we have $\chi_{s\perp} \sim (q^2 - \omega^2/c^2)^{-1}$ with no damping term; however, it is quite reasonable to expect that including higher-order paramagnon fluctuation corrections in SDW theory will lead to a damping term rather similar to that obtained in the SS model. So far, therefore, the spin-fluctuation properties of the magnetic transition in the SS model are essentially identical to those obtained in the SDW theory of the $z = 1$ transition in type-A systems.

However, differences do appear when we consider the fermionic excitations. The type-A transition in SDW theory has a large electron Fermi surface wholly within the magnetic Brillouin zone, which changes little between the two phases on either side of the critical point. On the contrary, the Fermi surface of the SS model, on the ordered side of the critical point, consists of elliptical hole pockets at the boundary of the magnetic Brillouin zone. The fate of the elliptical hole pockets in the quantum-disordered phase of the SS model is not at all clear.¹² Below we present a reasonable, but speculative, scenario: We propose that in this phase, the Fermi surface is large (i.e., encloses a volume equal to the total number of electrons), as it was in both types of SDW transitions. However, the quasiparticle residue is very anisotropic so that at the transition to Néel state, the residue vanishes everywhere except for the regions which surround the hole pockets in the ordered phase.²⁸ This implies that the critical theory of the $z = 1$ transition in the SS model in Ref. 12 remains essentially correct. If this large Fermi surface intersects the magnetic Brillouin zone boundary in the disordered phase, then there will be a *finite* $T = 0$ damping γ at $q = 0$ in the quantum-disordered phase; however, γ will vanish faster than the gap Δ as one approaches the critical point, thus behaving as a dangerously irrelevant perturbation (i.e., a perturbation which is irrelevant very near transition but leads to a new physics at some dis-

tance away from the transition). Alternatively, the large Fermi surface with anisotropic quasiparticle residue may be entirely within the magnetic Brillouin zone; in this case the properties of the quantum-disordered phase will be very similar to those of a type-*A* transition in SDW theory.

IV. MODEL FIELD THEORY FOR TYPE *B*

This section will present explicit computations of the scaling functions of Sec. II in a model field theory appropriate for a mean-field type-*B* transition. The motivation for the model follows the logic of Hertz:¹⁵ The action S for paramagnon fluctuations in the quantum-disordered phase should have a propagator which reproduces the mean-field, type-*B*, spin susceptibility in (3.9). The partition function of S in Matsubara imaginary time then has a form

$$\begin{aligned} Z &= \int \mathcal{D}\vec{n}(x, \tau) \delta(\vec{n}^2(x, \tau) - 1) \exp\{-S[\vec{n}(x, \tau)]\}, \\ S &= \frac{1}{2g} T \sum_{\omega_n} \int \frac{d^2q}{4\pi^2} |\vec{n}(q, \omega_n)|^2 (q^2 + \omega_n^2/c_0^2 + \gamma|\omega_n|). \end{aligned} \quad (4.1)$$

The action is written in Fourier space, and ω_n is a Matsubara frequency. The vector field $\vec{n}(x, \tau)$ represents the local orientation of the antiferromagnetic order parameter; we will allow \vec{n} to have N components to allow a subsequent large N calculation. We have chosen to implement a fixed-length constraint $\vec{n}^2 = 1$ to mimic interactions between the paramagnon modes. This restriction is, however, not crucial and identical scaling results would be obtained in a model with a more conventional $(u/2N)(\vec{n}^2)^2$ interaction. The scaling results appear a little more directly in the fixed-length model. The first two terms in S are those found in the usual $O(N)$ nonlinear σ model. The last term arises from the damping induced by the fermion particle-hole pairs. The action S was first explicitly written down by Liu and Su⁸—they used a two-component microscopic model in which the spins and fermions are locally independent degrees of freedom, and then integrated the fermions out.

We expect S to provide a reasonable description of the temperature-dependent crossovers in the quantum-disordered phase. However, close enough to the critical point in the quantum-disordered phase, the validity of S appears to us to not have been established convincingly. It is likely that it will be necessary to account for fluctuations involving incipient formation of the spin-density-wave gap over portions of the Fermi surface—note that these gaps form at precisely the same points on the Fermi surface which are responsible for the low-frequency damping on the disordered side; these issues will be examined in more detail in subsequent work. It is also easy to see explicitly that S certainly breaks down on the magnetically ordered side of the transition. On

this side, the spin-fluctuation propagator changes substantially when q and ω become smaller than the order parameter N_0 [see, e.g., Eq. (3.7)], and these changes *cannot* be implemented by simply introducing a condensate of the order parameter into S .

In the remainder of Sec. IV we will focus exclusively on the model S . We will describe its asymptotic critical behavior, and compute its scaling functions in a large N expansion.

It is also interesting to note in passing that the $|\omega|$ dissipation in S is similar to that used in the macroscopic quantum tunneling literature.²⁹ There the emphasis is on the consequence of this dissipation on the quantum mechanics of a single, heavy particle, whereas we are studying its effects on a field theory describing a large number of degrees of freedom.

Let us first discuss some general scaling properties of S from the vantage point of the primary, $z = 1$ fixed point. This fixed point clearly lies on the submanifold of the parameter space with $\gamma = 0$. The term proportional to γ is clearly a relevant perturbation at the primary fixed point. Moreover, $|\omega_n|$ is nonanalytic in frequency and leads to a long-range $-1/\tau^2$ interaction in space-time. Theories with long-range interactions of this type are familiar in the context of finite-temperature classical phase transitions³⁰ and many results can be transferred over. In particular, the $|\omega_n|$ term gets no singular loop renormalizations, and the scaling dimension of γ is exactly $1 - \eta$. Therefore, at $T = 0$, the response functions will therefore be scaling functions of the ratio $\gamma/(g - g_c)^{(1-\eta)\nu}$. The fully renormalized parameters introduced earlier will have a different dependence on the bare coupling constants depending on the value of this ratio; this dependence can be deduced by standard scaling arguments. For $\gamma \ll (g - g_c)^{(1-\eta)\nu}$ we will have dependences characteristic of the primary fixed point for which

$$\mathcal{Z} \sim \text{const}, \quad \Delta \sim (g - g_c)^\nu, \quad \Gamma \sim \gamma(g - g_c)^{\eta\nu}. \quad (4.2)$$

In the opposite limit $\gamma \gg (g - g_c)^{(1-\eta)\nu}$ we are much closer to the critical point and dominated by secondary behavior; in this case we have

$$\begin{aligned} \mathcal{Z} &\sim \text{const}, \quad \Delta \sim (g - g_c)^{1/2} \gamma^{(1-1/2\nu)/(1-\eta)}, \\ \Gamma &\sim \gamma^{1/(1-\eta)}. \end{aligned} \quad (4.3)$$

Explicit results for scaling functions of S can be obtained in a $1/N$ expansion. The methods are very similar to those discussed at length in Ref. 4 for $\gamma = 0$; one only has to add a $\gamma|\omega_n|$ to each propagator. We will therefore merely present the final results. At $N = \infty$ we found

$$\Phi_s^{N=\infty}(\bar{q}, \bar{\omega}, \bar{\Delta}, \bar{\Gamma}) = \frac{1}{\bar{q}^2 - \bar{\omega}^2 - i\bar{\Gamma}\bar{\omega} + m^2(\bar{\Delta}, \bar{\Gamma})}. \quad (4.4)$$

We chose the definition of the parameters earlier so that as $T \rightarrow 0$, $m = \bar{\Delta}$. For finite T , m is given implicitly as a universal function of $\bar{\Delta}$ and $\bar{\Gamma}$ by

$$\begin{aligned} \frac{\bar{\Gamma}}{2} \ln \left(\frac{\bar{\Delta}^2}{m^2} \right) + \bar{\Gamma} \phi \left(\frac{m}{\bar{\Gamma}} \right) - \bar{\Gamma} \phi \left(\frac{\bar{\Delta}}{\bar{\Gamma}} \right) \\ = 2 \int_0^\infty \frac{d\Omega}{e^\Omega - 1} \arctan \left(\frac{\bar{\Gamma} \Omega}{m^2 - \Omega^2} \right), \end{aligned} \quad (4.5)$$

where the value of the arctangent runs from 0 to π as Ω runs from 0 to ∞ , and

$$\phi(x) = \begin{cases} (4x^2 - 1)^{1/2} \tan^{-1}(4x^2 - 1)^{1/2} & \text{for } x \geq 1/2, \\ \frac{(1 - 4x^2)^{1/2}}{2} \ln \frac{1 - (1 - 4x^2)^{1/2}}{1 + (1 - 4x^2)^{1/2}} & \text{for } x \leq 1/2. \end{cases} \quad (4.6)$$

Despite appearances to the contrary, $\phi(x)$ is analytic for all real x , including $x = 1/2$. In the limit $\bar{\Gamma} \rightarrow 0$, (4.5) reproduces the $z = 1$ result of Ref. 4: $m = 2 \sinh^{-1}(e^{\bar{\Delta}/2}/2)$. In the opposite limit $\bar{\Gamma} \rightarrow \infty$ we expect properties of the secondary fixed point—we see from (2.5) that $m^2/\bar{\Gamma}$ should be a function of $\bar{\Delta}^2/\bar{\Gamma}$. This is exactly what we find from (4.5) which reduces to

$$\begin{aligned} \frac{m^2 - \bar{\Delta}^2}{\bar{\Gamma}} + \frac{m^2}{\bar{\Gamma}} \ln \frac{\bar{\Gamma}^2}{m^2} - \frac{\bar{\Delta}^2}{\bar{\Gamma}} \ln \frac{\bar{\Gamma}^2}{\bar{\Delta}^2} \\ = 2 \int_0^\infty d\Omega \frac{\tan^{-1}[\Omega/(m^2/\bar{\Gamma})]}{e^\Omega - 1} \end{aligned} \quad (4.7)$$

for large $\bar{\Gamma}$. The only violation of the $\bar{\Delta}^2/\bar{\Gamma}$ scaling in

the above comes from the logarithms, which was also expected; the arguments of the logarithms, however, remain universal. Analysis of the solution of (4.7) for m can be shown to yield the same crossovers (including and upto log-log corrections) as those discussed in Ref. 5 in the quantum-disordered region for $d = 2$, $z = 2$.

The calculations of the $1/N$ corrections also follow Ref. 4. The evaluation requires substantial numerical computations which are currently being carried out. However, we also know from Ref. 4 that, at least at $\bar{\Gamma} = 0$, these corrections were quite small, and that the $N = \infty$ results were satisfactory for most purposes [a notable exception is $\chi''(\omega)$ for $\Gamma \ll \Delta$, for which $1/N$ corrections are relevant]. The agreement of the large Γ , $N = \infty$ results with those of Ref. 5 is also encouraging.

A number of experimentally measurable quantities can be deduced from the above results. For example, the antiferromagnetic correlation length is

$$\xi = \frac{\hbar c}{mT}, \quad (4.8)$$

and the NMR relaxation rates (for a review, see, e.g., Ref. 31), up to known prefactors associated with hyperfine coupling constants, are^{7,32}

$$\frac{1}{T_1} \propto \frac{\bar{\Gamma}}{m^2}, \quad \frac{1}{T_{2G}} \propto \frac{1}{mT}. \quad (4.9)$$

We plot in Fig. 4 the universal crossover function for $\hbar c/\xi\Delta$ as a function of $k_B T/\Delta$ for various values of Γ/Δ . For $\Gamma \ll \Delta$ we have two main regimes of temperature:

$$\begin{aligned} \text{(a) } k_B T \gg \Delta : \quad \xi &= 2 \ln \left(\frac{\sqrt{5} + 1}{2} \right) \frac{\hbar c}{k_B T}; \\ \text{(b) } k_B T \ll \Delta : \quad \xi &= \frac{\hbar c}{\Delta} \left\{ 1 - 2 \frac{k_B T}{\Delta} \left[\exp \left(-\frac{\Delta}{k_B T} \right) + \frac{\pi}{6} \frac{k_B T}{\Delta} \frac{\Gamma}{\Delta} \right] \right\}. \end{aligned} \quad (4.10)$$

Regime (a) is the $z = 1$ quantum-critical behavior^{1,4} and (b) is the $z = 1$ quantum-disordered behavior. At very low temperatures, the damping gives rise to a power law rather than exponential behavior of the correlation length; these power laws are the same as for quantum-disordered, $z = 2$ behavior, discussed below. However, unlike for “pure” $z = 2$ relaxational behavior, $\chi''(\omega)$ has a knee at $\omega \sim \Delta \gg T$. Based on the presence of such knee [shown in Fig. 1(b)], we identify this regime as quantum disordered, $z = 1$.

For $\Gamma \gg \Delta$ there are three subregimes:

$$\begin{aligned} \text{(a')} \quad k_B T \gg \Gamma : \quad \xi &= 2 \ln \left(\frac{\sqrt{5} + 1}{2} \right) \frac{\hbar c}{k_B T}; \\ \text{(b')} \quad \frac{\Delta^2}{\Gamma} \ll k_B T \ll \Gamma : \quad \xi &= f_1 \left(\frac{k_B T}{\Gamma} \right) \frac{\hbar c}{(\Gamma k_B T)^{1/2}}; \\ \text{(c')} \quad k_B T \ll \frac{\Delta^2}{\Gamma} : \quad \xi &= \frac{\hbar c}{\Delta} \left[1 - \frac{\pi^2}{12 \log(\Gamma/\Delta)} \left(\frac{k_B T}{\Delta} \right)^2 \left(\frac{\Gamma}{\Delta} \right)^2 \right], \end{aligned} \quad (4.11)$$

with $f_1(x)$ a very slowly (logarithmically) varying, numerically calculable function of order unity. Now regime (a') is $z = 1$ quantum critical, (b') is $z = 2$ quantum critical, and (c') is $z = 2$ quantum disordered. In the neg-

ligibly small subregime of (b') where $\ln \ln(\Gamma/k_B T) \gg 1$, while maintaining $k_B T \gg \Delta^2/\Gamma$ we have $\xi = \hbar c/(\pi \Gamma k_B T)^{1/2} [\ln(\Gamma/k_B T)/\ln \ln(\Gamma/k_B T)]^{1/2}$. Most of the above asymptotic results are not very useful at any

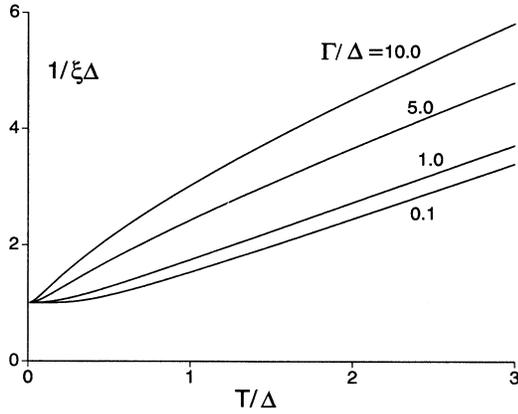


FIG. 4. Universal crossover behavior of the inverse antiferromagnetic correlation length $1/\xi$ (measured in units of $1/\hbar c$) as a function of the temperature T , for a variety of values of Γ/Δ . The plot is the $N = \infty$ result contained in Eqs. (4.5) and (4.8). There are no arbitrary scale factors, and the scales of both axes are universal and meaningful.

realistic value of Γ/Δ , and the exact values plotted in Fig. 4 should be used for experimental comparisons.

The above regimes are sketched in Fig. 6 below, which will be discussed in Sec. V.

A. Uniform susceptibility

Consider now the response of the nearly antiferromagnetic Fermi liquid to a uniform magnetic field. Unlike the staggered susceptibility which diverges at the transition, the uniform susceptibility χ_u is expected to remain finite. As a result, the crossover behavior of χ_u is more complicated and corrections to the leading scaling result now play a significant role.

The following results are for type-*B* systems; however, they can also be applied to type-*A* systems after imposing $\gamma = \Gamma = 0$.

We need the modifications to the action S , in the presence of an external field H . These corrections are equivalent to evaluating the effective three-point and four-point couplings between the antiferromagnetic order parameter and the uniform magnetization after integrating out the fermions; however, as we are mainly interested in the case of a uniform field, we will just carry the exact field dependence through in all the fermion propagators (see the Appendix).

In insulating antiferromagnets, the coupling to the field was given exactly by a symmetry-related argument³³ with no additional coupling constants appearing. These symmetry arguments must now be applied with more caution because of the presence of gapless fermionic excitations in the quantum-disordered phase. In particular, we need to account for the continuous change in the Fermi-liquid ground state in the presence of H associated

with the expanding (shrinking) of the up- (down-) spin Fermi surface.

Consider, first, the effect of H on the $|\omega_n|$ term in S . At small ω_n , this term arises from the absorption due to particle-hole pairs with momentum Q , made up of electrons and holes just above and below the Fermi surface. A field H will shift the bottom of the band of the up and down spins in opposite directions. In a model with a flat density of states, such a shift should have *no* effect on the particle-hole spectrum. So for small H , there is no change in the $|\omega_n|$ term. At larger H there will be a change, because there is always some structure in the density of states: This will show up as a field dependence in the value of the coupling γ : $\gamma \rightarrow \gamma(1 + \lambda \vec{H}^2)$ where λ is a small coupling. This is verified in an explicit computation of the field dependence of S discussed in the Appendix.

The computation in the Appendix also finds a precession term

$$\frac{i}{gc_0^2}(1 + \alpha) \int d^2x d\tau \vec{H} \cdot \vec{n} \times \frac{\partial \vec{n}}{\partial \tau}, \quad (4.12)$$

where we have absorbed in the definition of H a factor $gG\mu_B/\hbar$ ($gG\mu_B$ is the gyromagnetic factor). The prefactor of (4.12) has been written such that the coupling $\alpha = 0$ in the insulating limit; in this limit the precession rate of the spins is known exactly and hence (4.12) has no new coupling constants. In the doped system, there is a correction to the precession rate from internal fields generated by the polarization of the fermions, and this is represented by the new coupling α . If the system is not too strongly doped, we can expect that α is not too large.

Finally, there is a term (also derived in the Appendix), $\sim (\vec{n} \times \vec{H})^2$, which imposes an energetic preference for the relative orientation of the antiferromagnetic order parameter and the magnetic field.

All three terms discussed above induce nonuniversal corrections in the $T = 0$ value of χ_u . These may be interpreted as corrections to the fermionic Pauli susceptibility from higher-order interactions between the paramagnon modes.

Things do simplify, however, when we consider the T -dependent part of χ_u . Then the precession coupling in (4.12) turns out to be the most important in many cases. The T -dependent corrections from the H^2 dependence of γ can be shown by standard scaling arguments to be subdominant near the primary fixed point; near the secondary fixed point this term turns out to have the same T dependence as that due to (4.12), but with a prefactor of $(\gamma/t)^2$. Finally, the last $(\vec{n} \times \vec{H})^2$ term yields no temperature dependence in a model with a fixed-length constraint. In a soft-spin theory, scaling arguments show that this term is also subdominant near the primary fixed point; near the secondary fixed point, and going beyond $N = \infty$ limit for soft-length spins, one obtains a T dependence similar to that due to the H^2 dependence of γ .

We see, therefore, that if γ is sufficiently large, all three terms contribute roughly equally to the uniform suscepti-

bility. The temperature dependence of χ_u is then nonuniversal, and reliable predictions for experiments are difficult. To keep our discussion simple, and also because we think that this is most realistic experimentally, we will only present explicit results for χ_u for the case when γ is not too large. We will therefore only consider the T dependence of χ_u induced by the term (4.12) which dominates near the primary fixed point.

We now consider the consequence of (4.12) on the T dependence of χ_u . Application of our earlier scaling arguments and the results of Ref. 4 allow us to deduce the following crossover scaling function for χ_u :

$$\Phi_u^{N=\infty}(\bar{\Delta}, \bar{\Gamma}) = \frac{\bar{\Gamma}}{\pi^2} \left[\frac{1}{2} \ln \left(\frac{\bar{\Delta}^2}{m^2} \right) + \frac{2m^2 - \bar{\Gamma}^2}{4m^2 - \bar{\Gamma}^2} \phi \left(\frac{m}{\bar{\Gamma}} \right) - \frac{2\bar{\Delta}^2 - \bar{\Gamma}^2}{4\bar{\Delta}^2 - \bar{\Gamma}^2} \phi \left(\frac{\bar{\Delta}}{\bar{\Gamma}} \right) + 2 \int_0^\infty \frac{d\Omega}{e^\Omega - 1} \frac{\Omega^3}{(m^2 - \Omega^2)^2 + \bar{\Gamma}^2 \Omega^2} \right], \quad (4.14)$$

where m has to be determined from (4.5) as a function of $\bar{\Delta}$ and $\bar{\Gamma}$, and the function $\phi(x)$ was defined in (4.6). A plot of the universal contribution from Φ_u to χ_u is shown in Fig. 5, with the nonuniversal prefactor $1 + \alpha'$ dropped. There are several different regimes, similar to those found in the correlation length. Dropping the $1 + \alpha'$, we state the leading behavior of $\chi_u(T)$ in these regimes. For $\Gamma \ll \Delta$ we have the two regimes of temperature:

$$\begin{aligned} \text{(a) } k_B T \gg \Delta : \quad \chi_u(T) - \chi_u(0) &= \frac{\sqrt{5}}{\pi} \ln \left(\frac{\sqrt{5} + 1}{2} \right) \frac{k_B T}{c^2}; \\ \text{(b) } k_B T \ll \Delta : \quad \chi_u(T) - \chi_u(0) &= \frac{\Delta}{\pi c^2} \left[\exp \left(-\frac{\Delta}{k_B T} \right) + \frac{\pi}{6} \frac{\Gamma}{\Delta} \left(\frac{k_B T}{\Delta} \right)^2 \right]. \end{aligned} \quad (4.15)$$

For $\Gamma \gg \Delta$ the three subregimes are

$$\begin{aligned} \text{(a')} \quad k_B T \gg \Gamma : \quad \chi_u(T) - \chi_u(0) &= \frac{\sqrt{5}}{\pi} \ln \left(\frac{\sqrt{5} + 1}{2} \right) \frac{k_B T}{c^2}; \\ \text{(b')} \quad \frac{\Delta^2}{\Gamma} \ll k_B T \ll \Gamma : \quad \chi_u(T) - \chi_u(0) &= f_2 \left(\frac{k_B T}{\Gamma} \right) \frac{k_B T}{c^2}; \\ \text{(c')} \quad k_B T \ll \frac{\Delta^2}{\Gamma} : \quad \chi_u(T) - \chi_u(0) &= \frac{\Gamma}{6 \ln(\Gamma/\Delta)} \left(\frac{k_B T}{\Delta c} \right)^2, \end{aligned} \quad (4.16)$$

with f_2 a function similar to the function f_1 above in (4.11). Regimes (a), (b), (a'), (b'), and (c') are the same as those described for the correlation length [notice that in regime (b), the damping term gives rise to a power-law behavior at very low T]. Again, as with the correlation length, most of the asymptotic expressions are not useful for realistic Γ/Δ and numerically determinable values of the crossover function should be used. Within regime (b'), it is possible, as was the case with ξ , to have a negligibly small subregime where $\ln \ln(\Gamma/k_B T) \gg 1$ while $k_B T \gg \Delta^2/\Gamma$; here we find $\chi_u(T) - \chi_u(0) = (k_B T/\pi c^2) [\ln \ln(\Gamma/k_B T)] / \ln(\Gamma/k_B T)$.

B. Specific heat

The leading contribution to the free energy density of S is strongly divergent in the ultraviolet $\sim \Lambda^3$, where Λ is an upper cutoff in momentum space. In the model with $\gamma = 0$, it was found in Ref. 4 that a single subtraction of the $T = 0$ value of the free energy density was sufficient to make the T -dependent remainder finite. There were no terms diverging with powers of Λ smaller than 3—

$$\chi_u(T) - \chi_u(T = 0) = (1 + \alpha') \frac{k_B T}{c^2} \Phi_u(\bar{\Delta}, \bar{\Gamma}), \quad (4.13)$$

where α' is a nonuniversal constant, which vanishes when $\alpha = 0$. The function Φ_u is a completely universal scaling function which we will compute below at $N = \infty$. The only nonuniversalities of the above result are therefore related to the coupling α' , and the value of $\chi_u(T = 0)$ which is dominated by the Pauli susceptibility of the fermions.

The $N = \infty$ computation of Φ_u for S can be performed as in Ref. 4 we find

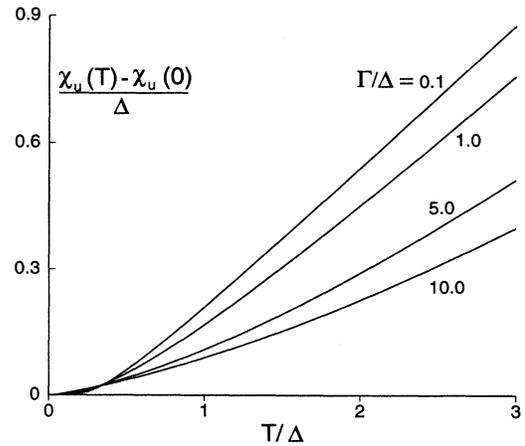


FIG. 5. As in Fig. 4, but now with results for the temperature-dependent part of the uniform spin susceptibility $\chi_u(T) - \chi_u(0)$ measured in units of $(g\mu_B/\hbar c)^2$. The nonuniversal constant α' in (4.13) has been set equal to 0. The plot is the $N = \infty$ result contained in Eqs. (4.5) and (4.14). Apart from the nonuniversal scale factor associated with α' , the scales of both axes are universal and meaningful.

a direct consequence of the hyperscaling properties of the original theory. For finite γ , we find here that this simple property does not hold: A single subtraction still leaves a term that is only logarithmically dependent on the upper cutoff. This is perhaps not surprising as the secondary fixed point violates hyperscaling. While most of this violation is actually cutoff by the crossover to the primary fixed point at high energies, some manages to survive.

We will restrict ourselves here to presenting the T dependence of the free energy density \mathcal{F} of the action S in a $N = \infty$ calculation. We found

$$\mathcal{F}(T) = \mathcal{F}(0) + \frac{(k_B T)^3}{(\hbar c)^2} \Psi(\bar{\Delta}, \bar{\Gamma}), \quad (4.17)$$

where the dimensionless function Ψ , instead of being fully universal, has a logarithmic dependence on Λ/T :

$$\begin{aligned} \Psi(\bar{\Delta}, \bar{\Gamma}) = & \frac{N}{4\pi^2} \int_0^\infty \frac{d\Omega}{e^\Omega - 1} \left[(m^2 - \Omega^2) \tan^{-1} \left(\frac{\bar{\Gamma}\Omega}{m^2 - \Omega^2} \right) + \frac{\bar{\Gamma}\Omega}{2} \ln \left(\frac{(m^2 - \Omega^2)^2 + \bar{\Gamma}^2 \Omega^2}{(\hbar c \Lambda / T)^4} \right) \right] \\ & - \frac{N\bar{\Gamma}}{48\pi^2} \left[\left(3m^2 - \frac{\bar{\Gamma}^2}{2} \right) \ln \frac{\bar{\Delta}^2}{m^2} + m^2 - \bar{\Delta}^2 + (4m^2 - \bar{\Gamma}^2) \phi \left(\frac{m}{\bar{\Gamma}} \right) + (2\bar{\Delta}^2 - 6m^2 + \bar{\Gamma}^2) \phi \left(\frac{\bar{\Delta}}{\bar{\Gamma}} \right) \right], \end{aligned} \quad (4.18)$$

where m is determined from (4.5) as a function of $\bar{\Delta}, \bar{\Gamma}$ and the function ϕ is defined in (4.6).

The asymptotic limits of the contribution of Ψ to the specific heat $C_V = -T\partial^2 \mathcal{F} / \partial T^2$ were determined in a manner similar to ξ and χ_u . For $\Gamma \ll \Delta$ we have

$$\begin{aligned} \text{(a) } k_B T \gg \Delta : C_V &= \frac{12\zeta(3)N}{5\pi} k_B \left(\frac{k_B T}{\hbar c} \right)^2; \\ \text{(b) } k_B T \ll \Delta : C_V &= \frac{N\Gamma k_B^2 T}{6(\hbar c)^2} \ln \frac{\hbar c \Lambda}{\Delta}. \end{aligned} \quad (4.19)$$

For $\Gamma \gg \Delta$ the three subregimes are

$$\begin{aligned} \text{(a')} k_B T \gg \Gamma : C_V &= \frac{12\zeta(3)N}{5\pi} k_B \left(\frac{k_B T}{\hbar c} \right)^2; \\ \text{(b')} \frac{\Delta^2}{\Gamma} \ll k_B T \ll \Gamma : C_V &= N f_3 \left(\frac{k_B T}{\Gamma} \right) \frac{\Gamma k_B^2 T}{(\hbar c)^2}; \\ \text{(c')} k_B T \ll \frac{\Delta^2}{\Gamma} : C_V &= \frac{N\Gamma k_B^2 T}{6(\hbar c)^2} \ln \frac{\hbar c \Lambda}{\Delta}, \end{aligned} \quad (4.20)$$

with f_3 a function similar to the function f_1 above in (4.11). Regimes (a), (b), (a'), (b'), and (c') are the same as those described for the correlation length. Within regime (b') above, it is possible, as was the case with ξ , to have a negligibly small subregime where $\ln \ln(\Gamma/k_B T) \gg 1$ while $k_B T \gg \Delta^2/\Gamma$; here we find $C_V = (N/12)[\Gamma k_B^2 T / (\hbar c)^2] \ln[(\hbar c \Lambda)^2 / k_B T \Gamma]$.

V. CONCLUSIONS

We begin this concluding section with a simple, qualitative discussion of the physical picture behind the computations in this and related, previous, works on the spin fluctuation properties of the not too strongly doped cuprate compounds. A discussion of the relationship of our ideas to experimental systems will follow. Finally we will discuss some open theoretical issues.

It is very useful to think in terms of the physics at different length and energy scales. On the whole, we may assume that the relevant energy scales decrease uniformly

with increasing length scales, and so the two can be used interchangeably to move between the different regimes. In what follows, we describe the sequence of crossovers as one moves from larger to smaller energy scales, or equivalently from smaller to larger length scales.

At the very largest energy scales, the behavior is dominated by lattice scale physics, which is inherently nonuniversal. At slightly smaller energies, provided the doping is not too large, one can neglect the effect of mobile carriers. It was argued in Ref. 4 that spin fluctuations at these scales are *quantum critical* and well described by properties of the $z = 1$ critical point in the $O(3)$ nonlinear σ model which separates the Néel-ordered and magnetically disordered phases. This proposal has been the source of some controversy in the literature, although some fairly convincing evidence has appeared in recent high-temperature series studies.³⁴ The excitations in this $z = 1$ quantum-critical regime are neither spin waves of the ordered state nor $S = 1$ magnons of the disordered state, but form a critical continuum.

The subsequent crossovers at smaller energies depend on the doping concentration. At very low doping, the presence of $T = 0$ long-range order in the Heisenberg antiferromagnet becomes apparent; there is then a crossover from $z = 1$ critical fluctuations at larger energies to the Goldstone spin-wave modes of the ordered state at lower energies. At a slightly larger doping, the $T = 0$ long-range order is destroyed. The crossover at smaller energy scales is then into the quantum-disordered state of the σ model, where the excitations are then gapped, triply degenerate, spin-1 magnons. At this same doping, and provided the geometry of the Fermi surface is of type B , there is a last crossover, at an even smaller energies, to a state where the mobile carriers induce an important $T = 0$ damping of the magnon excitations;^{15,5} this damping leads to to subgap absorption and power-law (in T) behavior of observables. All power laws are equivalent to those of the quantum-disordered, $z = 2$ regime, discussed below.

This sequence of crossovers has been described for the case in which the damping is not too large: $\Gamma < \Delta$ in the notation of Sec. II.

At even larger dopings we may have $\Gamma \geq \Delta$; then the intermediate quantum-disordered, $z = 1$, regime of a pure σ model is not realized, and the system undergoes crossovers from the $z = 1$ critical regime to the $z = 2$ critical regime, and then at even smaller energies to the $z = 2$ quantum disordered regime with relaxational behavior of spin excitations.

The scaling regimes and associated crossovers are sketched in Fig. 6. We emphasize that all phase boundaries between phases on this sketch are smooth crossovers, and their precise positions are therefore not meaningful, while the sequence of crossovers and overall structure of the diagram are meaningful. In terms of this phase diagram, the Shraiman-Siggia model (type *A*) corresponds to $\Gamma/\Delta = 0$ (Refs. 4, 12) (modulo dangerously irrelevant damping terms); models with primarily relaxational behavior of spin excitations (type *B*), similar to that observed in $\text{YBa}_2\text{Cu}_3\text{O}_7$, to $\Gamma/\Delta \gtrsim 1$ (Refs. 16, 5); and, finally, models with moderate $\Gamma/\Delta \lesssim 1$ exhibit spin pseudogap behavior similar to that observed in the underdoped Y-based materials, such as $\text{YBa}_2\text{Cu}_3\text{O}_{6.63}$ and $\text{YBa}_2\text{Cu}_4\text{O}_8$.^{6,7} The possible relationship of the phase boundaries to doping levels in the cuprate materials will be discussed in more detail in a subsequent publication.

Going to a finite T introduces an additional degree of complication which substantially changes the crossovers described above. However, all of these T -dependent crossovers are contained in the universal crossover function Φ_s in Eq. (2.3), which was computed in a large N expansion in this paper. The main results can be understood by keeping a simple rule of thumb in mind: The

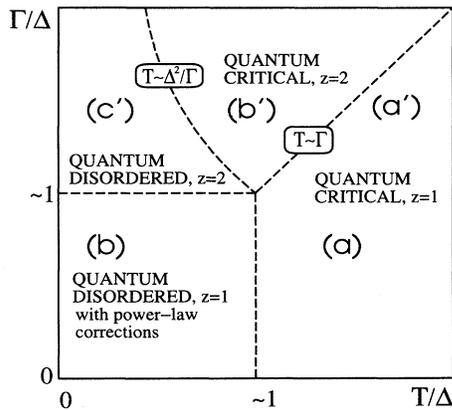


FIG. 6. Scaling regimes are sketched versus the normalized temperature, T/Δ , and the $T = 0$ parameter Γ/Δ which describes type-*A* to type-*B* crossover. While the asymptotic $T \rightarrow 0$ behavior is the same for all nonzero Γ/Δ , an important distinction between the quantum-disordered (QD), $z = 1$ regime at small Γ/Δ and the QD, $z = 2$ regime at large Γ/Δ is the presence of a knee [see Fig. 1(b)] in $\chi''(\omega)$ at $\omega \sim \Delta$ in the former case. We emphasize that the $T = 0$ parameters Δ and Γ may have a highly nontrivial, possibly nonmonotonic, doping dependence; therefore the boundaries between different regimes on this phase diagram may have a somewhat different structure compared to a phase diagram plotted as a function of doping.

primary effect of a finite T is to thermally quench the excitations at the energy scale $k_B T$, so that the crossovers at scales below $k_B T$ no longer occur. It is therefore possible to make experimental predictions for temperature dependences that are remarkably universal.³⁻⁵

We now briefly describe the relationship of our results to experiments on high- T_c oxides. We leave detailed quantitative comparisons for a separate publication which will also contain computations of $1/N$ corrections; here we underline only the main qualitative points.

$\text{La}_{2-x}\text{Sr}_x\text{CuO}_4$. The Fermi surface of La-based materials has not been measured experimentally. Apparently, they are close to type *A* (Ref. 20) in the terminology of Sec. I, and we expect to see only the features of the $z = 1$ behavior. However, these materials also show nontrivial spin correlations in the metallic phase which will certainly affect the spin fluctuations at low enough temperatures.³⁵ The primary effect of mobile carriers on the uniform susceptibility should be a temperature-independent additive contribution, which increases with doping. We have suggested earlier^{4,32,36} that this behavior may have been observed at a few percent Sr concentration.

$\text{YBa}_2\text{Cu}_3\text{O}_7$. Photoemission experiments²⁵ indicate that the Fermi surface is large and belongs to type *B*. Both our and earlier¹⁶ analyses of the NMR data are roughly consistent with type-*B* scaling behavior at $\Delta/\Gamma \sim 1$: The measured $1/T_1 T$ monotonically decreases with T , and the ratio $T_1 T/T_2^2$ is temperature independent.³⁷ A quantitative comparison with the measured uniform susceptibility³⁸ is difficult because nonuniversal contributions to susceptibility, neglected in (4.15), (4.16), are relevant for this fully doped material.

$\text{YBa}_2\text{Cu}_3\text{O}_{6.6}$ and $\text{YBa}_2\text{Cu}_4\text{O}_8$. For both of these materials, the measured Fermi surface is large and belongs to type *B*.^{39,40} The measured $1/T_1$ and uniform susceptibility decrease rapidly as temperature decreases below $T \simeq 150$ K.^{41,42,38} Such behavior is characteristic of the quantum-disordered $z = 1$ regime and we therefore expect that for these materials, $\Gamma/\Delta \ll 1$. Note, however, that we have shown explicitly that due to the finite damping, this decrease, particularly for the uniform susceptibility, is not purely exponential as it is in phenomenological “spin gap” models. We also note that it is important to include $1/N$ corrections⁴ in describing the behavior of $1/T_1 T$ in this regime. At high temperatures, this system is expected to crossover to $z = 1$ quantum-critical behavior, where $T_1 T$ is linear in T and $T_1 T/T_2 = \text{const}$;^{4,6} this behavior appears to have been observed in both $\text{YBa}_2\text{Cu}_3\text{O}_{6.6}$ and $\text{YBa}_2\text{Cu}_4\text{O}_8$.^{41,42}

We conclude this paper by recalling and raising some open theoretical questions. The explicit computation of temperature-dependent crossovers has been restricted to the quantum-disordered phase, and not too close to the quantum-critical point. We suggested that S may fail close enough to the critical point, and saw explicitly that it failed badly in describing the low-energy excitations on the ordered side. A drastic change in the action seems to be necessary, probably much more drastic than adding some additional dangerously irrelevant coupling which becomes important only on the ordered side. In fact, any

procedure which uses a truncated power series expansion in \bar{n} for the action appears doomed: No such expansion will correctly capture the appearance of gaps on portions of the Fermi surface. It seems that the only natural way of describing the crossover on the ordered side is to work in a theory in which the fermions are not completely integrated out. It is then possible to obtain an action which has only regular terms, and which holds on both sides of the transition. A possible form of the action involves two species of fermions (corresponding to pairs of points on the Fermi surface separated by Q) coupled to the n field. Computing crossovers for such an action remains an important open problem.

A complementary open problem applies to the analysis using the Shraiman-Siggia¹² model. In this case the ordered state and its crossovers are reasonably well studied. However, it is also clear that the model cannot apply in detail on the quantum-disordered side, and the crossovers between the ordered and disordered sides are not completely understood.

Finally, we make a few remarks about the effects of disorder. Hertz¹⁵ did begin to address the consequences of disorder, but his analysis is seriously incomplete in all cases. One sign of this is that his value of ν (which equals $1/2$ in all the cases considered by him) violates the inequality $\nu > 2/d$,⁴³ where $d < 4$ is the spatial dimension. It is easy to determine the source of this violation: The most important perturbation on a soft-spin version of an action like S is a random-“mass” term $\int d^d x d\tau \sim m^2(x) \bar{n}^2(x, \tau)$ which accounts for fluctuations in the local value of the critical coupling at which the magnetic order is destroyed. It can be easily shown that this term is relevant about the pure fixed point as long as $\nu < 2/d$. A study of S , with an additional random mass, along the lines of the work of Boyanovsky and Cardy⁴⁴ should be quite straightforward. However, this approach involves the potentially dangerous procedure of expanding about a problem with ϵ_τ time dimensions, and a method which works directly with $\epsilon_\tau = 1$ would be preferable.

Note added. Just before this paper was submitted, we received a paper before publication from Ioffe and Millis⁴⁵ which addressed the T dependence of the uni-

form susceptibility near the type- B fixed point. Their expression for χ_u in terms of correlators of \bar{n} is in general agreement with our discussion in Sec. IV A. However, they focused on the temperature dependence of χ_u at very low T ($\ln \Gamma/T \gg 1$) near the type- B fixed point; we chose to focus on the term important near the primary type- A fixed point—the latter term gives a universal (up to an overall prefactor) contribution to χ_u which should be dominant at all T unless Γ/Δ is very large.

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APPENDIX: SUSCEPTIBILITIES IN SDW THEORY AT $T = 0$

Staggered susceptibility

In SDW theory, the spin susceptibility is given by a ladder series of bubble diagrams. At $T = 0$, one fermion in the bubble should be above the Fermi surface and one below. In principle, the solution of the ladder series in the magnetically ordered state is a 2×2 matrix problem, as one has to consider bubbles with momentum transfer 0 and Q .^{21,10} For our considerations, it turns out that all terms with momentum transfer Q disappear at the critical point, and we have checked that they only account for small corrections to the expressions obtained below. We therefore neglect terms $\chi_0^{+-}(q, q + Q, \omega)$ with the momentum transfer Q in which case the total transverse dynamic susceptibility is given by

$$\chi^{+-}(q, \omega) = \frac{\chi_0^{+-}(q, \omega)}{1 - U\chi_0^{+-}(q, \omega)}, \quad (\text{A1})$$

where

$$\begin{aligned} \chi_0^{+-}(q, \omega) = & \frac{1}{2N} \sum'_k \left[1 - \frac{\epsilon_k^- \epsilon_{k+q}^- - N_0^2}{E_k^- E_{k+q}^-} \right] \left(\frac{n_{k+q}^d - n_k^c}{E_k^c - E_{k+q}^d - \omega} + \frac{n_k^d - n_{k+q}^c}{E_{k+q}^c - E_k^d + \omega} \right) \\ & + \frac{1}{2N} \sum'_k \left[1 + \frac{\epsilon_k^- \epsilon_{k+q}^- - N_0^2}{E_k^- E_{k+q}^-} \right] \left(\frac{n_{k+q}^d - n_k^d}{E_k^d - E_{k+q}^d - \omega} + \frac{n_k^c - n_{k+q}^c}{E_{k+q}^c - E_k^c + \omega} \right). \end{aligned} \quad (\text{A2})$$

Here and below we set $\hbar = 1$. We will now study the behavior of this result near the magnetic transition where N_0 becomes small.

Near $q = Q$, the coherence factors can be simplified to

$$\begin{aligned} 1 - \frac{\epsilon_k^- \epsilon_{k+q}^- - N_0^2}{E_k^- E_{k+q}^-} & \approx 2 - (\epsilon_k^- + \epsilon_{k+q}^-)^2 \frac{N_0^2}{2(E_k^-)^4} \\ & = 2 - O((Q - q)^2), \end{aligned}$$

$$1 + \frac{\epsilon_k^- \epsilon_{k+q}^- - N_0^2}{E_k^- E_{k+q}^-} \approx (\epsilon_k^- + \epsilon_{k+q}^-)^2 \frac{N_0^2}{2(E_k^-)^4} = O((Q - q)^2). \quad (\text{A3})$$

We first consider the behavior of dynamic susceptibility at the antiferromagnetic momentum, $q = Q$. Here $\epsilon_k^- + \epsilon_{k+q}^- = 0$, $E_k^d + E_{k+q}^c = 2E_k^-$, and one obtains using the self-consistency condition

$$\chi_0^{+-}(Q, \omega) = \frac{1}{U} + \frac{1}{N} \sum_k' (n_k^d - n_k^c) \left(\frac{1}{2E_k^- - \omega} + \frac{1}{2E_k^- + \omega} - \frac{1}{E_k^-} \right). \quad (\text{A4})$$

Expanding this expression in ω and substituting into (A1) we obtain

$$[\chi^{+-}(Q, \omega)]^{-1} = \frac{U^2 \omega^2}{4N} \sum_k' \frac{n_k^d - n_k^c}{(E_k^-)^3}. \quad (\text{A5})$$

The difference between type-*A* and -*B* transitions now becomes transparent. In the first case, the integration over k is restricted to a region which is located entirely inside the magnetic Brillouin zone. Accordingly, E_k^- remains finite when N_0 tends to zero, and at the critical point we have $[\chi^{+-}(Q, \omega)]^{-1} \sim \omega^2$. For type *B*, however, the allowed region of momentum integration includes the vicinity of $(x_0, \pi - x_0)$ and symmetry-related points where the Fermi surface crosses the magnetic Brillouin zone boundary. At each of these points, $E_k^- = N_0$; i.e., the denominator in (A5) diverges as $N_0 \rightarrow 0$. Expanding E_k^- near these points and performing the momentum integration, we obtain after some algebra

$$[\chi^{+-}(Q, \omega)]^{-1} = -D \frac{U^2}{t^2} \left(\frac{\omega^2}{\sqrt{4N_0^2 - \omega^2}} + \frac{\omega^2}{c^2} \right), \quad (\text{A6})$$

where $D = t/8\pi|t' \sin x_0 \sin 2x_0|$, and the ω^2 term comes from the integration over the regions far from $(x_0, \pi - x_0)$. We see that the leading term in the expansion in ω is now ω^2/N_0 . As one approaches the critical point, the first term in (A6) becomes purely imaginary, and at $N_0 = 0$ one has

$$[\chi^{+-}(Q, \omega)]^{-1} = -D \frac{U^2}{t^2} \left(i\omega + \frac{\omega^2}{c^2} \right). \quad (\text{A7})$$

We now consider the form of the susceptibility at finite q which we will understand as a deviation from Q . The calculations here are straightforward but tedious, and so we skip some of the details. First, we found that at $\omega = 0$, all potential terms of the form q^2/N_0 cancel each other for both type-*A* and -*B* transitions. The expansion of $[\chi^{+-}(q, 0)]^{-1}$ over q is therefore regular for both cases. Second, we found that when both ω and q are finite, the second piece in (A2) has an imaginary part which for type-*A* transition behaves as

$$\text{Im}(\chi^{+-})^{-1} \sim \omega q N_0^2. \quad (\text{A8})$$

We see that the damping of magnetic excitations disappears as $N_0 \rightarrow 0$.

For type-*B* transitions, the imaginary part of the susceptibility is a function of q/N_0 and ω/N_0 . For $q, \omega \ll N_0$, we obtained

$$\text{Im}(\chi^{+-})^{-1} = \bar{D} \frac{U^2}{tN_0} \psi(|q_x + q_y|, |q_x - q_y|, \omega), \quad (\text{A9})$$

where $\bar{D} = t/[8\pi|t' \sin 2x_0|]$, and ψ looks particularly simple at $t' \ll t$,

$$\psi(|q_x + q_y|, |q_x - q_y|, \omega) = \frac{1}{2} \text{Re} \left(\frac{(q_x + q_y)^4}{[(q_x + q_y)^2 - \bar{\omega}^2]^{3/2}} + \frac{(q_x - q_y)^4}{[(q_x - q_y)^2 - \bar{\omega}^2]^{3/2}} \right), \quad (\text{A10})$$

where $\bar{\omega} = \omega/(2t \sin x_0)$. We also obtained expression for ψ for arbitrary t'/t , but it is more cumbersome and we refrain from presenting it here. In the opposite limit, $q, \omega \gg N_0$, or equivalently when N_0 vanishes, the q/N_0 dependence in (A9) transforms into a constant term, and both pieces in (A2) contribute linear in ω terms into the imaginary part of χ^{+-} . The numerical factors for each of the two pieces contain combinations of θ functions of the form, e.g., $\theta(4t \sin x_0 |q_x + q_y| - |\omega + 4|t' \sin 2x_0|(q_x - q_y)|)$. However, we have checked explicitly that the sum of the two terms does not contain θ functions. Specifically, we obtained at $N_0 = 0$ and arbitrary ratio ω/q

$$\text{Im}(\chi^{+-})^{-1} = \bar{D} \frac{U^2}{t^2} \omega. \quad (\text{A11})$$

Combining the above results, one obtains the expressions presented in Eqs. (3.5), (3.8), and (3.9).

Uniform susceptibility

The uniform transverse spin susceptibility is defined as

$$\chi_{\perp} = (1/2)(g_G \mu_B/a)^2 \lim_{q \rightarrow 0} \chi^{+-}(q, \omega = 0), \quad (\text{A12})$$

where $g_G \mu_B$ is the gyromagnetic factor and a^2 is the volume per spin. In the $q \rightarrow 0$ limit, the coherence factors are simplified and we obtain $\chi_{\perp} = (g_G \mu_B/a)^2 \chi_0/[2(1 - U\chi_0)]$ where

$$\chi_0 = \frac{1}{N} \sum_k' \left(\frac{N_0}{E_k^-} \right)^2 \left(\frac{n_k^d - n_k^c}{E_k^-} \right) + \lim_{q \rightarrow 0} \frac{1}{N} \sum_k' \left(\frac{\epsilon_k^-}{E_k^-} \right)^2 \times \left(\frac{n_{k+q}^d - n_k^d}{E_k^d - E_{k+q}^d} + \frac{n_k^c - n_{k+q}^c}{E_{k+q}^c - E_k^c} \right). \quad (\text{A13})$$

As one approaches the critical point, the first term in (A13) disappears and χ_{\perp} takes the familiar random-phase-approximation (RPA) form of the magnetic susceptibility in a Fermi liquid, which is finite at $T = 0$. For the type-*A* transitions, the first term vanishes as $O(N_0^2)$, and for the type-*B* transitions it vanishes as $O(N_0)$. The uniform susceptibility then has forms presented in Eqs. (3.6) and (3.11). Notice, however, that the full χ_0 (and hence, χ_{\perp}) behaves as $\chi_0 = \text{const} + O(N_0^2)$ for both types of transitions: For the type-*B* transition, linear in N_0 contributions from the first and second terms in (A13) cancel each other.

Staggered susceptibility in a finite magnetic field

To obtain the temperature-dependent correction to the uniform susceptibility from the fluctuation and interaction of the magnon modes, we will need the form of the long-wavelength action for antiferromagnetic fluctuations in the presence of the external magnetic field. To derive this form, we need to understand how the staggered susceptibility is modified in such a field. In this subsection, we will determine this modification in the SDW formalism for the ordered state. We then extend SDW calculations up to a critical point which in turn will allow us to obtain the field-dependent part of the action in a

disordered phase.

To be definite, we assume that the field is directed along the x axis (antiferromagnetic order parameter is along the z axis). We also introduce new operators as linear combinations of quasiparticle operators with up and down spins,

$$\phi_k = \frac{a_{k,1} + a_{k,2}}{\sqrt{2}}, \quad \psi_k = \frac{a_{k,1} - a_{k,2}}{\sqrt{2}} \quad (\text{A14})$$

(the indices 1 and 2 correspond to up and down spins, respectively). The quadratic part of the Hubbard Hamiltonian now takes a form

$$\begin{aligned} \mathcal{H}_\epsilon = & \sum_k' (\epsilon_k - \bar{H}) \phi_k^\dagger \phi_k + (\epsilon_{k+Q} + \bar{H}) \psi_{k+Q}^\dagger \psi_{k+Q} - N_0 (\phi_k^\dagger \psi_{k+Q} + \psi_{k+Q}^\dagger \phi_k) \\ & + \sum_k' (\epsilon_{k+Q} - \bar{H}) \phi_{k+Q}^\dagger \phi_{k+Q} + (\epsilon_k + \bar{H}) \psi_k^\dagger \psi_k - N_0 (\phi_{k+Q}^\dagger \psi_k + \psi_k^\dagger \phi_{k+Q}), \end{aligned} \quad (\text{A15})$$

where $\bar{H} = H/2$. Clearly, the magnetic field splits conduction and valence bands for up and down spins, and so we now have four different branches of fermionic excitations. However, we see that the pairs of operators (ϕ_k, ψ_{k+Q}) and (ϕ_{k+Q}, ψ_k) in (A15) are decoupled from each other, and therefore the diagonalization is still a 2×2 problem. Performing the standard manipulations we obtain

$$\mathcal{H}_{\text{SDW}} = \sum_k' E_k^\alpha \alpha_k^\dagger \alpha_k + E_k^\beta \beta_k^\dagger \beta_k + E_k^\gamma \gamma_k^\dagger \gamma_k + E_k^\delta \delta_k^\dagger \delta_k, \quad (\text{A16})$$

where we introduced

$$E_k^{\alpha,\beta} = \epsilon_k^+ \pm E_k^{--}, \quad E_k^{\gamma,\delta} = \epsilon_k^+ \pm E_k^{-+}, \quad (\text{A17})$$

and $E_k^{--} = \sqrt{N_0^2 + (\epsilon_k^- - \bar{H})^2}$, $E_k^{-+} = \sqrt{N_0^2 + (\epsilon_k^- + \bar{H})^2}$. The self-consistency condition now takes the form

$$1 = \frac{U}{2} \sum_k' \left(\frac{n_{\beta_k} - n_{\alpha_k}}{E_k^{--}} + \frac{n_{\delta_k} - n_{\gamma_k}}{E_k^{-+}} \right). \quad (\text{A18})$$

The computation of susceptibilities proceeds in the same way as without a field, the only new feature being the appearance of a cross-polarization term χ_0^{xz} which makes the RPA ladder summation a 2×2 matrix problem even if we neglect, as we did earlier, small terms with a momentum transfer Q . Specifically, we found

$$\begin{aligned} \chi_{yy}(q, \omega) &= \frac{\chi_0^{yy}(q, \omega) [1 - 2U\chi_0^{zz}(q, \omega)] - 2U|\chi_0^{zy}(q, \omega)|^2}{[1 - 2U\chi_0^{yy}(q, \omega)][1 - 2U\chi_0^{zz}(q, \omega)] - 4U^2|\chi_0^{zy}(q, \omega)|^2}, \\ \chi_{yz}(q, \omega) &= \frac{\chi_0^{yz}(q, \omega)}{[1 - 2U\chi_0^{yy}(q, \omega)][1 - 2U\chi_0^{zz}(q, \omega)] - 4U^2|\chi_0^{zy}(q, \omega)|^2}. \end{aligned} \quad (\text{A19})$$

Below, we restrict the consideration to an analysis of the frequency dependence of the susceptibilities at $q = Q$. In this case, the bare susceptibilities are

$$\begin{aligned} \chi_0^{yy}(Q, \omega) &= \frac{1}{4N} \sum_k' (n_{\beta_k} - n_{\alpha_k}) \left(\frac{1}{2E_k^{--} - \omega} + \frac{1}{2E_k^{-+} + \omega} \right) + (n_{\delta_k} - n_{\gamma_k}) \left(\frac{1}{2E_k^{-+} - \omega} + \frac{1}{2E_k^{--} + \omega} \right), \\ \chi_0^{zz}(Q, \omega) &= \frac{1}{4N} \sum_k' (n_{\beta_k} - n_{\alpha_k}) \left(\frac{\epsilon_k^- - \mu_B H}{E_k^{--}} \right)^2 \left(\frac{1}{2E_k^{--} - \omega} + \frac{1}{2E_k^{-+} + \omega} \right) \\ &\quad + (n_{\delta_k} - n_{\gamma_k}) \left(\frac{\epsilon_k^- + \mu_B H}{E_k^{-+}} \right)^2 \left(\frac{1}{2E_k^{-+} - \omega} + \frac{1}{2E_k^{--} + \omega} \right), \\ \chi_0^{zy}(Q, \omega) &= i \frac{1}{4N} \sum_k' (n_{\beta_k} - n_{\alpha_k}) \left(\frac{\epsilon_k^- - \mu_B H}{E_k^{--}} \right) \left(\frac{1}{2E_k^{--} - \omega} - \frac{1}{2E_k^{-+} + \omega} \right) \\ &\quad - (n_{\delta_k} - n_{\gamma_k}) \left(\frac{\epsilon_k^- + \mu_B H}{E_k^{-+}} \right) \left(\frac{1}{2E_k^{-+} - \omega} - \frac{1}{2E_k^{--} + \omega} \right). \end{aligned} \quad (\text{A20})$$

Performing the computations, we found that

$$\begin{aligned}\chi_0^{yy}(Q, \omega) &= \frac{1}{2U} + c_1 \frac{\omega^2}{t^3} \left(1 + \frac{\bar{H}^2}{\bar{H}_0^2} \right), \quad \chi_0^{zz}(Q, \omega) = \chi_0^{yy}(Q, \omega) - c_2 N_0^2/t^3, \\ \chi_0^{zy}(Q, \omega) &= ic_3 \omega \bar{H}/t^3,\end{aligned}\tag{A21}$$

where $c_i = c_i(t'/t)$ and $H_0 \sim t$. For the type-*A* transition, we found that all three coefficients remain finite at the critical point. Elementary considerations then show that the dominant field dependence in the denominators of (A20) comes from $(\chi_0^{zy})^2$ and is in the form $\omega^2 H^2$. For the *n*-field action, this implies that the extra term in the presence of *H* has a form of Eq. (4.12) (to see this, one simply has to compute correlator $n^z n^y \propto \chi^{zy}$). For a disordering transition of type *B*, the situation is more complex. The most relevant point is that c_3 is still finite at the transition which implies that there is no term $\sim \text{sgn}(\omega)$ in the coupling to the field. However, we also found that c_1 and c_2 behave as $c_1 = (1/2)D[(t/\sqrt{4N_0^2 - \omega^2}) + \dots]$, $c_2 = D[(t/2N_0) + \dots]$ where D was defined after (A6) and the ellipses stand for nonsingular (field-dependent) terms. Right at the critical point, $c_1 = [(i/2)Dt/\omega + \dots]$, and we have $\chi_0^{yy,zz} = (1/2U) - i\gamma\omega(1 - \bar{H}^2/\bar{H}_0^2)/(4U^2) + O(\omega^2(1 - H^2/H_1^2))$. We see therefore that for *B*-type transitions, γ acquires a field-dependent part $\gamma \rightarrow \gamma(1 - \bar{H}^2/\bar{H}_0^2)$. Substituting this expression into (A20), we find an extra field dependence in the denominator which has the same $\omega^2 \bar{H}^2$ form as the dependence imposed by $(\chi_0^{zy})^2$, but with extra prefactor $(\gamma/t)^2$. This implies that for the $z = 2$ transition, there are two different relevant terms in the *n*-field action which describe the coupling of the antiferromagnetic fluctuations to the field. We discuss the relative importance of these terms in Sec. IV A. Finally, the effective action indeed contains a term $\bar{H}^2 - (\bar{H}\bar{n})^2 = (\bar{n} \times \bar{H})^2$ which can be extracted from the longitudinal susceptibility $[\chi^{xx}(0, 0)]^{-1} \propto H^2$. However, because of the constraint, this term contributes only to uniform susceptibility at $T = 0$ which is nonuniversal and is not considered in this paper.

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- ¹ S. Chakravarty, B.I. Halperin, and D.R. Nelson, Phys. Rev. B **39**, 2344 (1989).
- ² B. Andraka and A.M. Tsvetik, Phys. Rev. Lett. **67**, 2886 (1991); A.M. Tsvetik and M. Reizer, Phys. Rev. B **48**, 9887 (1993).
- ³ S. Sachdev and J. Ye, Phys. Rev. Lett. **69**, 2411 (1992).
- ⁴ A.V. Chubukov and S. Sachdev, Phys. Rev. Lett. **71**, 169 (1993); **71**, 2680(E) (1993); A.V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B **49**, 11919 (1994).
- ⁵ A.J. Millis, Phys. Rev. B **48**, 7183 (1993).
- ⁶ A. Sokol and D. Pines, Phys. Rev. Lett. **71**, 2813 (1993).
- ⁷ V. Barzykin, D. Pines, A. Sokol, and D. Thelen, Phys. Rev. B **49**, 1544 (1994).
- ⁸ Y.-L. Liu and Z.-B. Su (unpublished).
- ⁹ B.I. Shraiman and E.D. Siggia, Phys. Rev. B **42**, 2485 (1990).
- ¹⁰ A.V. Chubukov and D.M. Frenkel, Phys. Rev. B **46**, 11884 (1992).
- ¹¹ B.O. Wells *et al.*, Phys. Rev. Lett. **74**, 964 (1995).
- ¹² S. Sachdev, Phys. Rev. B **49**, 6770 (1994).
- ¹³ B.I. Shraiman and E.D. Siggia, Phys. Rev. Lett. **60**, 740 (1988); **61**, 467 (1988).
- ¹⁴ A.V. Chubukov and K. Musaelian, J. Phys. Condens. Matter **7**, L153 (1995); Phys. Rev. B **51**, 12605 (1995).
- ¹⁵ J.A. Hertz, Phys. Rev. B **14**, 525 (1976).
- ¹⁶ A.J. Millis, H. Monien, and D. Pines, Phys. Rev. B **42**, 197 (1990).
- ¹⁷ A. Sokol, in *The Los Alamos Symposium 1993—Strongly Correlated Electronic Materials*, edited by K. S. Bedell *et al.* (Addison-Wesley, Reading, MA, 1994), pp. 490-493.
- ¹⁸ E.K. Riedel and F.J. Wegner, Phys. Rev. B **9**, 294 (1974).
- ¹⁹ D.R. Nelson, Phys. Rev. B **11**, 3504 (1975).
- ²⁰ Q. Si, Y. Zha, K. Levin, and J.P. Lu, Phys. Rev. B **47**, 9055 (1993), and references therein.
- ²¹ J.R. Schrieffer, X.G. Wen, and S.C. Zhang, Phys. Rev. B **39**, 11663 (1989).
- ²² A.V. Chubukov and K. Musaelian, Phys. Rev. B **50**, 6238 (1994); J. Phys. Condens. Matter **7**, 133 (1995).
- ²³ M.S. Hybertsen *et al.*, Phys. Rev. B **41**, 11068 (1990).
- ²⁴ R.G. Gooding, K.J.E. Vos, and P.W. Leung, Phys. Rev. B **49**, 4119 (1994).
- ²⁵ R. Liu *et al.*, Phys. Rev. B **45**, 5614 (1992); **46**, 11056 (1992).
- ²⁶ S. Trugman, Phys. Rev. Lett. **65**, 500 (1990).
- ²⁷ B.I. Halperin and P.C. Hohenberg, Phys. Rev. **188**, 898 (1969); D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin/Cummings, Reading, MA, 1975).
- ²⁸ Similar ideas were expressed by C.M. Varma (unpublished).
- ²⁹ A.O. Caldeira and A.J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
- ³⁰ A. Aharony, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M.S. Green (Academic, New York, 1976), Vol. 6, p. 358, and references therein.
- ³¹ C.P. Slichter, in Proceedings of the 1993 Los Alamos Symposium (in press).
- ³² A.V. Chubukov, S. Sachdev, and A. Sokol, Phys. Rev. B **49**, 9052 (1994).
- ³³ S. Sachdev, Z. Phys. B **94**, 469 (1994).
- ³⁴ A. Sokol, R.L. Glenister, and R.R.P. Singh, Phys. Rev. Lett. **72**, 1549 (1994); N. Elstner, R.L. Glenister, R.R.P. Singh, and A. Sokol (unpublished).
- ³⁵ S.-W. Cheong *et al.*, Phys. Rev. Lett. **67**, 1791 (1991).
- ³⁶ A.V. Chubukov and S. Sachdev, Phys. Rev. Lett. **71**, 3615 (1993).
- ³⁷ T. Imai *et al.*, Phys. Rev. B **47**, 9158 (1993).
- ³⁸ For a review, see D.C. Johnston, J. Magn. Magn. Mater. **100**, 218 (1991), and references therein.
- ³⁹ J.M. Tranquada *et al.*, Phys. Rev. B **46**, 5561 (1992).
- ⁴⁰ The type-*B* Fermi surface has also been observed in Bi-2212 materials; see J. Ma *et al.* (unpublished); Z.-X. Shen, Physica B **197**, 632 (1994), and references therein.

- ⁴¹ M. Takigava *et al.*, Phys. Rev. B **43**, 247 (1991); **49**, 4158 (1994).
- ⁴² H. Zimmermann, M. Mali, D. Brinkmann, J. Karpinski, E. Kaldis, and S. Rusiecki, Physica C **159**, 681 (1989); T. Imai, K. O'Hara, C.P. Slichter, K. Yoshimura, M. Katoh, K. Kosuge, J.L. Cobb, and J.T. Markert (unpublished and private communication); R. Stern, M. Mali, J. Roos, and D. Brinkmann (unpublished).
- ⁴³ A.B. Harris, J. Phys. C **7**, 1671 (1974); J.T. Chayes, L. Chayes, D.S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986).
- ⁴⁴ D. Boyanovsky and J.L. Cardy, Phys. Rev. B **26**, 154 (1982).
- ⁴⁵ L.B. Ioffe and A.J. Millis (unpublished).