# Strong-coupling limit of Eliashberg theory

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We study the strong-coupling limit of the Eliashberg theory of superconductivity, where the coupling strength $\lambda$ goes to infinity and the critical temperature gets large compared to a typical phonon energy. This limit is of interest because it is both universal and simple, and we may hope to obtain from this study a deeper understanding of the conventional strong-coupling regime of superconductivity. Our work on this problem is both analytical and numerical. At $T=0$, we find that the excitation spectrum is discrete. We interpret physically the excited states as bound states due to a type of polaronic effect. We show that one can solve the Eliashberg equations essentially analytically by working fully on the real frequency axis. At finite temperature we find a thermal smearing of the $T=0$ structure. Since the critical temperature is small compared to the zero-temperature gap, thermal effects can be treated as a kind of perturbation over almost all the temperature range. In this spirit, we give a simple approximate solution which reproduces almost quantitatively the exact numerical results.

## I. INTRODUCTION

The Eliashberg theory of superconductivity[1–3] is of basic theoretical and practical importance. On the theoretical side, it stands for one of the few cases where we have a well-controlled solution for a nontrivial many body problem corresponding to a fairly realistic description of a physical system. On the experimental side it serves as a very accurate theory of traditional phonon-mediated superconductivity which describes very well all experimental data; conversely it can be used to extract microscopic information from experiment.[4] Furthermore Eliashberg theory is one of the competing theories for the description of high-$T_c$ superconductors, since it is the basic framework of all theories where pairing is due to the exchange of low-energy bosons. It is therefore worthwhile to have a physical understanding of this theory which is as good as possible.

In order to gain a deeper understanding of a complex theory one naturally has to study simple limiting cases. The best understood limit of the Eliashberg theory is the weak-coupling limit, where the coupling strength $\lambda$ goes to zero and the critical temperature is small compared to a typical phonon energy. In this limit the Eliashberg theory turns into the BCS theory, which is perfectly well understood. Here we are interested in the opposite limit, namely the strong-coupling limit, where the coupling strength $\lambda$ goes to infinity and the critical temperature gets large compared to a typical phonon energy. This strong-coupling limit has received rather little attention[5,6] in early applications of the Eliashberg theory mainly because the low-$T_c$ superconductors, the only ones known until recently, correspond to the weak-coupling limit of the theory. Strong-coupling effects were considered at best as mere corrections to this limit. The situation has changed somewhat with the discovery of high-$T_c$ superconductors which stimulated investigations of a variety of unusual mechanisms of superconductivity, including superconductivity in systems with very strong electron-phonon interaction.[7,8] But naturally, as it is well known, the situation for the high-$T_c$ superconductors is still unclear. Actually in real situations we would expect $\lambda$ to be at most of order a few times unity, so the strong-coupling limit seems to us practically irrelevant. Moreover it is a common expectation that structural instabilities, or more generally a crossover to a different physical behavior, would occur before we reach such a limit, although these expectations need to be substantiated.

Nevertheless, though the strong-coupling limit does not apply to any real superconducting material, it is of interest because it is both universal and simple, as we will see. This is our excuse for performing such a study which could otherwise look pretty much as an academic work. A good theoretical understanding of strong-coupling theory goes necessarily through a perfect knowledge of what happens when we take formally the strong-coupling limit. In addition we may hope that an expansion around this strong-coupling limit already gives good semiquantitative results, and thereby allows a physical understanding of the strong-coupling theory in the domain within experimental reach. Indeed the expansion parameter turns out to be $\lambda^{-1/2}$. A reasonable value for the coupling strength of a real strong-coupling superconductor could be $\lambda \sim 3$, which means $\lambda^{-1/2} \sim 0.6$. There are many examples in physics where a first-order expansion is still good for a small parameter of order 0.6. Therefore an expansion around the strong-coupling limit is not unreasonable. Indeed we will see at the end of the paper an example where in the strong-coupling limit the behavior is very similar to the one known in the strong-coupling regime, which leads us to believe that it is universal and does not depend on the specific spectrum which has been studied.

The strong-coupling limit has been studied recently by Marsiglio and Carbotte[7] (references to earlier studies can be found in their paper). They found in their numerical

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studies a very structured energy dependence of the gap function and of the density of states, which was unexpected. Specifically at $T=0$ they found very strong spikes as a function of energy, and not at all the usual "smooth structures." Near $T_c$ there is still some structure left at roughly the same frequencies, although it is very much attenuated. The authors interpret these features in the density of states as excitations involving broken pairs. The strong-coupling limit has also been discussed recently by Karakazov, Maksimov, and Mikhailovsky[8] who presented an approximate analytical study of the $T=0$ case, which leads to anomalous δ-function spikes in the gap function and in the density of states, in qualitative agreement[9] with Ref. 7.

The purpose of this paper is to present a detailed analysis of the strong-coupling limit. We study this problem both analytically and numerically. In some sense our paper makes a bridge between the findings of Marsiglio and Carbotte, and those of Karakazov, Maksimov, and Mikhailovsky. As we will see we obtain a complete analytical understanding of the problem. This analysis should lead to a deeper understanding of the anomalous excitation spectrum of a superconductor in the strong-coupling limit. At $T=0$, we find that the excitation spectrum is discrete. These excited states correspond to simple poles for the Green's function located on the real axis. We interpret physically these excited states as bound states due to a self-trapping of an excited quasiparticle in the off-diagonal pairing field. In other words, the discreteness of the spectrum is the consequence of a type of polaronic effect. We will show that, because of the existence of these simple poles, one can solve the problem essentially analytically by working fully on the real frequency axis. This analytic solution is of interest since the real frequency Eliashberg equations are nonlinear singular integral equations which makes them impossible to solve analytically in the general case (one has to obtain the solution numerically). At finite temperature we find a thermal smearing of this structure, which can be physically interpreted as the consequence of the decay of the $T=0$ excited states, due to thermally excited phonons. But because the critical temperature is small, thermal effects can be treated as a kind of perturbation over almost all the temperature range, which explains the observation of Marsiglio and Carbotte that the structures survive up to $T_c$. We make use of this remark and find a simple approximate solution which reproduces almost quantitatively the exact numerical results.

The paper is organized as follows. First we introduce in Sec. II the strong-coupling limit of Eliashberg equations. We review the method of Marsiglio, Schossmann, and Carbotte[10] for the analytic continuation of the imaginary axis Eliashberg equations toward the real axis. Their equations lead in the strong-coupling limit to an ordinary nonlinear differential equation, which serves as the mathematically simplest and most transparent formulation of the problem. This equation is the basis of our analysis. In Sec. III we consider the $T=0$ case where the differential equation takes a rather simple form, which leads straightforwardly to the rigorous conclusion that the excitation spectrum is discrete. We discuss the physical interpretation of this result. We then show that the problem can be solved entirely on the real frequency axis, without any input from the imaginary axis solution. Finally in Sec. IV we analyze the more complicated situation found at finite temperature.

## II. THE DIFFERENTIAL EQUATION

A first point to make when we look at the strong-coupling limit is that the shape of the phonon spectrum or more precisely of the Eliashberg function $\alpha^2F(\omega)$ becomes unimportant. This makes the strong-coupling limit universal: there is no free parameter left. As we stressed in the Introduction, this universality is a very interesting feature of this limit. This result is easily seen from the imaginary axis Eliashberg equations, namely[2]

$$\omega_n(Z_n-1)=\lambda\pi T\left\langle\sum_m \frac{\Omega^2}{(\omega_n-\omega_m)^2+\Omega^2}\frac{\omega_m}{(\omega_m^2+\Delta_m^2)^{1/2}}\right\rangle , \tag{1}$$

$$\Delta_n Z_n=\lambda\pi T\left\langle\sum_m \frac{\Omega^2}{(\omega_n-\omega_m)^2+\Omega^2)}\frac{\Delta_m}{(\omega_m^2+\Delta_m^2)^{1/2}}\right\rangle , \tag{2}$$

where $\omega_n=(2n+1)\pi T$ is the Matsubara frequency, $\Delta_n$ and $Z_n$ the respective values of the gap function and of the renormalization function at this frequency and $\Omega$ a phonon frequency. In these equations the bracket $\langle\cdots\rangle$ is an abbreviation for the following average over phonon frequencies $(2/\lambda)\int d\Omega\cdots\alpha^2F(\Omega)/\Omega$ where $\alpha^2F(\Omega)$ is the Eliashberg function. Naturally we have not taken into account any Coulomb repulsion or any impurity effect since in the strong-coupling limit they become irrelevant. When we let $\lambda$ go to infinity, all the relevant energy scales (like $T_c$ or the gap) will also go to infinity. It is therefore convenient to rescale the energies. Or equivalently we can let all the phonon frequencies $\Omega$ go to zero at the same time as we let $\lambda$ go to infinity, in order to keep these energy scales finite. When we eliminate $Z_n$ between Eqs. (1) and (2) (the term $m=n$ drops out) and take the above limit, we obtain the following equation:

$$\Delta_n\left[\omega_n+\lambda\langle\Omega^2\rangle\pi T\sum_{m\neq n}\frac{1}{(\omega_n-\omega_m)^2}\frac{\omega_m}{(\omega_m^2+\Delta_m^2)^{1/2}}\right]$$
$$=\lambda\langle\Omega^2\rangle\omega_n\pi T\sum_{m\neq n}\frac{1}{(\omega_n-\omega_m)^2}\frac{\Delta_m}{(\omega_m^2+\Delta_m^2)^{1/2}} \tag{3}$$

which depends only on the single energy scale $\lambda^{1/2}\langle\Omega^2\rangle^{1/2}$. For convenience we set this energy scale equal to unity. Therefore the strong-coupling limit is reduced to a universal parameterless problem. Hence we may just as well consider that we started with an Einstein spectrum with frequency $\Omega$ satisfying $\lambda\Omega^2=1$. This is what we will do in the rest of the paper. We note that the critical temperature is $T_c=0.1827$ with this convention, since it is given[5,6] by $T_c=0.1827\lambda^{1/2}\langle\Omega^2\rangle^{1/2}$ when we do not take our energy scale equal to unity. Similarly we find immediately that the gap is proportional to $\lambda^{1/2}\langle\Omega^2\rangle^{1/2}$ (provided that we find, as we will do, a result which is not zero or infinity with our reduced unit). Therefore we have the $\lambda$ dependence of the gap which is in agreement with Ref. 7.

Equation (3) can be solved for $\Delta_n$ numerically by iteration. The result is given on Fig. 6 of Marsiglio and Carbotte[7] [note that they take $(\lambda/2)^{1/2}\Omega$ as unity so all their energy scales should be reduced by a factor $\sqrt{2}$ before comparison with our results]. However we are interested in real frequency axis properties. A convenient starting point is the analytical continuation of Eq. (3) toward the real axis as found by Marsiglio, Schossmann, and Carbotte.[10] Their result can also be obtained from the real axis Eliashberg equations as pointed out by Karakazov, Maksimov, and Mikhailovsky.[8] Let us briefly rederive these equations for completeness.

In Eqs. (1) and (2) we are faced, in the right-hand side, with functions of $z=i\omega_n$ which we want to continue analytically. The general form for these functions is $\sum F(i\omega_m)[z-i\omega_m\pm\Omega]^{-1}$ with $F(z)=\hat{N}(z)\equiv z[\Delta^2(z)-z^2]^{-1/2}$ or $F(z)=\hat{D}(z)\equiv\Delta(z)[\Delta^2(z)-z^2]^{-1/2}$. The analytical continuation is not obtained by replacing merely $i\omega_n$ by $z$, because the result has simple poles for $z=i\omega_m\pm\Omega$ whereas it should be analytical for these values of $z$. This problem can be cured by subtracting a function which cancels on one hand all these singularities, and is zero on the other hand for all $z=i\omega_m$. The unique answer is that one should subtract $(1/2T)F(z\pm\Omega)\{(\tanh[(z\pm\Omega)/2T]-\coth(\pm\Omega/2T)\}$. In this way one obtains[10] that Eqs. (1) and (2) are continued analytically into

$$\omega[Z(\omega)-1]=\lambda\pi T\sum_m\frac{\Omega^2}{\Omega^2-(\omega-i\omega_m)^2}\frac{i\omega_m}{(\omega_m^2+\Delta_m^2)^{1/2}}+\frac{\lambda\pi\Omega}{2}\{[n(\Omega)+f(\Omega-\omega)]\hat{N}(\omega-\Omega)+[n(\Omega)+f(\Omega+\omega)]\hat{N}(\omega+\Omega)\}\ , \tag{4}$$

$$\Delta(\omega)Z(\omega)=\lambda\pi T\sum_m\frac{\Omega^2}{\Omega^2-(\omega-i\omega_m)^2}\frac{\Delta_m}{(\omega_m^2+\Delta_m^2)^{1/2}}+\frac{\lambda\pi\Omega}{2}\{[n(\Omega)+f(\Omega-\omega)]\hat{D}(\omega-\Omega)+[n(\Omega)+f(\Omega+\omega)]\hat{D}(\omega+\Omega)\}\ , \tag{5}$$

where $f(E)$ and $n(E)$ are the Fermi and Bose distribution functions, respectively. The interesting feature of these equations is that the terms with the Mastubara summations are actually perfectly regular when $\omega$ is on the real frequency axis. All the singular behavior is contained in the other terms. When these Matsubara summations are transformed in the standard way into contour integrals, and when these contours are deformed to the real axis, one obtains

$$\sum_m\frac{1}{\omega-i\omega_m\pm\Omega}F(i\omega_m)=\frac{1}{2\pi T}\int_{-\infty}^{\infty}d\omega'\tanh\frac{\omega'}{2T}\frac{\mathrm{Im}F(\omega')}{\omega-\omega'\pm\Omega}-\frac{1}{2T}\tanh\frac{\omega\pm\Omega}{2T}F(\omega\pm\Omega)\ . \tag{6}$$

When this result is carried into Eqs. (4) and (5), one obtains the equations used by Karakazov, Maksimov, and Mikhailovsky. As they showed, these equations reduce to the real axis Eliashberg equations when one makes use of the dispersion relation $\pi F(\omega)=\int d\omega'\mathrm{Im}F(\omega')/(\omega'-\omega)$.

Let us now come back to the strong-coupling limit. When we eliminate $Z(\omega)$ from Eqs. (4) and (5), we obtain for $D(\omega)\equiv\Delta(\omega)/\omega$:

$$\omega D(\omega)B(\omega,T,\Omega)-A(\omega,T,\Omega)=\frac{\pi}{2\Omega}\left[\frac{D(\omega-\Omega)-D(\omega)}{\sqrt{D^2(\omega-\Omega)-1}}[n(\Omega)+f(\Omega-\omega)]-\{\Omega\to-\Omega\}\right]\ . \tag{7}$$

In this equation we have made use of $\lambda\Omega^2=1$. We can take now the limit of $\Omega$ going to zero and we find the following second-order ordinary differential equation[11] for $D(\omega)$:

$$\frac{\pi}{2}\frac{T[D''-2D(D')^2/(D^2-1)]-D'\tanh(\omega/2T)}{(D^2-1)^{1/2}}=\omega B(\omega,T)D-A(\omega,T)\ , \tag{8}$$

where $A(\omega,T)=A(\omega,T,\Omega=0)$ and $B(\omega,T)=B(\omega,T,\Omega=0)$. Explicitly we have

$$A(\omega,T)=2\pi T\sum_{n=0}^{\infty}\frac{\Delta_n(\omega_n^2-\omega^2)}{(\omega_n^2+\Delta_n^2)^{1/2}(\omega_n^2+\omega^2)^2}\ , \tag{9}$$

$$B(\omega,T)=1+4\pi T\sum_{n=0}^{\infty}\frac{\omega_n^2}{(\omega_n^2+\Delta_n^2)^{1/2}(\omega_n^2+\omega^2)^2}\ . \tag{10}$$

Here $D'$ and $D''$ are the first and second derivative of $D(\omega)$ with respect to $\omega$.

This equation gets much simpler in the $T=0$ limit where we obtain merely a first-order differential equation:

$$\frac{\pi}{2}\frac{D'}{(D^2-1)^{1/2}}=A(\omega)-\omega B(\omega)D\ , \tag{11}$$

where we have set $A(\omega)=A(\omega,T=0)$ and $B(\omega)=B(\omega,T=0)$. It is interesting to note that the strong-coupling limit $\lambda\to\infty$, $\Omega\to0$ and the zero temperature limit $T\to0$ commute, which is not completely obvious at first. Indeed when we take the $T\to0$ limit first, we have no thermal phonon at all excited, while when we take the limit $\Omega \to 0$ first we will have an infinite number of thermal phonons. However their effect is nonsingular and still proportional to the temperature (that is to their number). Therefore when we let $T \to 0$ their effect disappears.

## III. THE ZERO TEMPERATURE CASE

The zero temperature equation Eq. (11) is even further simplified if we introduce the function $\varphi(\omega)$ defined by $D(\omega)=1/\sin[\varphi(\omega)]$. This leads to

$$\varphi' = \frac{2}{\pi}[\omega B(\omega) - A(\omega)\sin(\varphi)] \ . \quad (12)$$

On this very simple form all the properties of the solution can be read off immediately, though one needs a little numerical calculation in order to obtain the precise solution. Indeed we will see that $A(\omega)$ is negative, while $B(\omega)$ is positive. Moreover (see below) we have $\omega B(\omega) > \omega + |A(\omega)|$. From its definition we have, in the vicinity of $\omega=0$, $\varphi(\omega) \approx \omega/\Delta(0)$ and the boundary condition for $\varphi(\omega)$ is $\varphi(0)=0$. Then the differential equation implies immediately that $\varphi(\omega)$ increases regularly with $\omega$, with more precisely $\varphi(\omega) > \omega^2/\pi$. In particular $\varphi(\omega)$ is always real (in contrast to what we will find at $T \neq 0$). Therefore we obtain the surprising result that $\Delta(\omega)$ has an infinite set of poles for $\omega = x_n$ with $\varphi(x_n) = n\pi$ and $n = 1,2,\ldots$ and is otherwise real on the whole real frequency axis. Similarly $\omega/(\Delta^2(\omega)-\omega^2)^{1/2} = \tan(\varphi(\omega))$ is always real and the density of states $N(\omega)=\text{Im}\{\omega/[\Delta^2(\omega)-\omega^2]^{1/2}\}$ is zero, except for a set of delta functions corresponding to the poles of $\tan[\varphi(\omega)]$, located at $\omega = y_n$ with $\varphi(y_n)=(n-1/2)\pi$ and $n=1,2,\ldots$. Hence we come to the conclusion that the spikes found numerically at $T=0$ by Marsiglio and Carbotte for $N(\omega)$ as well as for $\text{Im}[\Delta(\omega)]$ are actually delta functions. The weight of these delta functions for $N(\omega)$ are easily obtained from the differential equation and we have explicitly

$$N(\omega) = \pi \sum_{n=1}^{\infty} P_n[\delta(\omega - y_n) + \delta(\omega + y_n)] \quad (13)$$

with

$$1/P_n = \varphi'(y_n) = (2/\pi)[y_n B(y_n) + (-1)^n A(y_n)] \ . \quad (14)$$

Similarly we find

$$\text{Im}[\Delta(\omega)] = \pi \sum_{n=1}^{\infty} Q_n[\delta(\omega - x_n) - \delta(\omega + x_n)] \quad (15)$$

with $1/Q_n = (-1)^{n+1}\varphi'(x_n)/x_n = (-1)^{n+1}(2/\pi)B(x_n)$.

In order to find a physical interpretation for this remarkable result, we note first that we can consider the diagonal part of the self-energy $\Sigma(\omega)=\omega[1-Z(\omega)]$ as the effective potential energy felt by an electron added to the system, due to the electron-phonon interaction. It is then convenient, since we are only looking for a qualitative interpretation, to omit the structureless terms (with Matsubara sums) in Eqs. (3) and (4) and to write the Eliashberg equations on the real axis (for $\omega > 0$) at $T=0$ as

$$-\Sigma(\omega) = \omega[Z(\omega)-1] = \frac{\pi}{2\Omega}\hat{N}(\omega - \Omega) \ , \quad (16)$$

$$\Delta(\omega)Z(\omega) = \frac{\pi}{2\Omega}\hat{D}(\omega - \Omega) \ . \quad (17)$$

We recall that $\hat{N}(\omega)=\omega/[\Delta^2(\omega)-\omega^2]^{1/2}$ and $\hat{D}(\omega)=\Delta(\omega)/[\Delta^2(\omega)-\omega^2]^{1/2}$. We note that in the normal state, we have merely $\hat{N}(\omega)=i$ and the imaginary part of $\Sigma(\omega)$ reduces to the standard result $-\pi/2\Omega$, corresponding physically to the very short lifetime of the excitation due to phonon emission.

Now in the superconducting state the effective potential $\Sigma(\omega)$ on the excitation depends on the off-diagonal field $\Delta(\omega)$. But from Eq. (17) the field $\Delta(\omega)$ depends itself on $\Sigma(\omega)$. Therefore we have a situation where an excitation interacts with a field it is itself creating (more precisely it is able to modify the value of this field). This leads to the possibility of self-trapping of this excitation in the off-diagonal pairing field when the interaction gets strong enough. This situation is quite similar to the polaronic effect where an electron gets trapped in the phonon field it has itself created. This can perhaps be seen more clearly if we take the Fourier transform of Eq. (16). This gives for the time dependence of the retarded effective potential

$$\Sigma(t) = -\frac{\pi}{2\Omega}e^{-i\Omega t}\hat{N}(t) \approx -\frac{\pi}{2\Omega}\hat{N}(t)(1-i\Omega t) \ , \quad (18)$$

where the last equality takes into account that we let the phonon frequency $\Omega$ go to zero. We see that in this limit the effective potential grows indefinitely when the retardation time increases. It is therefore reasonable to find self-trapping in such a potential. We note that this argument does not work in the normal state because $\hat{N}(\omega)$ is a constant which makes $\hat{N}(t)$ proportional to a delta function in time, that is instantaneous. But $\hat{N}(\omega)$ is no longer constant in the superconducting state which makes the strong retarded potential possible.

Naturally the trapping in this very strong pairing field leads to bound states and therefore it is not astonishing to find a discrete spectrum. We note that the energy of these states do not show any wave-vector dependence, which means that the corresponding effective mass of these states is infinite. This is again not so surprising since a polaron has also an infinite effective mass in the limit of infinitely strong coupling. The spatial extension of these bound states can be obtained from the wave-vector dependence of the Green's function. One obtains that they are localized on a length which is very short, of order of the coherence length divided by $\lambda^{1/2}$. This is again what is expected from very tightly bound states.

We can obtain explicitly from Eqs. (16) and (17) how the pairing field $\Delta(\omega)$ depends on $\Sigma(\omega)$. We find

$$\Delta^2(\omega) = \frac{1}{Z^2(\omega)}\left[\left(\frac{\pi}{2\Omega}\right)^2 + \omega^2[Z(\omega)-1]^2\right] \ . \quad (19)$$

When this is carried into Eq. (16), we find the equation showing how $\Sigma(\omega)$ or equivalently $Z(\omega)$ feeds back on itself, leading to self-trapping:

$$(\omega+\Omega)[Z(\omega+\Omega)-1]=\omega Z(\omega)\left[1+\left[\frac{2\Omega}{\pi}\right]^2\omega^2 Z(\omega)\right], \tag{20}$$

where we have taken into account that $Z(\omega)$ gets very large in the limit $\Omega\to 0$. When we expand $Z(\omega+\Omega)\approx Z(\omega)+\Omega\,\partial Z/\partial\omega$, [this is just the equivalent of the small $\Omega$ expansion in Eq. (18)], we find for $z(\omega)=2\Omega\omega Z(\omega)/\pi$ the simple equation

$$\frac{dz}{d\omega}=\frac{2}{\pi}\omega[1+z^2(\omega)] \tag{21}$$

which shows again simply that the growth of $Z(\omega)$ is due to $Z(\omega)$ itself. When $Z(\omega)$ diverges, the effective potential on the quasiparticle gets infinite and we obtain the bound states. Equation (21) is easily solved as $z(\omega)=\tan(\omega^2/\pi)$. This leads for the energy of the bound states to the approximate asymptotic solution found below. This is naturally due to the approximate nature of Eqs. (16) and (17).

Marsiglio and Carbotte have interpreted their peaked structure in terms of tightly bound pairs. They suggest that the second pole $y_2$ corresponds to the energy necessary to add an electron (requiring an energy equal to the gap $\Delta_0$) which breaks a pair at the same time (requiring $2\Delta_0$), in agreement with their finding that the second structure in the density of states lies essentially at $3\Delta_0$. The higher poles would be only due to nonlinearity. This proposal does not agree well with our results: we find (see below) that $y_2=3.04\neq 3y_1=3.48$, therefore there is no simple relation between the first and the second poles. Moreover all the poles are completely equivalent in our findings and we have no reason to consider that $y_n$ for $n>2$ are physically different from $y_2$. We have also to satisfy the physical requirement that the size of our bound states stays larger than the Fermi wavelength. This implies that $\lambda\Omega \ll E_F$ or equivalently $k_F\xi \gg \lambda^{1/2}$ which makes the coherence length necessarily much larger than the Fermi wavelength.

We come back now to our $T=0$ result and study it more precisely. In order to be more specific we have to look in more details at $A(\omega)$ and $B(\omega)$. First from the $T\neq 0$ expressions we obtain

$$A(\omega)=\int_0^\infty dx\frac{\Delta(ix)(x^2-\omega^2)}{[x^2+\Delta^2(ix)]^{1/2}(x^2+\omega^2)^2}=\frac{d}{d\omega}\int_0^\infty dx\frac{\omega\Delta(ix)}{(x^2+\omega^2)[x^2+\Delta^2(ix)]^{1/2}}, \tag{22}$$

$$B(\omega)=1+2\int_0^\infty dx\frac{x^2}{[x^2+\Delta^2(ix)]^{1/2}(x^2+\omega^2)^2} \tag{23}$$

so $A(\omega)$ and $B(\omega)$ are easily obtained numerically from the imaginary axis solution. Clearly both $A(\omega)$ and $B(\omega)$ are very regular functions. Since it is the derivative of a decreasing function, $A(\omega)$ is negative and clearly $|A(\omega)|$ decreases with $\omega$. From the above expression Eq. (22) we have, for large $\omega$, $A(\omega)\approx -a/\omega^2$ (with $a=\int dx\,\Delta(ix)[x^2+\Delta^2(ix)]^{-1/2}\approx 1.27$ from the imaginary axis solution) while $A(\omega)=\int(dx/x)(d/dx)\{\Delta(ix)[x^2+\Delta^2(ix)]^{-1/2}\}\approx -1.55$ for $\omega\to 0$. Formula Eq. (23) for $B(\omega)$ gives $B(\omega)\approx 1+1/\omega^2+O(\omega^{-4})$ for large $\omega$, while for $\omega\to 0$ one obtains $\omega B(\omega)\approx(\pi/2)1/\Delta(0)$.

Therefore for small $\omega$, the differential equation Eq. (12) reduces to $\varphi'=1/\Delta(0)$ which gives $\varphi(\omega)=\omega/\Delta(0)$ as it should from the definition of $\varphi(\omega)$. If we use this linear approximation to find $y_1$ which is also the gap $\Delta_0$, and $\Delta(0)\approx 0.75$ from the imaginary axis solution, we obtain $y_1=\Delta_0=(\pi/2)\Delta(0)\approx 1.18$. This is a very good approximation compared to the result $\Delta_0=1.16$ which we obtain numerically (in agreement with Ref. 7). Similarly this linear approximation gives $x_1=2.36$ for location of the first pole of $\Delta(\omega)$, in reasonable agreement with the result $x_1=2.20$ from the numerical integration of Eq. (12). We note that the existence of a fairly wide linear region for $\varphi(\omega)$ around $\omega=0$ could be expected. Indeed since $\varphi(-\omega)=-\varphi(\omega)$, there are no even terms and in particular no $\omega^2$ term in the expansion of $\varphi(\omega)$ in powers of $\omega$.

The behavior for large $\omega$ is also easily obtained since the $A(\omega)$ term becomes negligible in this limit. This leads to $\varphi(\omega)\approx(\omega^2+2\ln\omega)/\pi+C$. The dominant term in this limit, namely $\omega^2/\pi$, has already been obtained by Karakozov, Maksimov, and Mikhailovsky.[8] We find the constant $C$ by comparison with our numerical results and obtain $C\approx 1.05$. Very surprisingly, as it can be seen on Fig. 1, this asymptotic form for $\varphi(\omega)$ agrees with our numerical results [obtained by integrating Eq. (12) numerically] down to $\omega\approx 0.4$ with a 3.5% maximum relative error. In particular this expression for $\varphi(\omega)$ gives $\Delta_0=1.16$ for the gap, and $x_1=2.26$ for the first pole of $\Delta(\omega)$. Naturally we can also obtain from this asymptotic form the values of $x_n, y_n, P_n$, and $Q_n$ with very good precision. For completeness we have plotted $\Delta(\omega)$ in Fig. 2 obtained from the numerical calculation.

In our preceding solution of the $T=0$ case, we had to make use of the imaginary axis numerical solution in order to calculate $A(\omega)$ and $B(\omega)$. In the present case this solution is not very easy because the Eliashberg equations Eq. (3) become singular in this limit $T\to 0$ and $\lambda\to\infty$. However we can now make use of the specific analytical

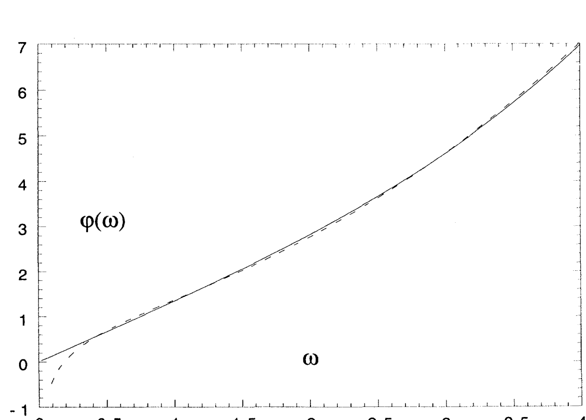


FIG. 1. Solid line: $\varphi(\omega)$ from numerical integration of Eq. (12); dashed line: $\varphi(\omega)=(\omega^2+2\ln\omega)/\pi+1.05$.

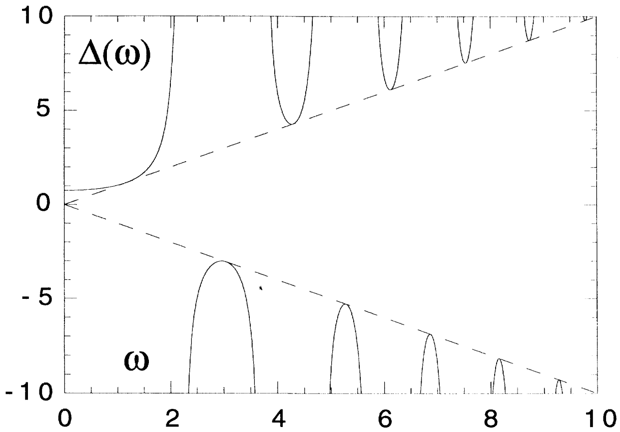


FIG. 2. $\Delta(\omega)$ as a function of $\omega$. The dashed lines are $\pm\omega$. Note that $|\Delta(\omega)| \geq \omega$.

properties of $\Delta(\omega)$ that we have found to bypass the imaginary axis solution and solve directly and easily the whole problem on the real axis, which is anyway of interest in itself. Indeed we can use the real axis expressions of $A(\omega)$ and $B(\omega)$. These can be obtained directly by a small $\Omega$ expansion of the real axis Eliashberg equations. They can also be obtained from Eqs. (22) and (23): we extend the integration in Eq. (22) over all the imaginary axis by parity, then separate the result into two integrals with each one having only a single pole at $ix = \pm\omega$ (the two contributions are actually equal by $x \to -x$). Then the integration contour is folded on the part of the real axis which does not contain the pole. Exactly the same procedure is used for Eq. (23). We find

$$A(\omega) = -\int_0^\infty dx \, \mathrm{Re} \frac{\Delta(x)}{[x^2-\Delta^2(x)]^{1/2}(x+\omega)^2}, \tag{24}$$

$$\omega B(\omega) = \omega + \int_0^\infty dx \, \mathrm{Re} \frac{x}{[x^2-\Delta^2(x)]^{1/2}(x+\omega)^2}. \tag{25}$$

When we substitute in these results the expression given above Eq. (13) for $N(\omega) = \mathrm{Re}\{\omega/[\omega^2-\Delta^2(\omega)]^{1/2}\}$ in terms of the $y_n$, taking into account that $\Delta(y_n) = (-1)^{n-1} y_n$, we obtain

$$A(\omega) = \pi \sum_{n=1}^{\infty} (-1)^n \frac{P_n}{(\omega+y_n)^2}, \tag{26}$$

$$\omega B(\omega) = \omega + \pi \sum_{n=1}^{\infty} \frac{P_n}{(\omega+y_n)^2}. \tag{27}$$

Since $y_n$ increases with $n$ and $P_n$ decreases with $n$, it is clear from these expressions for $A(\omega)$ and $B(\omega)$ that $A(\omega)$ is negative and that $\omega B(\omega) > \omega + |A(\omega)|$ as indicated above.

We can now solve Eq. (12) iteratively. We start from an initial guess $\varphi^{(0)}(\omega)$ for $\varphi(\omega)$ and obtain the corresponding set of poles $\{y_n^{(0)}\}$ by $\varphi^{(0)}(y_n^{(0)}) = (n-1/2)\pi$, with the corresponding weights $\{P_n^{(0)}\}$. We can then obtain $A^{(0)}(\omega)$ and $B^{(0)}(\omega)$ through Eqs. (26) and (27). Then we find the next order approximation $\varphi^{(1)}(\omega)$ for $\varphi(\omega)$ through Eq. (12) be merely integrating the right-hand side. Generally we find the approximation $\varphi^{(p+1)}(\omega)$ from the approximate $\varphi^{(p)}(\omega)$ by

$$\varphi^{(p+1)}(\omega) = \frac{2}{\pi}\int_0^\omega d\omega \{\omega B^{(p)}(\omega) - A^{(p)}(\omega)\sin[\varphi^{(p)}(\omega)]\} \tag{28}$$

then $\{y_n^{(p+1)}\}$ is given by $\varphi^{(p+1)}(y_n^{(p+1)}) = (n-1/2)\pi$. The set $\{P_n^{(p+1)}\}$ is obtained from Eq. (14) as

$$1/P_n^{(p+1)} = (2/\pi)[y_n^{(p+1)} B^{(p)}(y_n^{(p+1)}) + (-1)^n A^{(p)}(y_n^{(p+1)})]. \tag{29}$$

Then $A^{(p+1)}(\omega)$ and $B^{(p+1)}(\omega)$ are given from Eqs. (26) and (27) from $\{y_n^{(p+1)}\}$ and $\{P_n^{(p+1)}\}$, and the whole process is carried on to next order.

There is a small practical problem in carrying out this program. Naturally we want to cut off the summation in Eqs. (26) and (27) at some finite but large integer $N$. Because the alternating series Eq. (26) is rapidly converging, $A(\omega)$ is obtained quite precisely despite of the cutoff. However this makes more problems for Eq. (27). Fortunately we can sum up the series from $n=N$ up to infinity by making use of the Euler-MacLaurin formula, since we have for large $n$ the asymptotic expressions $y_n \approx \pi(n-1/2)^{1/2}$ and $P_n = \pi/(2y_n)$. This gives

$$\omega B(\omega) = \omega + \frac{1}{\omega+y_N} + \pi \sum_{n=1}^{N} \frac{P_n}{(\omega+y_n)^2} - \frac{\pi}{2}\frac{P_N}{(\omega+y_N)^2}. \tag{30}$$

We would have left out the term $1/(\omega+y_N)$ if we had stopped the summation at $n=N$. This term is important since it gives the $(2/\pi)\ln\omega$ term in $\varphi(\omega)$ for large $\omega$.

We have carried out the above iterative solution of the real axis problem, taking the dominant term in the asymptotic form $\varphi^{(0)}(\omega) = \omega^2/\pi$ as a starting guess, with correspondingly $y_n^{(0)} = \pi(n-1/2)^{1/2}$ and $P_n^{(0)} = \pi/(2y_n)$. The convergence is not very fast: 30 iterations give a good precision and 80 iterations an almost complete convergence [in particular starting from a better $\varphi^{(0)}(\omega)$ does not improve markedly the convergence rate]. We have taken $N$ such that $y_N \approx 50$ (i.e., $N=254$) and obtain in particular $\Delta_0 = y_1 = 1.161$, $P_1 = 0.709$, $y_2 = 3.043$, $P_2 = 0.482$. However it is already possible to obtain quite a very good solution by taking $N=3$ (this gives $\Delta_0 = 1.142$). Actually $N=2$ is reasonably accurate (it gives $\Delta_0 = 1.21$). It can be checked in particular that Eq. (26) (cutoff at $N=2$) and Eq. (30), with $y_{1,2}$ and $P_{1,2}$ given above, produce results for $A(\omega)$ and $B(\omega)$ almost undistinguishable graphically from the actual values. Even $N=1$ gives reasonable results. $A_1(\omega) = -\pi P_1/(\omega+y_1)^2$ is typically 10% off (where it is not small). And $B_1(\omega) = \omega + 1/(\omega+y_1) + 0.5\pi P_1/(\omega+y_1)^2$ is more than 10% off only for $x < 0.5$. We can see that the $N=1$ approximation is reasonably good by making a simple estimate of $y_1$ from $A_1(\omega)$ and $B_1(\omega)$, and making use of the quasilinearity of $\varphi(\omega)$ up to $y_1$. This linearity gives $\varphi'(y_1) = \pi/2y_1$ from $\varphi(y_1) = \pi/2$, and $P_1 = 1/\varphi'(y_1)$

$=2y_1/\pi$ (in very good agreement with numerics). Substituting this information into Eq. (12) together with $A_1(y_1)=-1/2y_1$ and $B_1(y_1)=y_1+3/4y_1$ gives $y_1=(\pi^2-5)^{1/2}/2\approx 1.10$ in satisfactory agreement with the numerical result.

In conclusion of this section we have obtained for the strong-coupling limit a full understanding of the solution of the real axis Eliashberg equations, which is quite interesting since these nonlinear singular integral equations are in the general case uneasy to master. Our results are actually rather close to a full analytical solution. As a final remark let us note that in our case $\Delta(\omega)$ is singular in the complex plane in the limit $|\omega|\to\infty$. Indeed by continuing analytically the imaginary axis solution we have $\Delta(\omega)\approx -1.27/\omega^2$ for $\omega\to\infty$ in the upper half complex plane. On the other hand, we have on the real axis an infinite set of poles with an accumulation point at infinity, and as we have seen, the asymptotic behavior of $\Delta(\omega)$ is quite different. In contrast the behavior of $\Delta(\omega)$ is regular at infinity in the generic situation, as well as for $T\neq 0$ in the limit $\lambda\to\infty$.

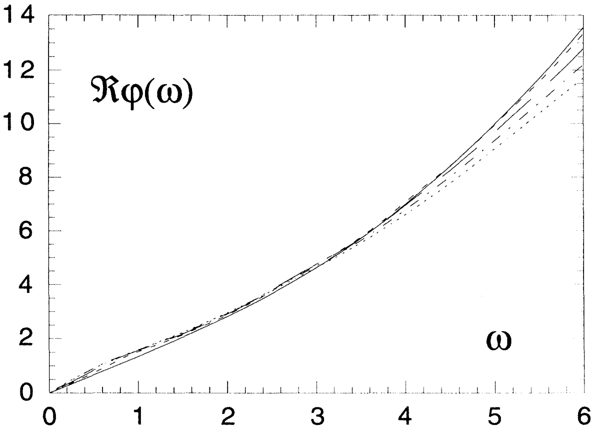


FIG. 3. Re$[\varphi(\omega)]$ from numerical integration of Eq. (31). Solid line: $T=0$, dashed line: $T/T_c=0.3$, long dashed line: $T/T_c=0.5$, dashed-dotted line: $T/T_c=0.7$, short dashed line: $T/T_c=0.9$.

## IV. THE $T\neq 0$ CASE

We come now to the nonzero temperature case. We will see that, compared to $T=0$ situation, the effect of the temperature can be described as a kind of perturbation. This will give us a complete understanding of the solution, both qualitatively and quantitatively. The basic reason for this simple result is that the ratio $\Delta_0/T_c=6.30$ is large in the strong-coupling limit. Therefore the temperature $T$ is always small compared to the characteristic energy of the $T=0$ system, namely $\Delta_0$, which allows us to consider the effect of $T$ as a perturbation.

As for $T=0$ we introduce $\varphi(\omega)$ defined by $D(\omega)=\Delta(\omega)/\omega=1/\sin[\varphi(\omega)]$. However $\varphi(\omega)$ is now a complex function of $\omega$ in contrast with the $T=0$ case. From Eq. (8) we obtain for $\varphi(\omega)$ the equation

$$-T[\varphi''+(\varphi')^2\tan\varphi]+\varphi'\tanh(\omega/2T)=\frac{2}{\pi}[\omega B(\omega,T)-A(\omega,T)\sin\varphi]\ , \quad (31)$$

where $A(\omega,T)$ and $B(\omega,T)$ are given by Eqs. (9) and (10). We want to understand in details the various features of the solutions. Naturally this equation can be integrated numerically. A quite remarkable feature of the numerical solution is that the results for Re$[\varphi(\omega)]$ show very little dependence on temperature up to $T_c$ in the range which is of interest for us as it can be seen in Fig. 3 (the behavior for large $\omega$ is actually unimportant since $\Delta(\omega)$ will be essentially zero in this range, except at low temperature where there is no temperature dependence anyway).

In order to understand this peculiar behavior, let us first consider Eq. (31) for small $T$ and $\omega$ small, typically of order $T$. At first it looks rather different from its $T=0$ counterpart. In particular the behavior of $A(\omega,T)$ and $B(\omega,T)$ is quite different for small $\omega$. Indeed we have

$$A(\omega,T)=\frac{\pi}{4T}\cosh^{-2}\left[\frac{\omega}{2T}\right]+A_{\rm reg}(\omega,T)\ ,$$

$$A_{\rm reg}(\omega,T)=2\pi T\sum_{n=0}^{\infty}\frac{[\Delta_n-(\omega_n^2+\Delta_n^2)^{1/2}](\omega_n^2-\omega^2)}{(\omega_n^2+\Delta_n^2)^{1/2}(\omega_n^2+\omega^2)^2} \quad (32)$$

and

$$\omega B(\omega,T)-\omega=\frac{\pi}{2\Delta_0}\left[\frac{\omega}{2T}\cosh^{-2}\left[\frac{\omega}{2T}\right]+\tanh\left[\frac{\omega}{2T}\right]\right]+\omega B_{\rm reg}(\omega,T)\ ,$$

$$B_{\rm reg}(\omega,T)=4\pi T\sum_{n=0}^{\infty}\frac{\omega_n^2}{(\omega_n^2+\omega^2)^2}\left[\frac{1}{(\omega_n^2+\Delta_n^2)^{1/2}}-\frac{1}{\Delta_0}\right]. \quad (33)$$

We see that $A_{\rm reg}(\omega,T)$ and $B_{\rm reg}(\omega,T)$ have commuting limits $\omega\to 0$ and $T\to 0$, while this is not the case for the other contributions to $A(\omega,T)$ and $B(\omega,T)$ that we have written explicitly. In particular $A(0,T)$ and $B(0,T)$ diverge in the limit $T\to 0$. However when we substitute these expressions into Eq. (31), we obtain

$$-\Phi''+\Phi'\tanh(X/2)+(\Phi/2)\cosh^{-2}(X/2)=\tanh(X/2)+(X/2)\cosh^{-2}(X/2) \quad (34)$$

where we have taken $X=\omega/T$ and $\Phi=T\varphi/\Delta_0$ as a variable and function in order to eliminate the $T$ dependence. We have kept the dominant terms and not written those terms which are negligible in the limit $T\to 0$. We see that the $T=0$ solution, namely $\Phi=X$ [i.e., $\varphi(\omega)=\omega/\Delta_0$], is still a solution of this equation, which means that the singular behavior of the different terms compensate each other. Actually the general solution of Eq. (34) is $\Phi=X+\alpha\sinh X+\beta(1+\cosh X)$, where $\alpha$ is real and $\beta$

imaginary, because $\varphi^*(\omega)=-\varphi(-\omega)$. Naturally $\alpha$ and $\beta$ go to zero for $T\to 0$ since we must recover the $T=0$ solution. Our numerical calculation merely shows that they are always indeed rather small (except naturally near $T_c$ for $\beta$).

Since we have obtained that, in the range $\omega$ of order $T$, $\varphi(\omega)$ is regular with $\varphi(\omega)\approx\omega/\Delta_0$ we can then look outside this range. Because $T$ is always small we can consider the two first terms of Eq. (31), namely $T[\varphi''+(\varphi')^2\tan(\varphi)]$ as a small perturbation and neglect it to lowest order. Similarly since $\tanh(\omega/2T)$ is markedly different from unity only in the small range $\omega<2T$, we can replace it by 1. We are left with

$$\varphi'=\frac{2}{\pi}[\omega B(\omega,T)-A(\omega,T)\sin\varphi]\ . \tag{35}$$

We extend the lower boundary of integration down to $\omega=0$ since the range $\omega<2T$ where it is not correct is small. To lowest order we can take $\mathrm{Im}\varphi(0)=0$. Therefore Eq. (35) will produce, as at $T=0$, a $\varphi(\omega)$ which is real. When this equation is integrated numerically we find remarkably that the result is almost independent of temperature. We have not plotted them because they are barely distinguishable on a graph. This can be partially understood in the following way: beyond the range $\omega<2T$, which plays a little role because of its small size, the right-hand side of Eq. (35) is dominated by the $B(\omega,T)$ term, which is not very temperature dependent. Indeed for $\omega>1$, it is very well approximated by its temperature-independent asymptotic form $B(\omega,T)=\omega+1/\omega$. Moreover there is a partial compensation between the slight increase of $B(\omega,T)$ with $T$ and the decrease of the term $-A(\omega,T)\sin\varphi$. We can check that the solution of Eq. (35) is nearly temperature independent by looking at the solution for $T=T_c$ which is merely $\varphi(\omega)=2/\pi\int d\omega\,\omega B(\omega,T_c)$. The result is quite close from the $T=0$ result. Moreover for $\omega<4$ (and higher values are rather unimportant for our purposes as we mentioned already), the results from Eq. (35) are themselves very close to the exact numerical results for $\mathrm{Re}\varphi(\omega)$ from Eq. (31) that we considered at the beginning of this paragraph. We can summarize this part by saying that $\mathrm{Re}\varphi(\omega)$ is nearly temperature independent because to lowest order in $T$, Eq. (31) reduces to Eq. (35), except for the unimportant range $\omega<2T$. And the solution of Eq. (35) itself is almost temperature independent because the $B(\omega,T)$ term dominates, and it is weakly temperature dependent. Therefore the $T=0$ solution for $\varphi(\omega)$ is a very good zeroth order solution $\varphi_0(\omega)$ for Eq. (31).

We consider now the perturbation due to the temperature $T$. It is clear that the term $-T\varphi''$ produces only a small and regular perturbation since $\varphi_0(\omega)$ is not far from being linear in our range of interest. On the other hand, we see immediately that the term $-T(\varphi')^2\tan\varphi$ produces an important perturbation since $\tan\varphi$ is infinite when $\omega=y_n$ where $\varphi_0(y_n)=(n-1/2)\pi$. For these values of $\omega$ we can take $\tanh(\omega/2T)=1$. When we integrate Eq. (31) in the vicinity of $y_n$, we obtain that, because of this pole of $\tan\varphi$, the term $-T(\varphi')^2\tan\varphi$ produces a jump $\pi T\varphi_0'(y_n)$ in the imaginary part of $\varphi$. Therefore with this simple approximation we find

$$\mathrm{Im}\varphi(\omega)=\pi T\sum_n \varphi_0'(y_n)\theta(\omega-y_n) \tag{36}$$

which has the shape of a staircase. When this simple expression is compared to the exact numerical results for $\mathrm{Im}\varphi(\omega)$ from Eq. (31), one finds a quite good semiquantitative agreement. Indeed for $\omega\approx y_1$, $\mathrm{Im}\varphi(\omega)$ has the shape of a step which is increasingly smooth with increasing temperature. For the higher poles the steps are completely smoothed out (except at very low temperature). As it can be seen for $T/T_c=0.5$ from Fig. 4, the height of the first step is well reproduced as well as the average shape of $\mathrm{Im}\varphi(\omega)$ for increasing $\omega$.

Our zeroth order handling of the term $-T(\varphi')^2\tan\varphi$ looks rather rough. We can do a better job by integrating exactly Eq. (31) in the vicinity of $y_n$. This procedure will become exact in the limit $T\to 0$. In this vicinity we can consider $A(\omega)$ and $B(\omega)$ as constant, and if we set $\omega=y_n+TX$ and $\varphi(\omega)=(n-1/2)\pi+CT\Phi$ where $C=(2/\pi)[y_nB(y_n)+(-1)^nA(y_n)]$, we obtain the reduced equation

$$\Phi''-\frac{\Phi'^2}{\Phi}-\Phi'=1 \tag{37}$$

which we cannot solve exactly analytically; but this is easily done numerically. Naturally we find that the step is smooth over a range of order $T$, but we obtain that the overall height of the step is very near 3, practically identical to our rough estimate of $\pi$. Actually the numerical solution shows that $\Phi''$ is always very small in Eq. (37). If this term is neglected, the equation can be solved analytically and the height of the step becomes exactly $\pi$. We see that our rough estimate is actually much better than what we could have expected.

Finally we can make use of the above remark that the $-T\varphi''$ term is unimportant qualitatively and quantitatively to obtain a simplified version of Eq. (31) which nevertheless gives essentially the correct quantitative answer for all temperatures. Indeed if we omit the second derivative in Eq. (31) and take again $\tanh(\omega/2T)=1$, we get

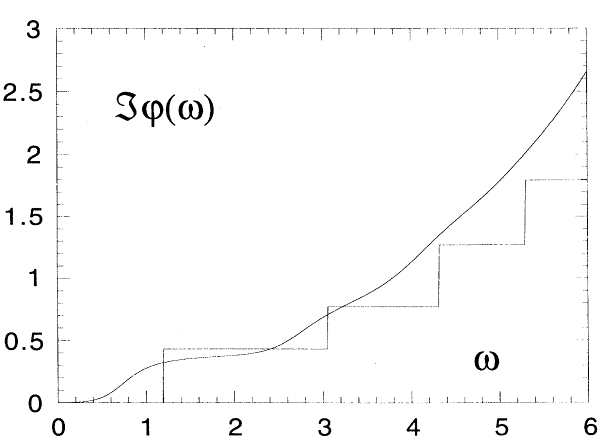


FIG. 4. $\mathrm{Im}[\varphi(\omega)]$ from numerical integration of Eq. (31) and the staircase approximation Eq. (36) for $T/T_c=0.5$.

$$\varphi' = \frac{2\varphi_0'(\omega)}{1+\sqrt{1-4T\varphi_0'(\omega)\tan\varphi}} , \tag{38}$$

where, consistently with the spirit of our perturbation handling, we have replaced the right-hand side of Eq. (31) by its $T=0$ value, namely $\varphi_0'(\omega)$ [a very good approximation is $\varphi_0'(\omega)=1.33$ for $\omega<0.75$ and the asymptotic expression $\varphi_0'(\omega)=2(\omega+1/\omega)/\pi$ elsewhere]. There is actually rather little change if we do not do this and keep the right-hand side as it is. The boundary condition for this first-order differential equation is $\mathrm{Re}\varphi(0)=0$. The value of $\mathrm{Im}\varphi(0)$ can be obtained by an extrapolation of the imaginary axis results. When we make a first-order expansion of Eq. (38) with respect to $T$, we get back to our staircase answer for $\mathrm{Im}\varphi(\omega)$. On the other hand, when Eq. (38) is integrated numerically, we find for $\Delta(\omega)$ results in quite good agreement with the integration of the full Eq. (31), as it can be seen in Figs. 5 and 6 for $T/T_c=0.3$ and 0.7. For $T/T_c=0.3$ the agreement is excellent [any small value for $\mathrm{Im}\varphi(0)$ will give the same result]. For $T/T_c=0.7$ the small discrepancy could be essentially corrected by giving to $\varphi_0'(\omega)$ a small temperature dependence.

To conclude this section let us remark that the energy $\Delta(T)$ where $N(\omega)$ is maximum changes very weakly as a function of temperature, as it can be seen on Fig. 7. This is of interest because such a behavior has already been noticed theoretically in the strong-coupling regime,[7,12,13] in relation with experimental results in high-$T_c$ superconductors where a "gap" has been tentatively observed with almost no temperature dependence. Indeed since there is no true gap in the excitation spectrum at nonzero temperature, this is the location $\Delta(T)$ of the maximum of $N(\omega)$ which is actually observed experimentally and which is very often called the "gap." The fact that such a behavior is found in the strong-coupling limit shows that this behavior is completely generic and not linked to a specific assumption made in the strong-coupling regime. We note also that the small variation of $\Delta(T)$ occurs essentially at low temperature and in this range $\Delta(T)$ is a linear function of $T$. This behavior is rather surprising since in the weak coupling or in the strong-coupling regime, $\Delta(T)$ is essentially temperature independent at low temperature. The present linear variation is clearly linked physically to the fact that phonons can be very easily excited thermally at nonzero temperature in the strong-coupling limit (their number is proportional to

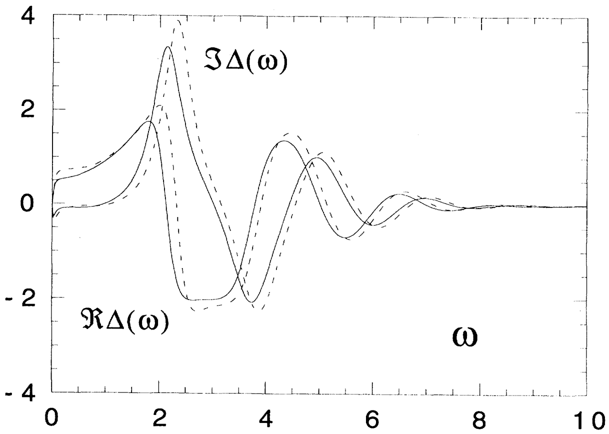


FIG. 6. $\mathrm{Re}[\Delta(\omega)]$ and $\mathrm{Im}[\Delta(\omega)]$ from numerical integration of Eq. (31) (solid line) and from the approximate equation Eq. (38) [dashed line), for $T/T_c=0.7$.

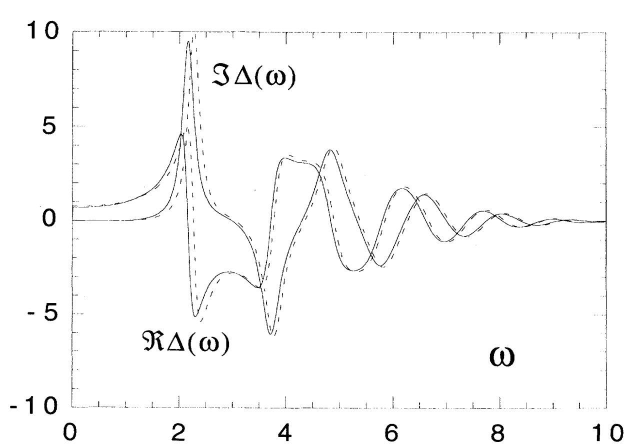


FIG. 5. $\mathrm{Re}[\Delta(\omega)]$ and $\mathrm{Im}[\Delta(\omega)]$ from numerical integration of Eq. (31) (solid line) and from the approximate equation Eq. (38) (dashed line), for $T/T_c=0.3$.

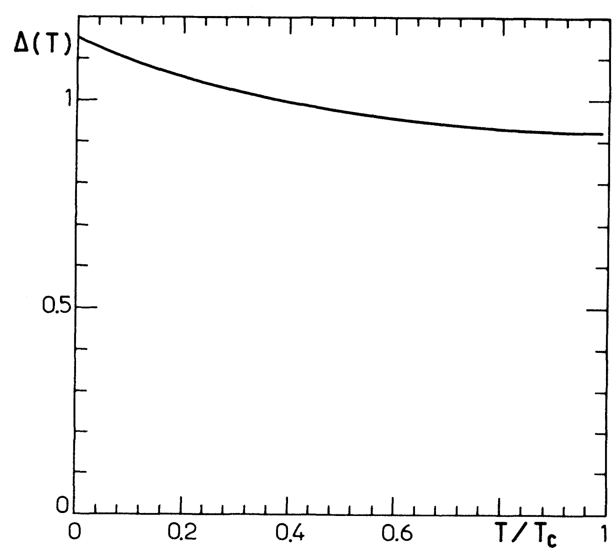


FIG. 7. Frequency location $\Delta(T)$ of the maximum of the density of states $N(\omega)$ as a function of reduced temperature.

$T/\Omega$). Therefore one never obtains at low temperature a regime where only virtual phonons are present. This linear behavior is actually directly linked to the corresponding linear decrease of $\Delta_0$ with temperature, which can be obtained from the imaginary axis equations.


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