Self-sustained Aharonov-Bohm flux in mesoscopic rings: Continuum hard-core boson model

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We perform analytical calculations to study the persistent current I of a q-charged hard-core boson gas in mesoscopic rings enclosing a magnetic flux Φ, and find that this current is periodic in Φ with the period $\Phi_0 = hc/q$. More interestingly, in the presence of a single δ-function impurity, it has been found that there generally exists a spontaneous Aharonov-Bohm flux as long as the current-induced flux is included.

Mesoscopic metallic rings exhibit a spectacular thermodynamic property: they carry a persistent nondissipative current due to the Aharonov-Bohm (AB) effect[1] when they are threaded by a magnetic flux $\Phi = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$, where $\boldsymbol{A}$ is the vector potential.[2–8] This current is a periodic function of the flux Φ with the period $\Phi_0 \equiv hc/e$. In this case the flux Φ which drives the persistent current I is the sum of the externally applied flux $\Phi_{\rm ext}$ and the flux Φ_I induced by the current itself, $\Phi = \Phi_{\rm ext} + \Phi_I$. Most theoretical discussions neglect the second term,[7,8] which is justified for the experimental structures realized so far. Recently, Wohlleben *et al.*[9] addressed a possibility that in the ballistic regime a mesoscopic metallic ring can self-sustain a persistent current in the absence of an external magnetic flux. Note that in metallic systems the charged carriers—electrons—are fermions. To the best of our knowledge, there have been no investigations on the persistent current of a q-charged hard-core boson gas in mesoscopic rings. Although a hard-core boson is fermionlike in some sense, the parity effect due to the finite number of hard-core bosons N will play a crucial role in the existence of a spontaneous AB flux in the ground state. In this paper, we study the persistent current of a hard-core boson gas, via the AB effect, in mesoscopic rings. As an interesting prediction, in the presence of a single impurity with δ-function potential, the spontaneous AB flux does appear in these systems regardless of the boundary condition being antiperiodic or not and N being *even* or *odd*, which are, respectively, in marked contrast with the impurity-free situation and the case of metallic rings.

Let us first consider N *impenetrable* q-charged bosons in a ring with radius R (circumference $L = 2\pi R$), vanishingly small circular cross section $\pi(d/2)^2$, lying in the xy plane. Here the magnitude of R should be chosen so that bosons retain the phase memory throughout the ring. A solenoid passes through the opening of the ring and carries the static magnetic flux $\Phi_{\rm ext}$, which can be continuously varied. This is a typical problem described by the Hamiltonian of a one-dimensional (1D) hard-core boson gas in the presence of the AB flux, which is defined by the following conditions.

(i) The wave function is symmetrical with respect to the interchange of particle coordinates (Bose-Einstein statistics).

(ii) The wave function satisfies the cyclic boundary condition, which in the presence of the AB flux reads in part

$$\Psi(x_1 + L, x_2, \ldots, x_N) = e^{i2\pi f}\Psi(x_1, x_2, \ldots, x_N) , \quad (1)$$

with a similar condition for the derivative. Here $f = \Phi/\Phi_0$ is the AB flux in units of the flux quantum $\Phi_0 \equiv hc/q$. Equation (1) shows clearly that all physical quantities will be the periodic function of Φ with the period Φ_0.

(iii) The wave function vanishes whenever two particle coordinates coincide.[10]

Because the wave function must vanish if any two particles touch, the full configuration space can be divided into $N!$ subspaces of the type R_1: $(0 \le x_1 \le x_2 \le \cdots \le x_N \le L)$. Inside R_1, Ψ satisfies a *free* particle Schrödinger equation, but since $\Psi = 0$ on the boundary of R_1, Ψ must be a determinantal wave function, namely,

$$\Psi(x_1, x_2, \ldots, x_N) = \det|e^{ik_n x_j}| \quad (2)$$

in R_1. On the other hand, condition (i) requires that Ψ should not change sign when two coordinates are interchanged. Consequently, in any region which is a permutation of R_1,

$$\Psi = (-1)^P \det|e^{ik_n x_j}| , \quad (3)$$

where $(-1)^P$ denotes the sign of the permutation, and is defined as the parity of the number of transpositions of two variables which brings any other region to the region R_1. Under the modified boundary condition (ii), Eq. (1), the k's are found to be

$$k_n = \begin{cases} (2\pi/L)(n+f) & \text{for odd } N \\ (2\pi/L)(n+f+1/2) & \text{for even } N , \end{cases} \quad (4)$$

where $n = 0, \pm1, \pm2, \ldots,$ and f is in the range $[-1/2, 1/2)$. Thus the energy level in the presence of AB flux is given by

$$E_n = \begin{cases} \frac{\hbar^2}{2mR^2}(n+f)^2 & \text{for odd } N \\ \frac{\hbar^2}{2mR^2}(n+f+1/2)^2 & \text{for even } N , \end{cases} \quad (5)$$

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which is explicitly parity dependent due to the evenness or oddness of N. Then, at zero temperature, the total energy for a fixed number of particles is

$$E = \sum_{n=-n_0}^{n_0} \frac{\hbar^2 n^2}{2mR^2} + \frac{N\hbar^2}{2mR^2} f^2 \tag{6}$$

for $N = 2n_0 + 1$ with $n_0 = 0, 1, \ldots$, and

$$E = \sum_{n=-n_0}^{n_0-1} \frac{\hbar^2 (n+1/2)^2}{2mR^2} + \frac{N\hbar^2}{2mR^2} f^2 \tag{7}$$

for $N = 2n_0$ with $n_0 = 1, 2, \ldots$. Using the relation[2,11,12]

$$I_n = \frac{q}{2\pi\hbar} \frac{\partial E_n}{\partial f} , \tag{8}$$

one can obtain the persistent current in the ring

$$I = \frac{Nq\hbar}{2\pi m R^2} f , \tag{9}$$

which is a piecewise periodic function of the AB flux. In each periodic region, I varies linearly with Φ and there are discontinuous jumps when one period Φ_0 is over.

We then consider the AB flux induced by the persistent current itself. It is easy to see that the induced AB flux and the magnetic energy are,[9] respectively,

$$f_I = -\frac{\mathcal{L}c}{\Phi_0} I \tag{10}$$

and

$$E_B = \frac{\Phi_0^2}{2\mathcal{L}c^2} f^2 , \tag{11}$$

which means that when Φ_I becomes comparable to $\Phi_{\rm ext}$, the energy stored in the magnetic field of the system also changes appreciably, away from the value $\Phi_{\rm ext}^2/2\mathcal{L}c^2$ supplied by the external source to $\Phi^2/2\mathcal{L}c^2$. Here $\mathcal{L}$ is the self-inductance of the ring and has the classical expression for the ring with circular cross section geometry, $\mathcal{L} = \left(\frac{4\pi}{c^2}\right) R \left[\ln(16R/d) - \frac{7}{4}\right]$. In the absence of the external flux, $f_{\rm ext} = 0$, Eqs. (9) and (10) lead to a unique self-consistent solution: $f^{(s)} = 0$. On the other hand, the total energy of the whole system consists of two parts: the energy of particles in the ring E and the energy of the magnetic field E_B, i.e., $E_T = E + E_B$. It is very interesting to find that the total energy E_T reaches its minimum just at $f = f^{(s)} = 0$, which implies that unlike the case of metallic rings, the spontaneous current is absent in 1D mesoscopic rings composed of hard-core bosons, regardless of the number of particles N being even or odd. However, if we choose the antiperiodic boundary condition in the absence of the AB flux, $\Psi(x_1 + L, x_2, \ldots, x_N) = -\Psi(x_1, x_2, \ldots, x_N)$, which is not unreasonable when we insert a π-phase-shift junction in the ring; we could easily find a ground state with a spontaneous AB flux

$$f^{(s)} = \pm \left(2 + 8\pi^2 mR^2/N\mathcal{L}q^2\right)^{-1} , \tag{12}$$

which may be observable.

Now, it is very desirable to explore the effects of impurity scattering on the I-Φ characteristics because such a situation may be more realistic. For simplicity, we consider only a single impurity characterized by a δ-function potential, $V(x) = \gamma\delta(x)$ ($\gamma > 0$). Obviously, the solution to this model has the form

$$\Phi(x_1, x_2, \ldots, x_N) = (-1)^P \det \left| a_n e^{ik_n x_j} + b_n e^{-k_n x_j} \right| . \tag{13}$$

Using the transfer-matrix method, it is not difficult to obtain

$$\cos(kL) + \frac{\Omega}{k} \sin(kL) = (-1)^{N-1} \cos(2\pi f) , \tag{14}$$

where $\Omega = m\gamma/\hbar^2$. Note that, for the odd N case, the problem investigated here is quite similar to the one-dimensional Dirac comb,[13] with $2\pi f$ in Eq. (14) being replaced by the dimensionless Bloch wave vector KL. Figures 1 and 2 show, respectively, the numerical results for E-Φ and I-Φ characteristics at zero temperature for a ring with odd N. It can be clearly seen from Fig. 1 that the energy E for a fixed N has a minimum at a nonzero $|\Phi^{(0)}|$, which is due to the fact that the presence of the δ-function impurity in the ring eliminates the zero-valued k, and therefore for an *odd* number of hard-core bosons ($N = 2n_0 + 1$), either $+k_{n_0}$ or $-k_{n_0}$, is not filled while the currents in other occupied states with $\pm k$ are pairwise cancelled. Consequently, the spontaneous AB flux can be roughly expected near this point. More rigorously, this AB flux should be determined self-consistently, the value of which, $\Phi^{(s)}$, is just the nontrivial intersection of the

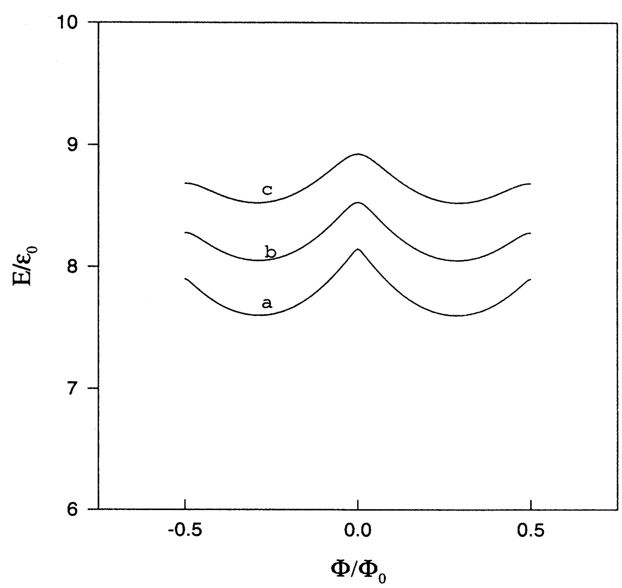

FIG. 1. Energy of particles versus the dimensionless flux f for an odd number of particles in the ring with the strength of a single δ-function impurity ΩL as (a) 0.5, (b) 2.0, and (c) 4.0, where $\varepsilon_0 = \hbar^2/2mR^2$.

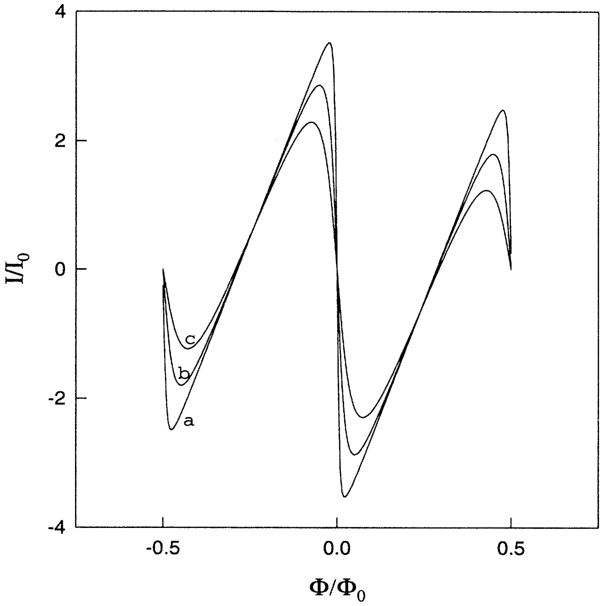

FIG. 2. Persistent current versus the dimensionless flux f for an odd number of particles in the ring in the presence of a single δ-function impurity. The value of ΩL is the same as that given in Fig. 1 and $I_0 = q\varepsilon_0/2\pi\hbar$.

curve described by Eq. (10) and that shown in Fig. 2. It is particularly remarkable that the total energy E_T reaches its minimum at $\Phi = \Phi^{(s)}$, i.e., $\frac{\partial E_T}{\partial \Phi}|_{\Phi=\Phi^{(s)}} = 0$, $\frac{\partial^2 E_T}{\partial \Phi^2}|_{\Phi=\Phi^{(s)}} > 0$, from which we can conclude that the AB flux state with the spontaneous current is stable, and it is in fact the ground state of the system. In addition, in the limit of $d \to 0$ ($\mathcal{L} \to \infty$), *the sustained persistent current approaches zero while the amplitude of the spontaneous AB flux is still finite* $\Phi^{(s)} \to \Phi^{(0)}$, which is just the nontrivial zero-point value of I in Fig. 2. On the other hand, one can also obtain the energy spectrum in the case of an *even* N by shifting a half Φ_0 along the Φ axis, and correspondingly the AB flux can also be self-sustained as well. Notable is that the curve of I-Φ characteristics becomes more smooth with increasing the strength of the δ-function potential (see Fig. 2).

Finally, we wish to discuss an issue addressed recently by Loss and Martin (LM).[14] In the framework of the Luttinger liquid model, LM concluded that the spontaneous persistent current is absent for interacting spinless fermions confined to a strictly one-dimensional mesoscopic ring (i.e., $d = 0$). The conclusion seems also valid for the case of $d = 0$ in our model. Therefore, it is understandable that the current-induced flux Φ_I is absent in their quantum-electrodynamics calculations performed for the strictly 1D model because Φ_I cannot be automatically included. However, in reality, the diameter of the wire d should not be exactly zero so that the current itself could induce a flux in quasi-1D rings according to the conventional physical considerations. In particular, it is our belief that the current-induced flux would appear automatically if quantum electrodynamics calculations are rigorously made on a real three-dimensional (3D) model even when $d \to 0$. More importantly, although the model Hamiltonian we use is strictly one-dimensional, the system considered here is actually three-dimensional [i.e., the ring with a finite cross section $\pi(d/2)^2$], and so is the current. Because of this, it appears reasonable and acceptable that the current-induced flux Φ_I, even in the limit of $I \to 0$, is incorporated into the 1D Hamiltonian as in the 3D case,[15] and Φ_I is self-consistently determined in the ground state.

In summary, we have presented analytical analyses on the persistent current I of a hard-core boson gas, via AB effect, in mesoscopic rings, and find this current is periodic in Φ with the period $\Phi_0 = hc/q$. More interestingly, in the presence of a single δ-function impurity, it has been rigorously shown that there generally exists a spontaneous AB flux as long as the self-induced flux is included, regardless of the number of hard-core bosons being *even* or *odd* and the boundary condition being antiperiodic or not. So far, we have only dealt with the limiting case of strongly correlated one-dimensional bosonic systems. There is no doubt that the soft-core interaction between bosons will further complicate the analysis.

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