Critical temperature of two coupled Ising planes

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It is shown that the method proposed by Wosiek [Phys. Rev. B **49**, 15 023 (1994)] for the calculation of the order-disorder critical temperature, which turns out to be exact when the critical temperature is determined by symmetry properties like self-duality, is, when applied to a system of two coupled Ising planes, the lowest-order approximation in a hierarchy which appears to converge to the exact result.

In a recent paper, Wosiek[1] proposed a quite simple method for the determination of the critical temperature of order-disorder transitions in Ising-like models. The method is based on the study of a "characteristic function" $\rho(\beta)$ (as customary $\beta = 1/k_B T$, and k_B and T are Boltzmann's constant and absolute temperature, respectively) which, for a d-dimensional model is defined by

$$\rho(\beta) = \lim_{L\to\infty} \left(\frac{(\mathrm{Tr}\mathcal{T})^2}{\mathrm{Tr}\mathcal{T}^2} \right)^{L^{-(d-1)}}, \tag{1}$$

where $\mathcal{T}$ is the transfer matrix of the model under investigation (periodic boundary conditions are assumed in all directions). The critical temperature is located as the maximum of ρ,

$$\beta_c = \beta_{\max}. \tag{2}$$

Wosiek shows that Eq. (2) yields the exact critical temperature for several planar Ising and Potts models and then applies the method to a system of two coupled Ising planes, obtaining

$$\beta_{\max} = 0.2656\ldots, \tag{3}$$

leaving as an open question, to be answered by means of Monte Carlo simulations or high-temperature expansions, whether this value is the exact one or not.

Here we show, by a simple generalization of the Wosiek method, that the result in Eq. (3) is but the lowest-order approximation in a hierarchy which appears to converge to the exact result. We propose to consider, instead of the characteristic function $\rho(\beta)$, its straightforward generalization

$$\rho(\beta; n_1, n_2) = \lim_{L\to\infty} \left[\frac{(\mathrm{Tr}\mathcal{T}^{n_1})^{n_2}}{(\mathrm{Tr}\mathcal{T}^{n_2})^{n_1}} \right]^{L^{-(d-1)}}, \qquad n_1 < n_2, \tag{4}$$

which reduces to Eq. (1) for $n_1 = 1$, $n_2 = 2$. It is easy to check, for example on the square lattice Ising model, that the maximum of $\rho(\beta; n_1, n_2)$ corresponds to the exact critical temperature, independent of n_1 and n_2. Furthermore, considering the logarithmic derivative, one obtains, as a condition for the maximum,

$$u_{n_1}(\beta_c) = u_{n_2}(\beta_c), \tag{5}$$

where u_n is the internal energy of n coupled $(d-1)$-dimensional systems, which is the obvious generalization of Eq. (20) in Ref. 1, referred to by Wosiek as the condition obtainable by "applying our proposition to higher moments of the transfer matrix."[1]

Let us now turn our attention to the system of two coupled Ising planes studied by Wosiek. In this case it is readily seen that the maximum of our generalized characteristic function depends on n_1 and n_2. In Table I we report the results obtained for several pairs (n_1, n_2) and compare them with the corresponding results obtained by the Lipowski-Suzuki method,[2] another transfer matrix method, the exact properties of which have been thoroughly investigated in Refs. 3,4. Notice that the best value $\beta_c = 0.276\,06$ given by the Lipowski-Suzuki method corresponds to $\tanh\,\beta_c = 0.269\,25$, which is in excellent agreement with the result of the high-temperature expansion,[5] $\tanh\,\beta_c = 0.2692 \pm 11$, while the sequence of approximations obtained by the generalized Wosiek method appears to converge to the previous estimates.

In conclusion, the Wosiek method is not exact (it is only the lowest-order approximation in a given hierarchy) for the system of two coupled Ising planes, and hence the same probably holds for any other system (including, e.g., the three-dimensional Ising model) in which the critical temperature is not determined by some symmetry property like self-duality.

TABLE I. $T_c(n_1, n_2)$ for the system of two coupled Ising planes, as given by the generalized Wosiek (GW) and Lipowski-Suzuki (LS) methods.

(n_1, n_2)	GW	LS
(1,2)	0.26561	0.27635
(2,3)	0.27040	0.27612
(3,4)	0.27452	0.27607
(4,5)	0.27562	0.27606

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