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Dressed S matrices in models with long-range interactions

Fabian H. L. Eßler*
Physikalisches Institut der Universität Bonn, Nussallee 12, 53115 Bonn, Germany
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The *dressed* scattering matrix describing scattering of quasiparticles in various models with long-range interactions is evaluated by means of Korepin's method [V. E. Korepin, Theor. Mat. Phys. **41**, 953 (1979)]. For models with $1/\sin^2(r)$ interactions, the S matrix is found to be a momentum-independent phase, which clearly demonstrates the ideal-gas character of the quasiparticles in such models. We then determine S matrices for some models with $1/\sinh^2(r)$ interactions and find them to be, in general, nontrivial. For the $1/r^2$ limit of the $1/\sinh^2(r)$ interaction we recover trivial S matrices, thus exhibiting a crossover from interacting to noninteracting quasiparticles. The relation of the S matrix to fractional statistics is discussed.

I. INTRODUCTION

Haldane[1] recently put forward an interpretation of the Haldane-Shastry (HS) model as a generalized ideal gas with fractional statistics (ideal gas of spinons). The Calogero-Sutherland (CS) model is another example of a system of free particles with fractional statistics.[2–6] All methods employed so far in exhibiting the ideal-gas character and the nature of the statistics are based on the knowledge of the exact wave functions[7] for the HS and CS models. For the multitude of other models with long-range interactions, in particular models solvable by the asymptotic Bethe ansatz (ABA),[8,9] exact wave functions are not known. It would be useful to have a method based merely on the ABA to decide whether or not those systems fall into Haldane's category of ideal gases with fractional statistics. The most direct way to determine whether a system of quasiparticles is an ideal gas is to evaluate the dressed scattering matrix describing scattering of the elementary excitations in the model. If it is a momentum-independent phase, then we are indeed dealing with an ideal gas. Furthermore, if the phase is not ± 1, the quasiparticles have fractional statistics in the sense that the phase of the wave function under interchange of two particles is neither bosonic $(+1)$ nor fermionic (-1). This follows from the observation that for noninteracting particles (i.e., for momentum-independent S matrices) the scattering phase is precisely equal to the phase picked up by interchanging the two particles.

The plan of this paper is as follows: In Secs. II and III we determine the S matrix for scattering of quasiparticles in SU(N) Haldane-Shastry chains. It is found to be a momentum-independent phase, which shows both the ideal-gas character and the fractional statistics of the quasiparticles. In Sec. IV we repeat this analysis for the case of the Calogero-Sutherland model. In Sec. V we discuss the generalization of our results to other models with $1/\sin^2(r)$ interactions. In Sec. VI we consider the $1/\sinh^2(r)$ CS model and its exchange generalizations and show that quasiparticles in these models are *interacting*, and become free in the $1/r^2$ limit only. In Sec. VII we summarize and discuss our results.

II. SU(2) HALDANE-SHASTRY CHAIN

The Hamiltonian of the SU(2) HS chain is given by[10,11]

$$H=2\sum_{i<j}\frac{1}{\left[\frac{N}{\pi}\sin\left[\frac{\pi}{N}(i-j)\right]\right]^2}(P_{ij}-1)\ , \tag{2.1}$$

where P_{ij} is a permutation operator exchanging the spins a sites i and j. Ha and Haldane[12] proposed to characterize eigenstates of (2.1) by means of sets of spectral variables k_α^n obeying the following set of Bethe-like equations:

$$Nk_\alpha^n=2\pi I_\alpha^n+\pi\sum_{m\beta}t_{nm}\,\mathrm{sgn}(k_\alpha^n-k_\beta^m)\ , \tag{2.2}$$

where k_α^n is the position of the center of a string of length n (α labels different strings of the same length), $t_{nm}=2\min(n,m)-\delta_{mn}$, and I_α^n are integers or half-odd integer quantum numbers with range

$$|I_\alpha^n|\le\frac{1}{2}\left[N-\sum_{m=1}^{\infty}t_{nm}M_m-1\right]\ , \tag{2.3}$$

where M_m is the number of strings of given length m. Thus $\sum_{n=1}^{\infty}nM_n=M$, where M is the total number of down-spins. The energy and momentum are given as $E=\sum_{n\alpha}\frac{1}{2}[(k_\alpha^n)^2-\pi^2]$ and $P=\sum_{n\alpha}(k_\alpha^n+\pi)$. The above equations are very similar to the Bethe equations for the spin-$\frac{1}{2}$ Heisenberg XXX (nearest neighbor) antiferromagnet.[13–16] Ha and Haldane proceed to show that ground state and excitations as well as the thermodynamics of the HS chain are described correctly by the above equations if one considers them true Bethe equations for an integrable system. The ground state is a filled Fermi sea, where all vacancies for the integers I_α^1 [allowed by (2.3)] are taken. More precisely we have $M=M_1=N/2$, and there are $N/2$ vacancies for the integers I_α^1, all of which are filled. This corresponds to filling all vacancies for the momenta k_α^1 between $-\pi$ and π. The Ha-Haldane equations take the form

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$$Nk_\alpha^1=2\pi I_\alpha^1+\pi\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{M}\operatorname{sgn}(k_\alpha^1-k_\beta^1)\ ,\quad \alpha=1,\ldots,M\ . \tag{2.4}$$

Subtracting (2.4) for α and $\alpha+1$, one obtains an equation for the density of k's $\rho_1(k_\alpha^1)=(1/k_{\alpha+1}^1-k_\alpha^1)$, which in the thermodynamic limit $N\to\infty$ turns into the following integral equation:

$$\frac{1}{2\pi}=\rho_1(k)+\int_{-\pi}^{\pi}dk'\delta(k-k')\rho_1(k')\ , \tag{2.5}$$

which can be solved trivially with the result $\rho_1(k)=1/4\pi$. This shows that the ground state is of a much simpler nature than for the Heisenberg antiferromagnet.

The elementary excitations or quasiparticles are identified as two spin-$\frac{1}{2}$ objects, called spinons, with dispersion (using the conventions of Ref. 12) $\epsilon(p)=p(\pi-p)$, $p\in[0,\pi]$. The situation is thus very similar to the nearest-neighbor Heisenberg model, where there are also two elementary excitations[14,15] carrying spin-$\frac{1}{2}$, but with dispersion $\epsilon_{\rm xxx}(p)=\pi\sin(p)$, $p\in[0,\pi]$. In order to compare the results of Ref. 14 with the Haldane-Shastry case, we should set $J=2$ in the Hamiltonian of Ref. 14. The similarity is not surprising due to the fact that (2.3) is the same for both models, and the ground states of both models are given by filling a Fermi sea of real spectral parameters. The SU(2) structure of excitations is the same for the HS chain[17] and the nearest-neighbor model: all excited states over the true ground state are *scattering states* of an *even number* of quasiparticles. A scattering state is characterized by having energy and momentum equal to the sum of the energies and momenta of its constituent quasiparticles. The simplest excited states are in the two-particle sector, and their SU(2)-representation theory is given simply by tensoring two fundamental representations (the quasiparticles transform in the fundamental representation $\frac{1}{2}$): $\frac{1}{2}\otimes\frac{1}{2}=1\oplus 0$. In other words there are four excited states in the two-particle sector, three of which form a SU(2) triplet and one a SU(2) singlet. Their energies and momenta are degenerate, and are given by the sums of the quasiparticle energies and momenta $E=\epsilon(p_1)+\epsilon(p_2)$ and $P=p_1+p_2$. In the four-particle sector we obtain the SU(2) representation content $\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}\otimes\frac{1}{2}=2\oplus 1\oplus 1\oplus 1\oplus 0\oplus 0$, and so on.

The dressed S matrix can also be obtained from Eqs. (2.2) and (2.3), if we treat the Ha-Haldane equations the same way we treat Bethe equations for Bethe-ansatz solvable models. This is not as straightforward as it may seem, as Bethe-ansatz equations have a *direct* connection to the *exact* eigenfunctions of the Hamiltonian, which is something missing for the case of the HS chain and the Ha-Haldane equations. However, this connection is not vital for determining the scattering phase shifts.[18,19] Adopting (2.2) as Bethe equations, Korepin's method[18] can then be applied in a way completely analogous to the nearest-neighbor XXX chain.[14,15] For a more detailed explanation of the method we use we refer to Refs. 18 and 20.

In order to determine the two-quasiparticle scattering matrix we need to determine the two-quasiparticle eigenstates of the Hamiltonian (the triplet and the singlet), which by construction are also eigenstates of the scattering operator we seek. We proceed again in a way analogous to the XXX case: the spin-triplet SU(2) highest weight state is obtained by choosing $M_1=(N/2)-1$. The allowed range of integers is $|I_\alpha^1|\le N/4$, which means that there are $(N/2)+1$ vacancies and thus two holes. We take the holes to have momenta k_1^h and k_2^h. The Ha-Haldane equations for this excitation are

$$N\tilde{k}_\alpha^1=2\pi\tilde{I}_\alpha^1+\pi\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N/2-1}\operatorname{sgn}(\tilde{k}_\alpha^1-\tilde{k}_\beta^1)-\pi\sum_{j=1}^{2}\operatorname{sgn}(\tilde{k}_\alpha^1-k_j^h)\ . \tag{2.6}$$

Here we have used the notation $\tilde{k}$ to indicate that the momenta of the particles in the Fermi sea are slightly different for the excited state as compared to the ground state.[20] Subtracting (2.6) from Eq. (2.4) for the ground state and taking the thermodynamic limit, we obtain an integral equation for the *shift function* $F_T(k)$ (Ref. 21) of the spin-triplet state [which is the limit of the finite-lattice quantity $F_T(k_\alpha^1)=(\tilde{k}_\alpha^1-k_\alpha^1)/(k_{\alpha+1}^1-k_\alpha^1)$]

$$F_T(k)=1-\int_{-\pi}^{\pi}dk'\delta(k-k')F_T(k') -\tfrac{1}{2}\sum_{j=1}^{2}\operatorname{sgn}(k-k_j^h)\ . \tag{2.7}$$

The solution to this equation is

$$F_T(k)=\tfrac{1}{2}-\tfrac{1}{4}\sum_{j=1}^{2}\operatorname{sgn}(k-k_j^h)\ . \tag{2.8}$$

The phase shift for the spin-triplet state is (for an explanation see, e.g., Ref. 20)

$$\delta_T(k_1^h,k_2^h)=2\pi F_T(k_1^h)\Big|_{k_1^h>k_2^h} =\pi-\frac{\pi}{2}\operatorname{sgn}(k_1^h-k_2^h)=\frac{\pi}{2}\ . \tag{2.9}$$

Here k_1^h must be larger than k_2^h [or vice versa, in which case $\delta_T(k_1^h,k_2^h)=2\pi F_T(k_2^h)|_{k_2^h>k_1^h}$] for scattering to occur. The spin-singlet excitation is constructed by taking $M_1=(N/2)-2$ and $M_2=1$. Now there are $N/2$ vacancies for the 1 strings and thus again two holes, whereas there is only one vacancy for the 2 string. Denoting the positions of the two holes again by k_j^h and the position of the 2 string by κ, we find that the Bethe equation for the 2 string leads to the condition $k_1^h>\kappa>k_2^h$ or $k_2^h>\kappa>k_1^h$, whereas the equations for the 1 string lead to an integral equation for the singlet shift function $F_S(k)$, which has the solution

$$F_S(k)=-\frac{\pi}{2}\sum_{j=1}^{2}\operatorname{sgn}(k-k_j^h)+\pi\operatorname{sgn}(k-\kappa)\ . \tag{2.10}$$

The leads to the following result for the singlet phase shift:

$$\delta_S(k_1^h,k_2^h)=2\pi F_S(k_1^h)|_{k_1^h>k_2^h}=\frac{\pi}{2}\ .$$

Our result for the complete dressed S matrix is thus

$$S_{\rm HS}(k_1^h,k_2^h)=i\,id\ . \tag{2.11}$$

This result ought to be compared with the exact S matrix for the nearest-neighbor Heisenberg model[14,15]

$$S_{XXX}(\mu)=-\frac{\Gamma\left[\frac{1+i\mu}{2}\right]\Gamma\left[1-\frac{i\mu}{2}\right]}{\Gamma\left[\frac{1-i\mu}{2}\right]\Gamma\left[1+\frac{i\mu}{2}\right]}\times\left[\frac{\mu}{\mu+i}id+\frac{i}{\mu+i}P\right], \tag{2.12}$$

where P is the 4×4 permutation matrix and $\mu=\lambda_1-\lambda_2$ is the difference of the spectral parameters of the two quasiparticles. We see that (2.11) is the $\mu\to\infty$ limit of the XXX S matrix. Note that this does not imply (and it is not true either) that the low-energy spinon-spinon scattering is the same in HS and XXX models: the low-energy region of the XXX chain is defined by taking $\lambda_j\to\pm\infty$, $j=1$ and 2, which still leaves the difference μ as a free parameter, the HS chain corresponds to the $\mu\to\infty$ limit of the XXX low-energy physics. This fact does not contradict the identification of the conformal limits of both XXX and HS chains with the $SU(2)_1$ Wess-Zumino-Witten (WZW) conformal field theory. Whereas the conformal limit of the XXX S matrix precisely coincides with the WZW result of Ref. 22 (with trivial LR scattering), the HS result (2.11) corresponds to soft scattering in the WZW model only (the conformal momenta are taken to be very small).

Like in the case of the XXX antiferromagnet the result (2.11) is *a priori* exact up to a possible overall constant factor, which stems from the fact that there are no one-spinon states and we thus cannot determine the one-particle phase shift directly.

Form (2.11) of the dressed S matrix indicates that the spinons behave like an ideal gas as the S matrix is both momentum independent and proportional to the identity. Furthermore, if we believe that there is no additional constant phase factor for the S matrix, the phase i can be interpreted as exhibiting the *semionic* character of the spinons as it is in *between* the phase shifts for ideal Bose and Fermi gases.

From the above discussion the following relation between SU(2) HS chain and XXX model emerges: the quasiparticles in both models are spin-$\frac{1}{2}$ spinons, and the SU(2) representation content is identical in both models. The difference is that the spinons in the HS model are *noninteracting,* whereas the spinons in the XXX chain are interacting. This follows directly from the form of the S matrices, and agrees with the picture previously put forward by Haldane.

III. SU(N) HALDANE-SHASTRY CHAIN

The SU(N) case can be dealt with in an analogous manner: the Bethe equations given by Haldane and Ha[12] are gain very similar to the ones of the nearest-neighbor SU(N) Sutherland model.[23] The quasiparticle interpretation[24] and exact S matrix[25,26] for the SU(N) Sutherland model have recently been derived and can be used to analyze the SU(N) HS chain (the important equations determining the ranges of integers are identical). For simplicity we only discuss the SU(3) case, the general case can be treated analogously. The Ha-Haldane equations for the SU(3) case read

$$\begin{aligned} Nk_\alpha^{(1)n}&=2\pi I_\alpha^{(1)n}+\pi\sum_{m\beta}t_{nm}\,\mathrm{sgn}(k_\alpha^{(1)n}-k_\beta^{(1)m})\\ &\quad-\pi\sum_{m\beta}\min(n,m)\,\mathrm{sgn}(k_\alpha^{(1)n}-k_\beta^{(2)m})\ ,\\ \pi\sum_{m\beta}\min(n,m)\,\mathrm{sgn}(k_\alpha^{(2)n}-k_\beta^{(1)m})&\\ &=2\pi I_\alpha^{(2)n}+\pi\sum_{m\beta}t_{nm}\,\mathrm{sgn}(k_\alpha^{(2)n}-k_\beta^{(2)m})\ . \end{aligned} \tag{3.1}$$

The ground state is obtained by choosing $M_1^{(1)}=2N/3$ and $M_1^{(2)}=N/3$ and filling all vacancies for the integers $I_\alpha^{(1)1}$ and $I_\alpha^{(2)1}$, which corresponds to filling two Fermi seas of spectral parameters $k_\alpha^{(1)1}$ and $k_\alpha^{(2)1}$ between $-\pi$ and π. In the thermodynamic limit we can describe the ground state by densities of spectral parameters $\rho_1^{(1)}(k)$ and $\rho_1^{(2)}(k)$ [like we did for the SU(2) case above] subject to a set of two coupled integral equations. The solution of these integral equations is straightforward as the integral kernels are again δ functions. We find that both densities are constant over the interval $[-\pi,\pi]$

$$\rho_1^{(1)}(k)=\frac{1}{3\pi}\ ,\quad \rho_1^{(2)}(k)=\frac{1}{6\pi}\ . \tag{3.2}$$

This extends straightforwardly to the general SU(N) case. Excitations over this ground state can be constructed in an analogous way to the SU(2) case above. One finds that the only dynamical objects are holes in the two Fermi seas of spectral parameters $k^{(1)1}$ and $k^{(2)1}$, i.e., only these holes carry energy and momentum, whereas longer strings (described by spectral parameters $k^{(j)n}$, $n>1$, $j=1$ and 2) contribute to neither energy nor momentum and are only counting degeneracies. All excited states [for the SU(N) case] can be interpreted as scattering states of $N-1$ types of quasiparticles subject to superselection rules. The quasiparticles are associated with a hole in one of the $N-1$ Fermi seas, respectively, and transform in the $N-1$ fundamental representations of SU(N). The SU(N) structure of the excited states as well as the superselection rules are the same as in the SU(N) Sutherland model; we refer to Ref. 26 for a detailed discussion with proofs of our assertions.

In the SU(3) case there are a total of six quasiparticles, three of which form the fundamental representations 3 and $\overline{3}$, respectively. The quasiparticles in 3 have energy and momentum

$$\epsilon_3(k)=\tfrac{1}{3}(\pi^2-k^2)\ ,\quad p_3(k)=\tfrac{2}{3}(\pi-k)\ ,$$

whereas the quasiparticles in $\overline{3}$ have energy and momentum

$$\varepsilon_{\overline{3}}(k)=\tfrac{1}{6}(\pi^2-k^2)\ ,\quad p_{\overline{3}}(k)=\tfrac{1}{3}(\pi-k)\ .$$

The superselection rules are that the number of quasiparticles of type 3 plus twice the number of quasiparticles of type $\bar{3}$ must be a multiple of (the integer number) 3. That means that the only two-particle states are given by $3\otimes\bar{3}=8\oplus 1$. In the three-particle sector only the states $3\otimes 3\otimes 3$ and $\bar{3}\otimes\bar{3}\otimes\bar{3}$ are allowed. It can be shown that all excited states are scattering states of quasiparticles subject to the superselection rules. Let us now turn to the evaluation of the phase shifts for the octet and singlet states in the two-quasiparticle sector. The octet is characterized by choosing $M_1^{(1)}=(2N/3)-1$ and $M_1^{(2)}=(N/3)-1$, which leads to one hole in the sea of $k^{(1)1}$'s and $k^{(2)1}$ with spectral parameters $k_h^{(1)}$ and $k_h^{(2)}$, respectively. The Ha-Haldane equations read

$$N\tilde{k}_\alpha^{(1)1}=2\pi\tilde{I}_\alpha^{(1)1}+\pi\sum_{\beta=1}^{2N/3}\text{sgn}(\tilde{k}_\alpha^{(1)1}-\tilde{k}_\beta^{(1)1})-\pi\sum_{\beta=1}^{N/3}\text{sgn}(\tilde{k}_\alpha^{(1)1}-\tilde{k}_\beta^{(2)1})-\pi\,\text{sgn}(\tilde{k}_\alpha^{(1)1}-k_h^{(1)})+\pi\,\text{sgn}(\tilde{k}_\alpha^{(1)1}-k_h^{(2)})\ , \tag{3.3}$$

$$\pi\sum_{\beta=1}^{\frac{2N}{3}}\text{sgn}(\tilde{k}_\alpha^{(2)1}-\tilde{k}_\beta^{(1)1})=2\pi\tilde{I}_\alpha^{(2)1}+\pi\sum_{\beta=1}^{N/3}\text{sgn}(\tilde{k}_\alpha^{(2)1}-\tilde{k}_\beta^{(2)1})+\pi\,\text{sgn}(\tilde{k}_\alpha^{(2)1}-k_h^{(1)})-\pi\,\text{sgn}(\tilde{k}_\alpha^{(2)1}-k_h^{(2)})\ .$$

Now, as for the case of the Hubbard model,[27,28] we have to deal with two shift functions $F_j(k_\alpha^{(j)1})=(\tilde{k}_\alpha^{(j)1}-k_\alpha^{(j)1})/(k_{\alpha+1}^{(j)1}-k_\alpha^{(j)1})$, $j=1$ and 2, which in the thermodynamic limit are found to obey a system of two coupled integral equations. The solution of these integral equations is elementary due to the occurrence of δ-function integral kernels:

$$F_1(k)=\tfrac{1}{6}[\text{sgn}(k-k_h^{(2)})-\text{sgn}(k-k_h^{(1)})]=-F_2(k)\ .$$

The dispersion for the octet states is found to be $E=\epsilon_3(k_h^{(1)})+\epsilon_{\bar{3}}(k_h^{(2)})$ and $P=p_3(k_h^{(1)})+p_{\bar{3}}(k_h^{(2)})$, in accordance with the quasiparticle interpretation. The octet phase shift is

$$\delta_8=-2\pi F_1(k_h^{(1)})+2\pi F_2(k_h^{(1)})+\pi=\frac{\pi}{3}\ ,$$

where we have used that $k_h^{(1)}>k_h^{(2)}$ for scattering to occur. The SU(3) singlet in $3\otimes\bar{3}$ is obtained by choosing $M_1^{(1)}=(2N/3)-2$, $M_2^{(1)}=1$, $M_1^{(2)}=(N/3)-2$, and $M_2^{(2)}=1$. The energy and momentum of the singlet are the same as for the octet. The shift functions can be determined analogously to the octet case, although the computation is slightly more difficult due to the presence of 2 strings. The result is

$$\delta_1=\frac{\pi}{3}=\delta_8\ ,$$

i.e., the singlet phase shift is the same (constant) as the octet phase shift. The phase shifts for scattering of quasiparticles of type 3 ($\bar{3}$) on quasiparticles of type 3 ($\bar{3}$) can be extracted from the three-particle states $3\otimes 3\otimes 3$ ($\bar{3}\otimes\bar{3}\otimes\bar{3}$), with the result that the phase shifts for scattering of 3 on 3 and $\bar{3}$ on $\bar{3}$ are also equal to $\pi/3$. This implies that the quasiparticles are an ideal gas with fractional statistics $\pi/3$. For the SU(N) case we conjecture the phase to be π/N. The interesting phenomenon is the decoloration of physical excitations: the superselection rules force the quasiparticles to combine to either mesons ($3\otimes\bar{3}$) or baryons ($3\otimes 3\otimes 3$ and $\bar{3}\otimes\bar{3}\otimes\bar{3}$). In this way the SU(3) HS chain [as well as the SU(3) Sutherland model[26]] is reminiscent of an ideal one-dimensional (1D) gas of (confined) quarks. As for the SU(2) case, the constant S matrix found for the HS model is precisely the limit $\mu\to\infty$ of the corresponding S matrix of the nearest-neighbor Sutherland model,[26] where μ is the difference of the spectral parameters of the two scattering quasiparticles.

IV. CALOGERO-SUTHERLAND MODEL

The Calogero-Sutherland model[8,9,29] is given by the following Hamiltonian:

$$H=-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+\sum_{j<k}\frac{2\lambda(\lambda-1)}{(x_k-x_j)^2}\ . \tag{4.1}$$

Ground state, excitations, and thermodynamics for the CS model are all constructed from the following set of asymptotic Bethe equations:[9]

$$\exp(-ik_jL)=\prod_{l\neq j}S(k_j,k_l)\ , \tag{4.2}$$

where $S(k)=-\exp[-i\pi(\lambda-1)\,\text{sgn}(k)]$ is the *bare S* matrix describing scattering of two *bare* particles over the bare (trivial) vacuum. Following the logic of the Bethe ansatz, Sutherland used (4.2) to construct the true ground state and *dressed* excitations over it, by filling the Fermi sea. By this we mean the following: taking the logarithm of (4.2), we arrive at the set of equations

$$Lk_j=2\pi I_j+(\lambda-1)\pi\sum_{l\neq j}\text{sgn}(k_j-k_l)\ ,\quad j+1,\ldots,N\ , \tag{4.3}$$

where I_j are all integers of half-odd integer numbers, which can be chosen as a set of quantum numbers that completely determines an eigenstate of the Hamiltonian. The ground state is characterized by filling all vacancies for the k_j's symmetrically around zero in the interval $[-k_F,k_F]$. The Fermi momentum is $k_F=\sqrt{\mu}$, where μ is the chemical potential. In analogy to the δ-function Bose gas,[30,31] *dressed* (particle hole) excitations can be constructed by removing one particle with rapidity k_h from the Fermi sea and placing it on a vacancy k_p outside the Fermi sea. The energy and momentum of a particle-hole excitation are given by $E_{\text{ph}}=(k_p^2-\mu)+[(\mu-k_h^2)/\lambda]$ and $P_{\text{ph}}=(k_p-k_F)+[(k_F-k_h)/\lambda]$. Equations (4.3) can also be used to determine the phase shifts for scattering of dressed excitations over the ground state. The computations are completely analogous to the ones for the δ-function Bose gas,[20,21] so that we will only give the results here. It is again straightforward to show that all phase shifts are momentum-independent constants, which proves the ideal-gas nature of the quasiparticles. In determining the constants we follow Ref. 20 (where the δ-function Bose gas was treated; see p. 23) and change the boundary conditions for one-particle and one-hole excitations (the situation here is quite analogous to the Bose gas case). We then find that particles do not receive any dressing through the ground state; i.e., they still behave like bare particles, which scatter off each other with the bare phase shift $\delta_{\rm pp}=-\pi\lambda$. This is in agreement with the results previously obtained in Ref. 2 by means of different methods. Particles scatter off holes with phase shift $\delta_{\rm ph}=\pi\lambda$, which means that to the scattering particle a hole is nothing but the absence of a bare particle. Last but not least, the hole-hole phase shift is $\delta_{\rm hh}=\pi/\lambda$.

We note that we recover the correct scattering phases for free fermions $e^{\delta_{\rm pp}}=e^{\delta_{\rm hh}}=-1$ for $\lambda=1$ and free bosons $e^{\delta_{\rm pp}}=0$ for $\lambda=0$ (in this limit there are no holes but only particles).

V. OTHER $1/\sin^2(r)$-TYPE MODELS

Other candidates for applying Korepin's dressed S-matrix method would be, for example, the $gl(n,1)$ supersymmetric t-J models with long-range exchange interactions,[32–34] or Kawakami's hierarchy of SU(N) electron models.[35] The asymptotic Bethe equations (ABE's) (Refs. 33–35) have to be complemented by a squeezed-string prescription[12] in order to give the correct degeneracies of the spectrum. If we take into account only the states given by the ABE's we find that all phase shifts will be constants, and the states described by the ABE's will thus describe mixtures of ideal gases with fractional statistics. This can be seen as follows: from the computations above it is clear that the ideal-gas character of the quasiparticles is caused by the δ-function kernel in the integral equations, or alternatively the sgn(x) kernels in the asymptotic Bethe equations. The occurrence of sgn(x) kernels in the ABA equations is generic feature of models with $1/\sin^2(r)$-type interactions, so that by analyzing only ABA states we conclude that all these models describe mixtures of noninteracting quasiparticles.

However, there are a number of open problems concerning the supersymmetric models: the squeezed-string prescription seems not to be available, but more importantly the ground state will in general *not* be a $gl(n,1)$ singlet and thus not a $Y[gl(n,1)]$ Yangian singlet. This is easily seen for the case of the long-range supersymmetric $[gl(2,1)]$ t-J model: in the 3^L-dimensional Hilbert space without doubly occupied sites there exists no $gl(2|1)$ singlet. Thus, very much like the case of the nearest-neighbor model,[36] the ground state will belong to a larger $gl(2,1)$ multiplet. This raises the question of how to interpret the other states in the multiplet containing the ground state in terms of a quasiparticle picture, which is necessary for identifying the model as a gas.

VI. INTERACTING QUASIPARTICLES: $1/\sinh^2(r)$ MODELS

Let us now demonstrate that not all models with long-range interactions describe noninteracting quasiparticles. To this end let us consider the $1/\sinh^2(r)$ CS model,[37] defined in terms of the Hamiltonian ($\lambda\geq 1$)

$$H=-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+\sum_{j<k}\frac{2\lambda(\lambda-1)}{\sinh^2\left[\dfrac{x_k-x_j}{a}\right]} . \qquad (6.1)$$

In the limit $a\rightarrow\infty$, (6.1) reduces to the CS model with coupling $2\lambda(\lambda-1)a^2$. The ABA as well as ground state and excitations were constructed in Ref. 37. The ABA equations are

$$Lk_j=2\pi I_j+\sum_{l\neq j}\theta(k_j-k_l) , \quad j+1,\ldots,N , \qquad (6.2)$$

where a has been set to 1, and where

$$\theta(k)=i\ln\left[\frac{\Gamma\left[1+\dfrac{ik}{2}\right]\Gamma\left[\lambda-\dfrac{ik}{2}\right]}{\Gamma\left[1-\dfrac{ik}{2}\right]\Gamma\left[\lambda+\dfrac{ik}{2}\right]}\right] . \qquad (6.3)$$

In terms of the variables k_j the effect of a is recovered by a rescaling $k_j\rightarrow ak_j$. Excitations over the ground state are (as in the CS case above) of particle-hole type. Their energy and momentum are $E_{\rm ph}=\epsilon(k_p)-\epsilon(k_h)$ and $P_{\rm ph}=p_{k_p}-p_{k_h}$, where k_p and k_h are the rapidities of the particle and hole, respectively, and where $\epsilon(k)$ and $p(k)$ are given in terms of the integral equations

$$\epsilon(k)=k^2-\mu-\frac{1}{2\pi}\int_{-B}^{B}dk'\epsilon(k')\theta'(k-k') ,$$

$$p(k)=k-\frac{1}{2\pi}\int_{-B}^{B}dk'p(k')\theta'(k-k') ,$$

where $\theta'(x)=(d/dx)\theta(x)$. Here μ is the chemical potential, and the integral boundary B is determined as a function of μ through the requirement that $\epsilon(\pm B)=0$. The computations of the S matrices for scattering of particles on holes, particles on particles, and holes on holes are again completely analogous to the ones for the Bose gas (see p. 23 of Ref. 20). The result is that all phase shifts can be expressed in terms of a function $\delta(\lambda,\mu)$ subject to the integral equation

$$\delta(k_1,k_2)+\frac{1}{2\pi}\int_{-B}^{B}dk\,\theta'(k_1-k)\delta(k,k_2)=\pi+\theta(k_1-k_2) . \qquad (6.4)$$

The S matrix for particle-hole scattering ($k_p>B, -B\leq k_h\leq B$, the constraint $k_p>B$ is only a matter of convenience) is given as

$$S_{\rm ph}(k_p,k_h)=e^{i\delta(k_p,k_h)} .$$

Similarly particle-particle and hole-hole S matrices are found to be

$$S_{\rm pp}(k_{p,2},k_{p,1})=e^{-i\delta(k_{p,2},k_{p,1})}\ ,$$
$$k_{p,2}>k_{p,1}>B\ ,$$
$$S_{\rm hh}(k_{h,2},k_{h,1})=e^{-i\delta(k_{h,2},h_{h,1})}\ ,$$
$$B\geq k_{h,2}>k_{h,1}\geq -B\ .$$

As noted above the CS limit is obtained by rescaling $k_j \rightarrow ak_j$ and then taking $a\rightarrow\infty$, keeping $2\lambda(\lambda-1)a^2=:2g(g-1)$ fixed. In this limit one obtains $\theta(k_1-k_2)\rightarrow\pi(g-1)\,{\rm sgn}(k_1-k_2)$,[37] and the expression for the S matrices reduces to the ones found for the CS model in Sec. IV (if g is identified with λ of Sec. IV), as can be seen directly from (6.4). In general the integral equation (6.4) can only be solved numerically, the result being a nontrivial function of k_1 and k_2.

Physically our results for the S matrices imply that the quasiparticles in the $1/\sinh^2(r)$ CS model are *interacting,* as the S matrices are momentum dependent and nontrivial. In the $1/r^2$ limit they become *noninteracting.*

A very interesting extension of the $1/\sinh^2(r)$ CS model is the $1/\sinh^2(r)$ CS model with exchange.[38–40] The Hamiltonian of the model is[41]

$$H=-\sum_{j=1}^{N}\frac{\partial^2}{\partial x_j^2}+\sum_{j<k}^{N}\kappa^2\frac{(\lambda^2-\lambda P_{jk})}{\sinh^2[(x_k-x_j)\kappa]} \tag{6.5}$$

where P_{jk} is a permutation operator exchanging the spins of the particles at positions x_j and x_k. We will consider only the simplest case of SU(2) spins. N is the number of particles in a box of length L, and we are interested in the limit $L\rightarrow\infty$ keeping the density N/L fixed. In the inverse square limit $\kappa\rightarrow 0$ the interaction becomes $\sum_{j<k}[\lambda(\lambda-P_{jk})]/(x_j-x_k)^2]$, and the model reduces to the CS model with inverse square exchange.[38,41,7,42] The ABA equations for (6.5) are[39]

$$e^{ik_jL}=\prod_{l\neq j}^{N}\frac{\Gamma\left[1-\frac{i(k_j-k_l)}{2}\right]}{\Gamma\left[1+\frac{i(k_j-k_l)}{2}\right]}\frac{\Gamma\left[\lambda+\frac{i(k_j-k_l)}{2}\right]}{\Gamma\left[\lambda-\frac{i(k_j-k_l)}{2}\right]}\times\prod_{s=1}^{N_\downarrow}\frac{k_j-\alpha_s+i\lambda}{k_j-\alpha_s-i\lambda}\ , \tag{6.6}$$
$$\prod_{j=1}^{N}\frac{\alpha_s-k_j+i\lambda}{\alpha_s-k_j-i\lambda}=-\prod_{j=1}^{N_\downarrow}\frac{\alpha_s-\alpha_t+2i\lambda}{\alpha_s-\alpha_t-2i\lambda}\ .$$

Here κ has been set to 1, and the effect of κ corresponds to a rescaling $k_j\rightarrow k_j/\kappa$ and $\alpha_s\rightarrow\alpha_2/\kappa$. All k_j's are real [complex k's do not lead to bound states in the bare scattering amplitudes on the right-hand side of the first equation in (6.6)], whereas the α_s's can form bound states of the form $\alpha_s^{n,j}=\alpha_s+i(n+1-2j)\lambda$ with $\alpha_s^n\in\mathbb{R}$. This is not surprising as the second set of equations in (6.6) is nothing but the set of Bethe equations for an inhomogeneous Heisenberg model. Inserting this string hypothesis into (6.6) and then taking the logarithm, we obtain

$$Lk_j=2\pi I_j+\sum_{l\neq j}\theta(k_j-k_l)-\sum_s\vartheta\left[\frac{k_j-\alpha_s^n}{n\lambda}\right]\ ,$$
$$0=2\pi J_s^n-\sum_l\vartheta\left[\frac{\alpha_s-k_l}{n\lambda}\right]+\sum_{(m,t)\neq(n,s)}\vartheta_{nm}\left[\frac{\alpha_s^n-\alpha_t^m}{\lambda}\right]\ , \tag{6.7}$$

where I_j and J_α^n are integer or half-odd integer numbers, $\theta(x)$ is given by (6.3), $\vartheta(x)=2\arctan(x)$, and

$$\vartheta_{nm}(x)=\begin{cases}\vartheta\left[\frac{x}{|n-m|}\right]+2\vartheta\left[\frac{x}{|n-m|+2}\right]+\ldots+2\vartheta\left[\frac{x}{n+m-2}\right]+\vartheta\left[\frac{x}{n+m}\right] & \text{if } n\neq m\\ 2\vartheta\left[\frac{x}{2}\right]+2\vartheta\left[\frac{x}{4}\right]+\ldots+2\vartheta\left[\frac{x}{2n-2}\right]+\vartheta\left[\frac{x}{2n}\right] & \text{if } n=m\ .\end{cases}$$

The range of J_α^n follows from (6.7) to be

$$|J_s|\leq\frac{1}{2}\left[N-\sum_{m=1}^{\infty}t_{nm}M_m-1\right]\ , \tag{6.8}$$

where M_m is the number of α strings of length m.

The construction of the ground state and excitations is rather similar to the less than half-filled Hubbard model.[43] The ground state is obtained by filling two Fermi seas of spectral parameters k_j and α_s^1. In the thermodynamic limit it is described in terms of two densities (of spectral parameters) $\rho(k)$ and $\sigma(\alpha)$ subject to the coupled integral equations

$$\rho(k)=\frac{1}{2\pi}-\frac{1}{2\pi}\int_{-A}^{A}dk'\theta'(k-k')\rho(k')+\frac{1}{2\pi}\int_{-\infty}^{\infty}d\alpha\frac{2\lambda}{\lambda^2+(k-\alpha)^2}\sigma(\alpha)\ ,$$
$$\sigma(\alpha)=\frac{1}{2\pi}\int_{-A}^{A}dk\frac{2\lambda}{\lambda^2+(k-\alpha)^2}\rho(k)-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\alpha'\frac{4\lambda}{4\lambda^2+(\alpha-\alpha')^2}\sigma(\alpha')\ ,$$

where $\int_{-A}^{A}dk\,\rho(k)=N/L$, and where the integration boundary A is a function of the chemical potential μ. The ground-state energy density is given by

$E_{GS}=\int_{-A}^{A}dk\,\rho(k)k^2$ ($\mu=dE_{GS}/dN$, which fixes A as a function of μ). We note that for $\kappa>0$, $\sigma(\alpha)>0$ on the whole real axis (the sea of α's is completely filled), whereas in the inverse square limit $\kappa\to 0$, $\sigma(\alpha)=0\forall|\alpha|>A$. This is in agreement with Ref. 35. There are two classes of low-lying excitations over the ground state: particle-hole excitations in the Fermi sea of k's, which are very similar to the excitations in the $1/\sinh^2(r)$ CS model (see above; the only difference is that now there will be a dressing through the second Fermi sea of λ's), and spin excitations in the second Fermi sea. We will constrain ourselves to a discussion of the spin excitations here. Inspection of (6.8) shows that the situation for $\kappa>0$ is very similar to the one for the HS chain treated in Sec. II: the simplest low-lying excitations are a spin-triplet $[M_1=(N/2)-1]$ and a spin-singlet $[M_1=(N/2)-2,\ M_2=1]$ two-hole excitation. In the inverse square limit the Fermi sea of α's is not completely filled, so that the simplest spin excitations are of particle-hole type. We consider only the case $\kappa>0$, as it is the far more interesting one. As in Secs. II and III we describe the excitations in terms of shift functions $F_1(k)$ and $F_2(\alpha)$ (the construction is very similar to the one of spin excitations in the Hubbard model, which was treated in detail in Ref. 27). After some manipulations we find, for the triplet,

$$F_1^T(k)=\frac{1}{2\pi}\sum_{p=1}^{2}\left[2\arctan\left[e^{\pi(k-\alpha_{h,p})2\lambda}\right]-\frac{\pi}{2}\right]-\frac{1}{2\pi}\int_{-A}^{A}dk' F_1^T(k')\Theta(k-k')\ ,$$
$$F_2^T(\alpha)=\frac{1}{2}-\frac{i}{2\pi}\sum_{p=1}^{2}\ln\left[\frac{\Gamma\left[\dfrac{1+i\frac{\alpha-\alpha_{h,p}}{2\lambda}}{2}\right]\Gamma\left[1-i\frac{\alpha-\alpha_{h,p}}{4\lambda}\right]}{\Gamma\left[\dfrac{1-i\frac{\alpha-\alpha_{h,p}}{2\lambda}}{2}\right]\Gamma\left[1+i\frac{\alpha-\alpha_{h,p}}{4\lambda}\right]}\right]+\frac{1}{4\lambda}\int_{-A}^{A}dk\frac{F_1^T(k)}{\cosh\left[\frac{\pi}{2\lambda}(\alpha-k)\right]}\ , \tag{6.9}$$

where $\alpha_{h,p}$ are the rapidities of the two holes and where

$$\Theta(x)=\mathrm{Re}\left\{\frac{1}{2\lambda}\Psi\left[\frac{1}{2}+i\frac{x}{4\lambda}\right]-\frac{1}{2\lambda}\Psi\left[1+i\frac{x}{4\lambda}\right]+\Psi\left[\lambda+i\frac{x}{2}\right]-\Psi\left[1+i\frac{x}{2}\right]\right\}.$$

Here $\Psi(x)$ is the digamma function. The energy and momentum of the spin-triplet are given by

$$E_{ST}(\alpha_{h,1},\alpha_{h,2})=\int_{-A}^{A}dk\,2kF_1^T(k)\ ,$$
$$P_{ST}(\alpha_{h,1},\alpha_{h,2})=\int_{-A}^{A}dk\,F_1^T(k)\ .$$

The scattering phase shift is given as $\delta_T(\alpha_{h,1},\alpha_{h,2})=2\pi F_2^T(\alpha_{h,1})$, with $\alpha_{h,1}-\alpha_{h,2}>0$. For the spin singlet we find

$$F_1^S(k)=F_1^T(k)\ ,$$
$$F_2^S(\alpha)=F_2^T(\alpha)+\frac{1}{\pi}\arctan\left[\frac{\alpha-\frac{\alpha_{h,1}+\alpha_{h,2}}{2}}{\lambda}\right]-\frac{1}{2}\ , \tag{6.10}$$

where $F_{1,2}^T$ are given by (6.9). From the first equality in (6.10) it follows immediately that the energy and momentum of the triplet and singlet are identical (as they must be). The singlet scattering phase shift is found to be $\delta_S(\alpha_{h,1},\alpha_{h,2})=2\pi F_2^S(\alpha_{h,1})$. The resulting two-particle S matrix describing scattering of spinons in the $1/\sinh^{2(r)}$ CS model with exchange is

$$S(\alpha_{h,1},\alpha_{h,2})=e^{2\pi i F_2^T(\alpha_{h,1})}\left[\frac{\nu}{\nu+i}id+\frac{i}{\nu+i}P\right]\ ,$$
$$\nu=\frac{\alpha_{h,1}-\alpha_{h,2}}{2\lambda}>0\ , \tag{6.11}$$

where P is the 4×4 permutation matrix and where $F_2^T(\alpha)$ is given by (6.9). The result we obtain is extremely similar to the spinon-spinon S matrix of the Hubbard model:[27,28,44] the rapidities are renormalized by a factor of 2λ ($2U$ in the Hubbard model) as compared to the pure XXX scattering matrix (2.12), and the common overall phase receives an additional contribution from the dynamical degrees of freedom (i.e., the Fermi sea of k's). We can rewrite (6.11) in terms of the XXX S matrix S_{XXX} given by (2.12) as

$$S(\alpha_{h,1},\alpha_{h,2})=S_{XXX}(\nu)\exp\left[\frac{2\pi i}{4\lambda}\int_{-A}^{A}dk\frac{F_1^T(k)}{\cosh\left[\frac{\pi}{2\lambda}(\alpha_{h,1}-k)\right]}\right]\ ,$$

where v is as in (6.11). In the limit $v\rightarrow\infty$ this reduces to the Haldane-Shastry result (2.11). We see that in (6.11) there are two distinct contributions: one from pure spin-spin scattering [given by $S_{XXX}(v)$], and one from coupling of spin and dynamical degrees of freedom (given by the second factor).

As was noted by Sutherland, Römer, and Shastry, it is possible to freeze out the dynamical degrees of freedom in (6.5) by taking the limit $\lambda\rightarrow\infty$.[39] In this limit the particles freeze into an equidistant lattice $x_j=j/d$ (recall that $d=N/L$ is the fixed density of particles), and the Hamiltonian (6.5) separates into $H_{\rm dyn}+2\lambda H_{\rm latt}$, where $H_{\rm dyn}$ is of the form (6.1) with coupling $\lambda(\lambda-1)$, and where

$$H_{\rm latt}=-\frac{1}{2}\sum_{j>k}\frac{1+P_{jk}}{\sinh^2\left[\dfrac{k-j}{d}\right]} . \tag{6.12}$$

Ground state and excitations of the lattice model (6.12) can be obtained by rescaling and expanding the spectral parameters in (6.7) according to $k_j=2\lambda\xi_j^{(0)}+\xi_j^{(1)}+(1/2\lambda)\xi_j^{(2)}+\cdots$ and $\alpha_s^n=2\lambda\beta_s^n+\cdots$, and then expanding the ABE's in inverse powers of λ. This procedure yields

$$\begin{aligned}
&L\xi_j^{(0)}=\sum_l\theta_0(\xi_j^{(0)}-\xi_l^{(0)}) ,\\
&0=2\pi I_j-\sum_l\theta_0'(\xi_j^{(0)}-\xi_l^{(0)})\xi_l^{(1)}\\
&\qquad-\sum_{(n,s)}\vartheta\left[\frac{2}{n}(\xi_j^{(0)}-\beta_s^n)\right]+\sum_l\theta_1(\xi_j^{(0)}-\xi_l^{(0)}) ,\\
&0=2\pi J_s^n-\sum_l\vartheta\left[\frac{2}{n}(\beta_s^n-\xi_l^{(0)})\right]\\
&\qquad+\sum_{(m,t)\neq(n,s)}\vartheta_{nm}(2(\beta_s^n-\beta_t^m)) ,
\end{aligned}\tag{6.13}$$

where $\theta_0(x)=(x/2)\ln[1+(1/x^2)]+(i/2)\ln[(1-ix)/(1+ix)]$ and $\theta_1(x)=-(\pi/2)-(i/2)\ln[(1-ix)/(1+ix)]$. The first set of equations (6.13) is of order λ, and leads in the thermodynamic limit to an integral equation for the ground-state density of the dynamical part[39] $\rho(x)$ (defined to be the limit $N\rightarrow\infty$ of $\rho(\xi_j^{(0)})=1/[N(\xi_{j+1}^{(0)}-\xi_j^{(0)})]$). The second and third sets of equations (6.13) are of order 1 and can be used to construct ground state and excitations of the spin model (6.12). This has already been done [for the general SU(N)] case in Ref. 39. Our goal here is to determine the exact S matrix for the SU(2) case, for which we need to construct all two particle excitations in the framework of the F-function formalism. This is easily done as the integers J_s^n are actually the same as in the dynamical model treated above. Before we get to this let us review some results of Ref. 39 that we will need later on. The ground state of (6.5) in the limit $\lambda\rightarrow\infty$ is obtained by taking $M_1=N/2$ and $M_k=0\forall k>1$. In the thermodynamic limit $N\rightarrow\infty$ ($d=N/L$ fixed) the ABA equations turn into a set of three coupled integral equations[39]

$$\begin{aligned}
&\frac{1}{d}=\int_{-a}^{a}d\xi'\rho(\xi')\theta_0'(\xi-\xi') ,\\
&0=2\pi\rho(\xi)-\int_{-a}^{a}d\xi'\gamma(\xi')\theta_0''(\xi-\xi')\\
&\qquad+\int_{-a}^{a}d\xi'\rho(\xi')\theta_1'(\xi-\xi')-\int_{-\infty}^{\infty}d\beta\frac{4\sigma(\beta)}{1+4(\xi-\beta)^2} ,\\
&0=2\pi\sigma(\beta)-\int_{-a}^{a}d\xi\,\rho(\xi)\frac{4}{1+4(\xi-\beta)^2}\\
&\qquad+\int_{-\infty}^{\infty}d\beta'\sigma(\beta')\frac{2}{1+(\beta-\beta')^2} ,
\end{aligned}\tag{6.14}$$

where a is a function of the fixed density d, and where $\sigma(\beta)$ and $\gamma(\xi)$ are the infinite volume limits of the densities $1/[N(\beta_{s+1}^1-\beta_s^1)]$ and $\xi_j^{(1)}/[N(\xi_{j+1}^{(0)}-\xi_j^{(0)})]$. In order to describe only the ground state of (6.12) it is necessary to decouple the dynamical degrees of freedom by hand.[39] We note that for the excitations no such decoupling has to be carried out because the structure of (6.14) is such that the dynamical degrees of freedom decouple automatically. Equations (6.14) and (6.13) are all we need to determine the S matrix. Let us start with the spin-triplet phase shift. The spin-triplet excitation is obtained by taking $M_1=(N/2)-1$ and all other $M_k=0$. There are two holes with corresponding spectral parameters $\beta_{h,j}$, $j=1$ and 2 in the distribution of β's. The ABA equations (6.13) read (our convention is $\tilde{J}_s-J_s=\frac{1}{2}$)

$$\begin{aligned}
&0=\pi I_j-\sum_l\theta_0'(\xi_j^{(0)}-\xi_l^{(0)})\tilde{\xi}_l^{(1)}\\
&\qquad-\sum_s\vartheta[2(\xi_j^{(0)}-\tilde{\beta}_s)]+\sum_l\theta_1(\xi_j^{(0)}-\xi_l^{(0)})\\
&\qquad+\sum_{j=1}^{2}\vartheta[2(\xi_j^{(0)}-\beta_{h,j})]-\pi ,\\
&0=2\pi\tilde{J}_s-\sum_l\vartheta[2(\tilde{\beta}_s-\xi_l^{(0)})]+\sum_t\vartheta(\tilde{\beta}_s-\tilde{\beta}_t)\\
&\qquad+\pi-\sum_{j=1}^{2}\vartheta(\tilde{\beta}_s-\beta_{h,j}) .
\end{aligned}\tag{6.15}$$

Subtracting the corresponding ground-state equations from (6.15), we obtain coupled equations for the shift functions $F_2(\beta_j)=(\tilde{\beta}_j-\beta_j)/(\beta_{j+1}-\beta_j)$ and $F_1(\xi_j)=(\tilde{\xi}_j^{(1)}-\xi_j^{(1)})/(\xi_{j+1}^{(0)}-\xi_j^{(0)})$, which in the thermodynamic limit turn into coupled integral equations

$$\begin{aligned}
&F_2(\beta)=1-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\beta'\frac{2}{1+(\beta-\beta')^2}F_2(\beta')\\
&\qquad-\frac{1}{2\pi}\sum_{j=1}^{2}\vartheta(\beta-\beta_{h,j}) ,\\
&0=-\int_{-a}^{a}d\xi'F_1(\xi')\theta_0'(\xi-\xi')\\
&\qquad+\int_{-\infty}^{\infty}d\beta\frac{4}{1+4(\beta-\xi)^2}F_2(\beta)\\
&\qquad-\pi+\sum_{j=1}^{2}\vartheta[2(\xi-\beta_{h,j})] .
\end{aligned}\tag{6.16}$$

Note that in order to obtain (6.16) we used the ground-state equations (6.14). The equation for F_2 is readily solved by Fourier techniques:

$$F_2(\beta)=\frac{1}{2}-\frac{i}{2\pi}\sum_{p=1}^{2}\ln\left[\frac{\Gamma\left[\frac{1+i\frac{\beta-\beta_{h,p}}{2}}{2}\right]\Gamma\left[1-i\frac{\beta-\beta_{h,p}}{2}\right]}{\Gamma\left[\frac{1-i\frac{\beta-\beta_{h,p}}{2}}{2}\right]\Gamma\left[1+i\frac{\beta-\beta_{h,p}}{2}\right]}\right].$$

The triplet phase shift is $\delta_T=2\pi F_2(\beta_{h,1})$ with $\beta_{h,1}-\beta_{h,2}>0$, and is *identical* to the triplet phase shift in the nearest-neighbor *XXX* model. The excitation energy of the triplet states is

$$E_T=\int_{-a}^{a}d\xi\, 2\xi F_1(\xi)$$
$$=-\sum_{j=1}^{2}\int_{-a}^{a}d\xi\frac{e(\xi)}{2\cosh[\pi(\xi-\beta_{h,j})]},$$
$$\xi^2-\mu=\int_{-a}^{a}d\xi' e(\xi')\theta_0'(\xi-\xi'),$$

where $e(\xi)$ is the classical ground-state energy density of Sutherland, Römer, and Shastry.[39] Repeating the above steps for the spin singlet $[M_1=(N/2)-2$ and $M_2=1]$ we find that the excitation energy is the same as for the triplet, and the phase shift is $\delta_S=\delta_T+2\arctan(\beta_{h,1}-\beta_{h,2})-\pi$, which results in an S matrix identical to the nearest-neighbor Heisenberg *XXX* S matrix (2.12) with $\mu=\beta_{h,1}-\beta_{h,2}$. This shows that the spinons in the nearest-neighbor Heisenberg model and its $1/\sinh^2(r)$ analog are very similar: in both models they are interacting with the same S matrix; the only difference is the dispersion. Our result for the S matrix, furthermore, leads to the conclusion that the conformal limit of the $1/\sinh^2(r)$ model (6.12) is given by the $SU(2)_1$ WZW conformal field theory.

VII. DISCUSSION

In this paper we have determined the dressed scattering matrices for several models with long-range interactions by applying a method invented by Korepin for models solvable by (normal) Bethe ansatz. We would like to stress that this method can be applied to *any* model for which the asymptotic Bethe ansatz can be formulated. Our result show very directly that models with $1/\sin^2(r)$ interaction are ideal gases with fractional statistics. Long-range models with $1/\sinh^2(r)$ interactions describe *interacting* elementary excitations and are close in nature to integrable nearest-neighbor models. Our analysis in Sec. VII can readily be generalized from SU(2) to SU(N). The structure of the ABE relevant for the spin degrees of freedom is that of an inhomogeneous SU(N) Sutherland model.[23] On the basis of our results for SU(2) we conjecture that the resulting dressed S matrix for the SU(N) spin chain with $1/\sinh^2(r)$ hopping is identical to the one for the nearest-neighbor model. The fact that elementary excitations in $1/\sinh^2(r)$ models are interacting in basically the same way as in their nearest-neighbor analogs indicates that the evaluation of correlation functions may be rather difficult than for the $1/\sin^2(r)$ case, in which elementary excitations are free.

Finally we would like to point out a close relation between fractional statistics and the fractional charge previously observed in many solvable models. As was first observed by Korepin for the case of the massive thirring model (MTM),[18] elementary excitations over the true ground state will in general carry a fractional charge. Here the charge is the eigenvalue of the fermion number operator defined in terms of the (fermionic) quantum fields entering the Hamiltonian. The relation to fractional statistics is most easily seen for the simple example of the SU(2) *XXX* model: the analog of charge is the third component of the spin. A one-particle excitation over the bare vacuum corresponds to flipping one spin, and thus carries charge, i.e., spin 1. By construction this excitation has bosonic statistics. From the discussion above we see that flipping one spin over the true (antiferromagnetic) ground state leads to a *two*-spinon excitation, and that one spinon thus carries charge $\frac{1}{2}$, and carries fractional statistics. Analogously we can deduce that the quasiparticles with fractional charge in the MTM ought to be thought of as objects of fractional statistics as well.

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*Electronic address: fabman@avzw03.physik.uni-bonn.de

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