Thermodynamics of the exactly solvable two-chain and multichain quantum spin models

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The equations of the thermodynamic Bethe ansatz for two-chain and multichain quantum spin 1/2 systems are constructed. It is shown that the system is in the antichiral state for any temperature range and for any value of the external magnetic field (except for the ferromagnetic ground-state case). The elementary spin excitations carry nonzero chirality and are gapless. Those excitations are confined into pairs (singlets and triplets), with opposite sign of chiralities, so the state of the system remains antichiral at any temperature. The reason for such chiral behavior is the existence of the topological term of the Wess-Zumino nature in the Hamiltonian. We conjecture that the quantum spin (1/2) multichain system with the Hamiltonian without the topological terms has excitations with gaps.

The interest in the theoretical description of quantum many-body low dimensional systems has been renewed last decade. That interest is connected with the two most intriguing questions in solid state physics of recent years—the questions about the nature of the quantum Hall effect and the metal oxide compounds. Though the basic problems of the quantum Hall effect have already been solved (see Ref. 1), the questions about the metal oxides are still open. Following Anderson,[2] a great number of theoreticians express their hope that the nature of the high-T_c superconductivity is connected not with the well-known BCS electron-phonon coupling but with quantum features of the strong two-dimensional (2D) correlation of electrons there (e.g., with the interaction between the spin and charge degrees of freedom, spinons and holons). That is why the problem of the quantum description of the 2D spin 1/2 Heisenberg system, which is believed to be a magnetic model for some metal oxides, is one of the main problems in solid state theory now. There is a number of theoretical models of 2D superconductivity (see, e.g., Ref. 3). The exotic 2D superconductivity can be understood with the concept of anyons, quasiparticles carrying intermediate statistics between bosons and fermions.[4] Anyons are specific to the (2+1) quantum theories.[5] A way to describe anyons is in adding the Chern-Simons term to the Lagrangian of the system, which changes the statistics of the excitations.[4] One of the most interesting concepts in such anyonic superconductivity is the connection between the chiral spin ordering and the superconductivity.[6,3]

Most of the 2D spin quantum theories used the pertubation theory or any modifications of the mean field approximation.[3] But it is important to solve the quantum many-body problem in order to understand the features of 2D quantum strongly correlated electron systems. Several theoretical models gave the exact solutions to 2D spin or electron systems.[7–10] Some of them (devoted to the quantum spin 1/2 systems) carry some features of anyon physics: they have terms in the Hamiltonians which do not conserve T or P symmetries separately, but only the TP symmetry.[9,10] However, those theories are far from reality because the interchain coupling reveals itself in mesoscopic corrections only.

Spin properties of metal oxides are known to admit frustration; see, e.g., Ref. 11. That is why there are some theories devoted to the study of the triangular spin lattice with the antiferromagnetic Heisenberg interaction (see, e.g., Ref. 12), which is believed to show the main features of the spin behavior of the metal oxides.

We know some examples of real cuprates like $Sr_{2n-2}Cu_{2n}O_{4n-2}$ which contain weakly coupled arrays of metal-oxide-metal ladders (multiladders with n chains weakly coupled to each other) with triangular frustrating bonds. The conductivity for such systems is connected with those ladders.[13,14] Other examples of such ladderlike metal oxides are $(VO)_2P_2O_7$ and $La_{4+4n}Cu_{8+2n}O_{4n-2}$.[13,14] From the magnetic point of view those compounds are spin 1/2 antiferromagnetic multiladder arrays with frustration.

One of the questions to be answered is about the spin gap (see, e.g., Refs. 2 and 14): whether the spin excitations of the 2D Heisenberg spin 1/2 model have gaps and the quantum disordered ground state of Anderson emerges, or they are gapless and the ground state is ordered.

In our paper we have studied exactly the multichain spin system with T and P violation, using the quantum inverse scattering method.[15] We have built the thermodynamics of the system. We have shown that the system is in the antichiral state for any value of both the external magnetic field and temperature. Although the elementary excitations (gapless) behave similarly to classical instantons carrying nonzero spin chirality, only (confined) pairs with different signs of their chiralities contribute to the free energy, so they do not change the antichirality of the system. We have shown that the T and P violating terms in the Hamiltonian are the reason for the gapless behavior of the spin excitations. We have also conjec-

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tured that for the multichain (and, naturally, two-chain) quantum spin (1/2) frustrated system without the T and P symmetry violation elementary excitations have gaps.

Let us consider first the Hamiltonian of the simplest case of two spin 1/2 chains of the form (note that it is the simplest example of a double spin chain which reveals frustration;[14] see also the recent paper devoted to an experiment on the double chain system[16])

$$\mathcal{H} = \sum_n [2(\vec{\sigma}_{1,n}\vec{\sigma}_{2,n} + \vec{\sigma}_{1,n}\vec{\sigma}_{2,n+1}) + \theta^2(\vec{\sigma}_{1,n}\vec{\sigma}_{1,n+1} + \vec{\sigma}_{2,n}\vec{\sigma}_{2,n+1}) + \theta\varepsilon^{ikl}(\sigma^i_{1,n} - \sigma^i_{2,n+1})\sigma^k_{1,n+1}\sigma^l_{2,n} - h(\sigma^z_{1,n} + \sigma^z_{2,n})] - E_f, \quad (1)$$

where $\sigma^i_{1,2,n}$ are the operators (the Pauli matrices) of the ith projection ($i = x, y, z$) of the spin operator of the first or second chain in site n, h is the external magnetic field, θ is the interchain coupling parameter, and E_f is the energy of the "ferromagnetic" state.

The third term in Eq. (1) has a nonusual form. It may originate from the spin-orbital coupling for low enough temperatures, for which the orbital degrees of freedom of electrons are frozen [θ is proportional to $\langle\varepsilon^{ikl}(L^i_{1,n} - L^i_{2,n+1})L^k_{1,n+1}L^l_{2,n}\rangle$, where $L^i_{1,2,n}$ is the operator of the ith projection of orbital momentum of the electron in the nth site on the first or second chain and the angular brackets denote the averaging]. One can see from Eq. (1) that the third term breaks both T and P symmetries separately, the TP symmetry being conserved. Only substitutions of 1 for 2 and $(n+1)$ for $(n-1)$ and vice versa do not change the Hamiltonian. We can understand the structure of that term in the classical long-wave limit. If $\mathbf{n}$ is the spin density [$\mathbf{n}^2 = 1$ for the site spin 1/2 (σ) value], one can see that the effect of the third term is equivalent to (for the phase with a zero value of the magnetization)

$$I_0 = \frac{\theta}{2\pi}\int \varepsilon^{ikl}\varepsilon_{\mu\nu}n^i\partial_\mu n^k\partial_\nu n^l d^2x, \quad \mu,\nu = 1,2. \quad (2)$$

Equation (2) is the topological charge (or Noether charge) for the chiral field $\mathbf{n}$, which is the time component of the conserved topological current. One can deform continuously two $\mathbf{n}$ fields into one another if and only if $I_{0,1} = I_{0,2}$.[17] For the classical field (I_0/θ) is an integer number[18] (or Pontrjagin index), that is the number of instantons in the system. On the other hand, one can recognize in (2) the topological current known as the Chern-Simons term,[17] which is specific for 2D systems. Note that in $D = 1$ or 3, we cannot construct a scalar (or pseudoscalar) with the properties of the topological charge, e.g., for 3D we have the vector I_α instead of I_0, and can construct the Hopf invariant, which has integer values only.[17] I_0 is the homotopy class characteristic $\pi_2(S^2) = \mathbf{Z}$.[18] The spin density distribution is topologically nontrivial if $\pi_2(S^2) \neq 0$. The term under consideration is the total time derivative, $\partial_0 I_0 = 0$,[19] so it does not change the classical equations of motion.

From the classical point of view the system with the Hamiltonian (1) is frustrated. The problem of the ground state for the spin system with frustration is one of the most intriguing issues in magnetic properties of solids.[3] Recently, real systems with trianglelike frustrating spin-spin interactions between the two chains with the site spin 1/2 [phosphates $VO(HPO_4)4H_2O$] were examined experimentally.[20] Another system which reveals some properties of the two-chain spin quantum model (it has topological spin excitations) is the two-plane quantum Hall effect.[21]

Using the quantum inverse scattering method[15] we have constructed the L operator $\mathcal{L}$ and the transfer matrix $\mathcal{T}$ of the system (see Ref. 22):

$$\mathcal{L}(\lambda) = L_{\sigma_{1,0}\sigma_{2,n}}(\lambda+\theta)L_{\sigma_{1,0}\sigma_{1,n}}(\lambda)L_{\sigma_{2,0}\sigma_{2,n}}(\lambda) \times L_{\sigma_{2,0}\sigma_{1,n}}(\lambda-\theta), \quad (3)$$

$$\mathcal{T}(\lambda) = T(\lambda)T(\lambda-\theta), \quad (4)$$

where $L(\lambda)$ and $T(\lambda)$ are the standard L operator and transfer matrix of the single spin 1/2 chain, λ is the spectral parameter, and $\sigma_{1,2,0}$ are the auxiliary quantum subspace indices. The eigenvalue of $\mathcal{T}$ is equal to

$$\Lambda(\lambda) = \left(\prod_{j=1}^{M}\frac{\lambda_j-\lambda+i}{\lambda_j-\lambda} + \frac{\lambda^N(\lambda+\theta)^N}{(\lambda+i)^N(\lambda+\theta+i)^N}\prod_{j=1}^{M}\frac{\lambda-\lambda_j+i}{\lambda-\lambda_j}\right) \times\left(\prod_{j=1}^{M}\frac{\lambda_j+\theta-\lambda+i}{\lambda_j+\theta-\lambda} + \frac{\lambda^N(\lambda-\theta)^N}{(\lambda+i)^N(\lambda-\theta+i)^N}\prod_{j=1}^{M}\frac{\lambda-\theta-\lambda_j+i}{\lambda-\theta-\lambda_j}\right), \quad (5)$$

where N is the number of sites on each chain, M is the number of down spins, and λ_j are the spin rapidities determined from the Bethe ansatz equations

$$\frac{\lambda_j^N(\lambda_j+\theta)^N}{(\lambda_j+i)^N(\lambda_j+\theta+i)^N} = \prod_{k=1}^{M}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i}, \quad k\neq j, \quad j = 1,2,\ldots,M. \quad (6)$$

The Hamiltonian (1) is the logarithmic derivative of the transfer matrix (4); another form of Eq. (1) without the magnetic field is

$$\mathcal{H} = \sum_n \{2(P_{\sigma_{1,n}\sigma_{2,n}} + P_{\sigma_{1,n}\sigma_{2,n+1}}) + \theta^2(P_{\sigma_{1,n}\sigma_{1,n+1}} + P_{\sigma_{2,n}\sigma_{2,n+1}}) + \theta[(P_{\sigma_{1,n}\sigma_{2,n}} + P_{\sigma_{1,n}\sigma_{2,n+1}}), (P_{\sigma_{1,n}\sigma_{1,n+1}} + P_{\sigma_{2,n}\sigma_{2,n+1}})]\} + \text{const}, \quad (7)$$

where P is the permutation operator and the square brackets denote the commutator; so the eigenvalue of the Hamiltonian is equal to

$$E = -2h(N-M) + 2\sum_{k=1}^{M}[\lambda_k^{-1}(\lambda_k+i)^{-1} + (\lambda_k+\theta)^{-1}(\lambda_k+\theta+i)^{-1}]. \quad (8)$$

Deriving the values of λ_j from Eqs. (6) and substituting them into Eq. (8), we obtain the system energy in the state with M spins down. To solve the set (6) we use the Hulthèn method;[23] in the thermodynamic limit $N \to \infty$, $M \to \infty$, $(M/N) = m$ being fixed, we have $(\lambda_j \to \lambda_j - \frac{i}{2})$

$$\pi\rho(\lambda) + \int_{-Q}^{Q} \frac{\rho(\mu)}{[(\lambda-\mu)^2+1]} d\mu = \tfrac{1}{2}\{(\lambda^2+\tfrac{1}{4})^{-1} + [(\lambda+\theta)^2+\tfrac{1}{4}]^{-1}\}, \quad (9)$$

$$E = -2h(N-M) - 2N\int_{-Q}^{Q} \rho(\lambda)\{(\lambda^2+\tfrac{1}{4})^{-1} + [(\lambda+\theta)^2+\tfrac{1}{4}]^{-1}\}d\lambda, \quad (10)$$

where the integration limits Q are determined from

$$m = \int_{-Q}^{Q} \rho(\lambda) d\lambda. \quad (11)$$

In the simple case $h = 0$, $m = 2$, $Q = \infty$, we have for the ground-state energy

$$E_0 = -2N\int_{-\infty}^{\infty} \frac{1+\cos(\omega\theta)}{1+\exp|\omega|} d\omega. \quad (12)$$

Equation (12) is an even function of the spin chirality parameter θ: we define the spin chirality of the system as the nonzero average value of the operator

$$\sum_n \varepsilon^{ikl}(\sigma_{1,n}^i - \sigma_{2,n+1}^i)\sigma_{1,n+1}^k\sigma_{2,n}^l.$$

In Ref. 22 we called that ground state the "antichiral" one.

For $h \ll 16\frac{(1+2\theta^2)}{(1+4\theta^2)}$, using the Wiener-Hopf method (see, e.g., Ref. 24) we obtain

$$E_0 = E_0|_{h=0} - N\left(\frac{h}{\pi}\right)^2. \quad (13)$$

For $h \geq 16\frac{(1+2\theta^2)}{(1+4\theta^2)}$, the ground-state energy is

$$E_0 = -2Nh.$$

We can see that the nonzero magnetic field does not change the θ dependence of the ground-state energy. It means that a magnetic field (less then the critical value) does not change the antichirality of the system or the number of excitations like instantons and anti-instantons which form the Dirac sea of the system.

For a simple doublet excitation energy (which is the quantum analog of a classical instanton solution in some sense) following Ref. 25 we have

$$E_d = E_0 + \pi\{\text{sech}(\pi\lambda_0) + \text{sech}[\pi(\lambda_0+\theta)]\}, \quad (14)$$

where λ_0 is the rapidity of the spinon. Two doublet excitations (spinons) carrying different chiralities (winding numbers with opposite signs) with both spins $S = \frac{1}{2}$ confine into the singlet and triplet states; see Ref. 25. (Note that singlet and triplet states can be formed by the confinement of strings with higher spin values but carrying the opposite signs of the spin chiralities; see below.)

The doublet excitation is gapless (the same is true for singlet and triplet states). The question whether the spin excitation of a two-chain spin 1/2 frustrated system is gapless is crucial for the understanding of antiferromagnetic and superconducting ordering in some metal oxides with a specific role of two-chain coupling.[14] Here we have shown that the exact excitations (doublets, singlets, and triplets) of the Hamiltonian (1) are gapless. But that Hamiltonian contains the θ vacuum (the topological charge) terms. The situation is analogous to Haldane's picture of 1D antiferromagnetic spin chains (see, e.g., Refs. 26 and 27). Haldane argued that the integer spin chain has a gap, and the half-integer one has not. But the only difference in the field theory description of integer and half-integer spin 1D quantum systems is the θ vacuum term in the Lagrangian, which does not change the classical properties but is essential for the quantum spin excitation problem. For half-integer antiferromagnetic chains $\theta \neq 0$, and it causes the gap to vanish.[27] Using reasoning analogous to Haldane's, we may suppose that existence of the θ term in the Hamiltonian (1) implies the gapless behavior of our system. The same difference takes place in the 2D classical Wess-Zumino model and the 2D classical nonlinear σ model,[7,8] where the Bethe ansatz chiral equations are similar to our Eqs. (6) (note that in the cited papers the numbers of excitations with the opposite signs of their chiralities may differ from each other, but not in our case) and spin excitations have masses; see also Ref. 28. That is why for the frustrated two-chain system with the Hamiltonian without topological charge terms we can conjecture that the quantum spin excitations do have gaps.

Doublet excitation changes the chiral properties of the system, because Eq. (13) is not an even function of θ. In Ref. 22 we suggested that those excitations (from the classical point of view they look like instantons,[29] and the total number of excitations like instantons and anti-instantons depends on the temperature) form a chiral spin liquid. To answer the question if it is so we have to construct the thermodynamics.

Let us build the thermodynamics, following the Refs. 30–32. Using the string hypothesis within the accuracy of $O(\exp(-\delta N))$, $\delta > 0$,

$$\lambda_j^{n,\alpha} = \lambda_j^n + i(n+1-2j), \quad j = 1,2,\ldots,n, \qquad (15)$$

where α determines the string and n means that the rapidity belongs to the bound state of the $n-\lambda$ string. We can derive the equations for the densities of quasiparticles and quasiholes $\rho_n(\lambda)$ and $\rho_n^h(\lambda)$, respectively:

$$n\pi^{-1}\{(4\lambda^2+n^2)^{-1} + [4(\lambda+\theta)^2+n^2]^{-1}\} = \rho_n^h(\lambda) + \sum_{m=1}^{\infty} A_{n,m}\rho_m(\lambda), \qquad (16)$$

where

$$A_{n,m} = [|n-m|] + 2[|n-m+2|] + \cdots + 2[n+m-2] + [n+m], \qquad (17)$$

$$[n]f(x) = \pi^{-1}\int_{-\infty}^{\infty} \frac{n}{n^2+4(x-y)^2}dy. \qquad (18)$$

The energy of the system is equal to

$$\frac{E}{N} = -2h + \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} \{2h - 16(4\lambda^2+n^2)^{-1} - 16[4(\lambda+\theta)^2+n^2]^{-1}\}\rho_n(\lambda)d\lambda, \qquad (19)$$

and the magnetization per site is

$$m^z = 1 - \sum_{n=1}^{\infty} n \int_{-\infty}^{\infty} \rho_n(\lambda)d\lambda. \qquad (20)$$

Minimizing the free energy of the system $F = E - T\mathcal{S}$, where $\mathcal{S}$ is the entropy of the system and T is the temperature,

$$\frac{\mathcal{S}}{N} = 2\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} [(\rho_n+\rho_n^h)\ln(\rho_n+\rho_n^h) - \rho_n\ln\rho_n - \rho_n^h\ln\rho_n^h]d\lambda \qquad (21)$$

with respect to $\rho_n(\lambda)$ we have (see Ref. 30)

$$n\{2h - 16(4\lambda^2+n^2)^{-1} - 16[4(\lambda+\theta)^2+n^2]^{-1}\} = T\ln\left(1+\frac{\rho_n^h}{\rho_n}\right) - T\sum_{m=1}^{\infty} A_{n,m}\ln\left(1+\frac{\rho_m}{\rho_m^h}\right). \qquad (22)$$

Solving Eqs. (16) and (22) and substituting the solution, e.g., into Eqs. (19)–(21) we can find the energy, magnetization, and entropy of the system. However Eqs. (16) and (22) are nonlinear integral equations, and we can solve them analytically only in two limiting cases of high and low temperatures. For the first case of high temperatures, $T \gg 1$, (h/T) being fixed, we have for the Fourier component of ρ_n,[32]

$$\rho_n(\omega) = \left[\frac{\exp[-n|\omega|]}{f(n-1)} - \frac{\exp[-(n+2)|\omega|]}{f(n+1)}\right] \times \frac{[1+\exp(-i\theta\omega)]}{f(1)f(n)}, \qquad (23)$$

where

$$f(n) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}, \qquad (24)$$

$$z = \exp(-h/T),$$

Substituting the solution (23) into Eqs. (19) and (20) we obtain

$$\frac{E}{N} = -2h\tanh(h/T) + 32\left(\frac{1+2\theta^2}{1+4\theta^2}\right)[\tanh^2(h/T) - 1], \qquad (25)$$

$$m^z = 2\tanh(h/T). \qquad (26)$$

From Eqs. (25) and (26) one can obviously see that the magnetization of the system in this limit is the well-known one of the free two spin 1/2 spins (note that we have σ instead of s). As to the energy (25), we see that it is an even function of θ. That is why we can conclude that for the high temperature limit our system is in the antichiral state too. This means that the higher spin strings of opposite chiralities are confined in pairs, too.

For the low temperature limit it is of use to change the variables:[30]

$$\varepsilon_n(\lambda) = T\ln\left(\frac{\rho_n^h}{\rho_n}\right).$$

The function $\varepsilon_n(\lambda)$ is the dressed energy of the n string. The zero temperature limit of Eq. (22) yields

$$\varepsilon_1 = 2h - 16\{4^{-1}(\lambda^2+1)^{-1} + [4(\lambda+\theta)^2+1]^{-1}\} - [2]\varepsilon_1^-, \qquad (27)$$

$$\varepsilon_n = 2h + [1]\varepsilon_{n-1}^+, \quad n = 2,3,\ldots, \qquad (28)$$

where $\varepsilon^- = \varepsilon$ if $\varepsilon < 0$, $\varepsilon^- = 0$ if $\varepsilon > 0$, $\varepsilon^+ = 0$ if $\varepsilon < 0$, $\varepsilon^+ = \varepsilon$ if $\varepsilon > 0$. From Eqs. (27) and (28) one can see that the ground state is formed by the Dirac sea filled by the excitations of $n = 1$ only. At low temperatures one has for the free energy for, e.g., $h < 16\frac{1+2\theta^2}{1+4\theta^2}$

$$F(T,h) = E_0 - N\frac{\pi T^2 V(Q_1)}{24W(Q_1)} + \cdots, \qquad (29)$$

where

$$V(x) + (2\pi)^{-1} \int_{-Q_1}^{Q_1} \frac{2V(y)}{(x-y)^2+1} dy = \pi^{-1}\{(4x^2+1)^{-1} + [4(x+\theta)^2+1]^{-1}\},$$

$$W(x) + (2\pi)^{-1} \int_{-Q_1}^{Q_1} \frac{2W(y)}{(x-y)^2+1} dy = 8\pi^{-1}\{x(4x^2+1)^{-2} + (x+\theta)[4(x+\theta)^2+1]^{-2}\}.$$

At low temperatures we have (see Ref. 33)

$$F(T,h) = E_0 - N(T^2/24) + \cdots \tag{30}$$

and the low temperature part of the specific heat is equal to $T/12$. Note that for the critical magnetic field region the low temperature corrections begin with the term proportional to $T^{3/2}$. We can see from Eq. (30) that for low temperatures the θ dependence of the energy is the same as for the ground state (for other values of the magnetic field the same is true). Earlier we have seen that for high temperature limit the system is in the antichiral state too. That is why we can say that our system is antichiral for any range of temperature and external magnetic field. This result is correct because the temperature does not change the winding numbers (the topological properties) of the system, so the system remains antichiral for nonzero temperatures as for the ground state. To change the antichirality of the system we have to change the periodic boundary conditions into free ones. The results of that study will be reported elsewhere.

We have explained the features of the studied quantum spin model with the simplest example of two coupled chains. For the L spin 1/2 chains (L is even) with both inter- and intrachain interactions we have

$$\mathcal{T}(\lambda) = T(\lambda-\theta_1)T(\lambda-\theta_2)\cdots T(\lambda-\theta_L), \tag{31}$$

$$\mathcal{H} = \sum_{r=1}^{L}\sum_{n}\Bigg[P_{\sigma_{r,n}\sigma_{r+1,n}} + \Bigg(\prod_{i,k}\theta_{i,k}\Bigg)P_{\sigma_{r,n}\sigma_{r,n+1}} + \sum_{p<q}\theta_{p,q}^{-1}\Bigg(\prod_{i,k}\theta_{i,k}\Bigg)[P_{\sigma_{p,n}\sigma_{q,n}},(P_{\sigma_{p,n}\sigma_{p,n+1}} + P_{\sigma_{q,n}\sigma_{q,n+1}})] + \cdots\Bigg] + \text{const}, \tag{32}$$

where $\theta_{i,k} = \theta_i - \theta_k$, $\theta_{L+k} = \theta_k$, and P is the permutation operator; in the first term we must replace $P_{\sigma_{L,n}\sigma_{1,n}}$ with $P_{\sigma_{L,n}\sigma_{1,n+1}}$, which means toroidal winding boundary conditions (see also Ref. 34); square brackets denote the commutator and we have omitted the higher order terms in permutation operators. The third term and the omitted terms in Eq. (31) have the same nature as the third term in Eq. (1) because they all do not change the classical equations of motion (they are the total derivatives), and they are the higher order topological lattice charges of the multichain spin quantum system in spin operators. The omitted terms are nonlocal, which is consistent with the nature of the (2+1) statistical Chern-Simons interaction.[4] The energy of the M spins down state is equal to

$$E = -h(LN-2M) + \sum_{r=1}^{L}\sum_{k=1}^{M}(\lambda_k+\theta_{r,1})^{-1}(\lambda_k+\theta_{r,1}+i)^{-1}, \tag{33}$$

where λ_k are the solutions to the set of equations

$$\prod_{r=2}^{L}\frac{\lambda_j^N(\lambda_j+\theta_{r,1})^N}{(\lambda_j+i)^N(\lambda_j+\theta_{r,1}+i)^N} = \prod_{k=1}^{M}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i}, \tag{34}$$

$$k \neq j, \quad j = 1,2,\ldots,M.$$

It follows from Eqs. (32)–(34) that in the limit $\theta_{i,k} \to 0$ we have the 1D spin 1/2 chain with LN sites, and if $\theta_{i,k} \to \infty$ we have L noninteracting spin 1/2 chains with N sites. We can solve Eqs. (34) in the limit $N \to \infty$, $M \to \infty$, $m = (M/N)$ being fixed. For the $h = 0$ case the ground-state energy is

$$E_0 = -N\sum_{r=2}^{L}\int_{-\infty}^{\infty}\frac{|1+\exp(i\omega\theta_{r,1})|^2}{1+\exp(|\omega|)}d\omega, \tag{35}$$

and the magnetization has zero value.

The doublet excitation (spinon) energy is equal to

$$E = E_0 + \pi\sum_{r=1}^{L}\text{sech}[\pi(\lambda_0+\theta_{r,1})].$$

All doublet excitations confine into singlets and triplets as for the two-chain case. Despite the chiral behavior of the single doublet (it causes nontrivial topology of the spin density and carries a nonzero chirality) the total topological properties of the system are unchanged: the system is in the antichiral state. A more rigorous statement is that the background (the Dirac sea) follows the excitations carrying nonzero chiralities (or nonzero topological charge) in such a way that the total spin system remains antichiral. The nonzero temperature properties are analogous to the former two-chain case. The set of equations describing the multichain frustrated spin system is

$$n\pi^{-1}\sum_{r=1}^{L}[4(\lambda+\theta_{r,1})^2+n^2]^{-1}$$

$$=\rho_n^h(\lambda)+\sum_{m=1}^{\infty}A_{n,m}\rho_m(\lambda), \quad (36)$$

$$n\left(2h-16\sum_{r=1}^{L}[4(\lambda+\theta_{r,1})^2+n^2]^{-1}\right)$$

$$=T\ln\left(1+\frac{\rho_n^h}{\rho_n}\right)-T\sum_{m=1}^{\infty}A_{n,m}\ln\left(1+\frac{\rho_m}{\rho_m^h}\right). \quad (37)$$

All the antichiral properties are conserved for the L spin chain case. The same conjecture as for two-chain case can be made: the exact excitation energy of the spinon is gapless; however, the excitations for the frustrated multichain quantum spin Hamiltonian without the topological terms have gaps. The same is valid, naturally, for higher spin strings.

The generalization of the results of this paper for the case of the multichain strongly correlated electron exactly solvable model with spin and charge chiralities (or T and P symmetry violation) will be reported later.

To conclude, we have studied the exact solutions of the two-chain and multichain frustrated spin 1/2 systems. We have shown that the ground state and the nonzero temperature states reveal antichiral properties for any values of the external magnetic field (except for the trivial "ferromagnetic" case in the ground state). The elementary excitations of the considered system are spinons (analogous to the classical instantons carrying nonzero spin chirality or nontrivial topology of the spin density distribution) but only pairs of such excitations with opposite spin chiral charges contribute to the free energy of the system. The same statement holds for the higher spin strings. The elementary excitations of the model are gapless, but for the multichain quantum spin 1/2 antiferromagnetic frustrated system without the topological terms in the Hamiltonian, breaking T and P symmetries separately, we can conjecture that the elementary excitations have gaps.

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