# Critical behavior of randomly pinned spin-density waves

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Monte Carlo simulations have been used to study a $Z_6$ version of the two-component random anisotropy model on a simple cubic lattice. For strong random anisotropy, there is a finite temperature second-order phase transition with critical exponents $\eta=0.01\pm0.03$ and $\alpha=-0.76\pm0.03$. The specific-heat amplitude ratio is $A/A'=1.00\pm0.02$. The low-temperature phase is characterized by an infinite susceptibility and a $|\mathbf{k}|^{-2.4}$ decay of two-spin correlations, but no long-range magnetic order. Although the Hamiltonian has only a twofold exact symmetry, it appears that the low-temperature phase contains two pairs of Gibbs states, which approximate a fourfold symmetry in the phase space. These results are generally in agreement with the existing experiments for the randomly pinned spin-density wave state in dilute YEr alloys, and with similar experiments on certain amorphous magnetic alloys.

## I. INTRODUCTION

The Overhauser theory[1] of spin-density waves (SDW) in dilute alloys is now 35 years old, and the experiments[2] that motivated it are even older. It is remarkable that the fundamental difficulties which occur when one attempts to carry this theory beyond the mean-field level have remained unresolved. These difficulties are frequently ignored, even though it is well-known that in the related problem of charge-density waves[3,4] a random pinning potential will always destroy the long-range phase coherence of a three-dimensional system. In this work we take a major step toward a satisfactory resolution of this situation for the case in which the SDW modulation vector, $\mathbf{Q}$, is uniquely determined by the crystal structure. Examples of this are yttrium alloys doped with rare-earth (RE) elements,[5–17] which have a hexagonal close-packed lattice structure.

The most straightforward case, and the one we will work with here, is a longitudinally polarized SDW, such as the one which occurs in YEr alloys.[10–14] We will assume that the amplitude fluctuations of the SDW are unimportant compared to the phase fluctuations. This is reasonable because the localized $f$-electron moment on each RE site induces a local polarization of the conduction electrons near it even in the high-temperature phase, so that only the phase coherence disappears at the critical temperature, $T_c$. Then, in a semiclassical lattice formulation, the theory may begin with the Hamiltonian

$$H_{\mathrm{SDW}}=-J\sum_{\langle ij\rangle}\cos[\theta_i-\theta_j-\mathbf{Q}\cdot(\mathbf{x}_i-\mathbf{x}_j)]-G\sum_{i'}S_{i'}\cos(\theta_{i'})\ . \quad (1)$$

The sum over $\langle ij\rangle$ is a sum over all nearest-neighbor pairs, and the sum over $i'$ is a sum over only the RE sites, which are assumed to be randomly distributed and stationary. The $\theta_i$ variable represents the phase of the conduction-electron SDW at site $i$, and the $S_{i'}$ variable gives the direction of the local RE moment at site $i'$. The $\mathbf{Q}$ vector, which spans the Fermi surface of the conduction electrons,[18] points along the hexagonal ($c$) axis[19] and is incommensurate with the Bravais lattice vector. We assume that $J>0$; this causes no loss of generality, as we could compensate for a change in the sign of $J$ by a change of $\mathbf{Q}$. $G$ may be of either sign, as long as the temperature is high enough so that the Kondo effect can be neglected. Due to the large crystal-field energy, each Er moment will be assumed to point along the $c$ axis, either up or down. Most of the other RE elements have Stevens factors of the opposite sign, so that their moments tend to lie in the $a$-$b$ plane. We take $S_{i'}=\pm1$, and absorb the magnitude of the Er moment in the coupling constant, $G$.

If we make the change of variables $\theta_i\rightarrow\theta_i-\mathbf{Q}\cdot\mathbf{x}_i-\phi$, Eq. (1) takes the form

$$H=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)-G\sum_{i'}S_{i'}\cos(\theta_{i'}-\mathbf{Q}\cdot\mathbf{x}_{i'}-\phi)\ . \quad (2)$$

The angle $\phi$ determines the overall phase of the SDW. Without the random pinning, the energy would be independent of $\phi$ when the $\mathbf{Q}$ vector was incommensurate with the underlying lattice. In that case, there is a continuous set of ground states parametrized by the value of $\phi$. In the presence of the random pinning, this continuous symmetry is destroyed. The only remaining exact symmetry of $H$ is time reversal, which sends each $S_{i'}$ to $-S_{i'}$, and each $\theta_i$ to $\theta_i+\pi$. This yields the twofold Kramers degeneracy.

A Hamiltonian of this type was studied analytically by Ioffe and Feigelman[20] some time ago. They concluded that it should have a low-temperature phase which has a lower critical dimension of 3, and a Curie-like magnetic susceptibility at low temperatures. This appears to be in disagreement with the reported experimental results,[10–14] which claim to find long-range antiferromagnetic order at low temperatures.

## II. RANDOM ANISOTROPY MODEL

Each Er local moment is strongly coupled to the conduction electrons in its vicinity, so that we are interested

 

in the case $|G| \gg J$. This is the limit of strong pinning, in which the value of $\theta_{i'}$ is almost completely determined by the value of $S_{i'}$. Dynamically, the relaxation of the $S_{i'}$ variables (the local moments) takes place on a slower time scale than the relaxation of the $\theta_i$ variables (the conduction electrons). This is largely responsible for the "glassy" behavior seen in the experiments. We now restrict ourselves to temperatures of $T=2zJ$ or less, where $z$ is the number of nearest neighbors. We can then integrate out the degrees of freedom represented by the $S_{i'}$ variables. We treat the Er moments as "slaved" degrees of freedom, which are determined if we know the $\theta_i$. The effective Hamiltonian takes the form

$$H=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)-\tilde{G}\sum_{i'}[\,|\cos(\theta_{i'}-\phi_{i'})|-1\,]\ , \tag{3}$$

where $\tilde{G}>0$, regardless of the sign of $G$ in Eq. (2). The $\phi_{i'}$ variables represent the locally favored value of the phase, as determined by the positions of the Er atoms, the **Q** vector, and possibly the overall phase angle $\phi$. Since the Er atoms are assumed to be immobile, the $\phi_i$ variables are quenched, not dynamical. In YEr it should be a good approximation to assume that the $\phi_i$ are all uncorrelated.

We then consider a coarse graining of the Hamiltonian, so that each box has, on the average, a few randomly arranged Er atoms in it. Because we have already assumed that $T \ll \tilde{G}$, it is then permissible to replace Eq. (3) by

$$H_{\mathrm{RAM2}}=-J\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)-\frac{\tilde{G}}{2}\sum_i[\cos^2(\theta_i-\phi_i)-1]\ . \tag{4}$$

Equation (4) is the two-component random anisotropy model,[21–26] which has been studied primarily in the context of amorphous magnetic alloys such as TbFe,[27] HoFe,[28] and ErCo.[29] The model is even better as a representation of dilute YEr alloys. This is true for two reasons: because there should be no question in this case that a two-component model is called for, and because in the ferromagnetic case the magnetic dipolar interactions (which we are neglecting) are important in determining the large-scale structure.

Pelcovits, Pytte, and Rudnick[22] argued that the analysis for the random-field case[3,4] could be carried over without essential change to the random anisotropy problem. These arguments appear to show that there should be a finite range of spin correlations and no true long-range order at any finite temperature in a three-dimensional system described by Eq. (4). Later work[25,26] indicates that, while these arguments are probably correct for spins with three (or more) components, the two-component case is more subtle.

From the high-temperature series expansions[25] we find that three is a temperature at which the magnetic susceptibility, $\chi_M$, diverges. The nature of this divergence, however, is not obvious from the series results. If one assumes that it is a simple power law, described by a critical exponent $\gamma_M$, then the apparent value of $\gamma_M$ which one obtains from the series for the simple cubic lattice is different from the apparent value from the series for the face-centered-cubic lattice. Also, these values are not well converged, meaning that they drift higher as more terms are added to the series. This problem is quite common for $\chi_M$ series in random magnets.[30] It often indicates that one is looking at a crossover, not the true asymptotic large-order behavior.

In the earlier computer simulations[26] there is another hint of the existence of some rather long length scale which must be reached before the asymptotic behavior can be seen. The long-wavelength behavior of the two-spin correlation function at $T=0$ was found to be approximately

$$C_2(\mathbf{k},T=0)=\frac{1}{L^3}\sum_{i,j}\exp[i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)]\cos(\theta_i-\theta_j)$$
$$=\mathrm{const}/|\mathbf{k}|^{2.4} \tag{5}$$

for $L\times L\times L$ lattices. However, the size dependence of the $T=0$ magnetization per site was found to obey

$$M(L)=1/[0.37\ln(L)+1] \tag{6}$$

when $L\leq 48$. Taken together, these two results indicate that $L=48$ is not large enough to be the asymptotic scaling region (assuming that it exists).

Given Eq. (5), scaling relations predict that for large $L$ we should see

$$M(L)=\mathrm{const}/L^{0.3}\ . \tag{7}$$

In the work reported here we will find that Eq. (6) seems to be breaking down for $L=64$, but Eq. (5) continues to hold. We will also find a phenomenon which is probably responsible for causing this crossover to occur. We are not able to determine explicitly whether Eq. (7) is obeyed for large $L$. Numerically, this would require working at even larger $L$, which would require a faster computer. The numerical results also suggest additional relations which may be amenable to analytical study.

Fisher[31] has given arguments which suggest the existence of a crossover length at which there is a breakdown of the perturbative renormalization-group calculations.[22] The mechanism invoked by Fisher to create this effect is multiple minima of the free energy which are not related by any symmetry of the Hamiltonian. Assuming the existence of this large crossover length, we can then attribute the disagreement between the theory of Ioffe and Feigelman[20] and the experiments[10–14] at least partly to a failure of the experiments to probe through the crossover scale. In its current state of development, however, the theory of Ioffe and Feigelman does not provide a complete explanation of either the experiments or our computer simulations.

There is an important conceptual issue which must be resolved before a true finite-temperature phase transition should be believed to exist in this model. According to the simple version of the Fisher-Huse droplet picture,[32] the pure Gibbs state of a random Hamiltonian of the sort we are considering would be essentially unique, except for the twofold Kramers degeneracy. This means that the singular behavior of the phase space at the critical point, $T_c$, should be a simple bifurcation. Then at $T_c$ the suscep-

tibility $\chi_M$ would be anisotropic, being larger along the axis of the bifurcation than perpendicular to that axis. This, however, is quite implausible. The probability distribution of the random $\phi_i$ is assumed to be isotropic. For (almost) any finite sample there will be a residual anisotropy due to statistical fluctuations, but the effects of this residual average anisotropy must disappear in the limit that $L \to \infty$. Below $T_c$, averaging over the Gibbs states must restore the isotropy of $\chi_M$.

This is an important difference between Eq. (4) and similar models in which the random anisotropy has a $p$-fold symmetry with $p \geq 3$. When $p \geq 3$ the symmetry of the Hamiltonian is still high enough to insure that $\chi_M$ will be isotropic at $T_c$. The result of this, as shown both by analytical[33–35] and numerical[36] calculations, is that for $p \geq 3$ the random anisotropy is irrelevant near $T_c$, and the critical behavior is that of a pure $XY$ model, even though the "Goldstone mode" is massive.[36]

According to the more detailed calculations of Bovier and Frohlich,[37] the arguments of Fisher and Huse[32] are not rigorous in more than two dimensions, and a low-temperature phase with power-law decay of spin correlations is possible in three dimensions. It is still a big step to show that more than one pair of pure Gibbs states actually exist at low temperatures for Eq. (4) in three dimensions. There is no doubt that for small $L$ almost all samples[26] have only one pair of low-temperature Gibbs states. What the Monte Carlo calculations reported here have found is that, of two thoroughly studied $L=64$ lattices, one has a single pair of low-temperature Gibbs states, and the other has two pairs which are arranged to create an almost perfect *fourfold* symmetry in the phase space.

## III. MONTE CARLO CALCULATION

Although it was used in the earlier study,[26] Eq. (4) is not a convenient choice for performing Monte Carlo simulations. In order to improve the efficiency of the computer simulations, it was decided to replace Eq. (4) by its related six-state clock model:

$$H_{\mathrm{RAZ6}} = -J \sum_{\langle ij \rangle} \cos\left[\frac{\pi}{3}(l_i - l_j)\right] - \frac{2D}{3} \sum_i \left[\cos^2\left[\frac{\pi}{3}(l_i - h_i)\right] - 1\right] . \quad (8)$$

In Eq. (8) each $l_i$ is a dynamical variable belonging to the group $Z_6$, and each $h_i$ is an independent quenched variable chosen from $Z_3$. Henceforth we will set the units of energy so that $J=1$. The factor of 2/3 which multiplies $D$ is included for later convenience. We only consider $D \geq 0$, although there is nothing improper with having $D<0$.

Equation (8) has the useful property that, if $D$ is chosen to be an integer, then the energy of every state is an integral multiple of 1/2. Thus for integer values of $D$ it becomes possible to write a Monte Carlo program to study Eq. (8) which uses integer arithmetic to calculate energies. This gives substantial improvements in performance over working with Eq. (4), for both memory size and speed. Monte Carlo calculations for Eq. (4) are only practical in the limit $\tilde{G}/J \to \infty$.

It is well known that the pure (i.e., $D=0$) six-state clock model is in the same universality class as the pure $XY$ model. It may be less appreciated that it is also a good quantitative approximation near $T_c$, even for nonuniversal quantities. For example, $T_c$ for the pure $XY$ model on the simple cubic lattice is[38] $2.2016 \pm 0.0004$, while the author obtained (with very little effort, while checking the Monte Carlo program) an estimate of $T_c = 2.215 \pm 0.005$ for the pure six-state clock model on this lattice. The use of discrete groups to approximate continuous groups is well established for lattice gauge theory Monte Carlo calculations.[39,40]

For small $D$ Eq. (8) is probably not a faithful approximation to Eq. (4) in three dimensions. It is believed (but not proven) that any nonzero value of $D$ will destabilize long-range ferromagnetic order in three dimensions for Eq. (4). For the discretized case, Eq. (8), it is known that for small values of $D$ a ferromagnetic phase must be stable at low temperatures in three dimensions, just as it is for the random-field Ising model.[41] The numerical results indicate that on a simple cubic lattice long-range ferromagnetic order becomes unstable for $D$ greater than about 5 in Eq. (8).

The Monte Carlo program used two linear congruential pseudorandom number generators. In order to avoid unwanted correlations, the random number generator used to select the $h_i$ was different from the one used for the spin-flip dynamics. For finite $D$ a heat bath method was used. At each step the value of a spin was reassigned to one of the six allowed states, weighted according to their Boltzmann factors and independent of the prior state of the spin. For $D=\infty$ each spin has only two allowed states, $l_i = h_i$ and $l_i = h_i + 3$. In this case it is more efficient to use a Metropolis algorithm. Some short runs of $D=\infty$ were made with the heat bath algorithm, as a check of the programs. Periodic boundary conditions were used throughout.

Only single-spin-flip dynamics were used, unlike the earlier work[26] on the $D=\infty$ limit of Eq. (4), in which pair spin flips were also included. Using only single spin flips improves the performance of the program near $T_c$, at the risk of a possible failure to equilibrate fully at low temperatures. The ground-state energy found for the $D=\infty$ limit of Eq. (8) in this work was approximately $E_0 = -1.504$ for both $L=64$ lattices. This is only slightly above the value of $E_0 = -1.506$ found in the earlier study of Eq. (4) using the pair spin flips. The dominant contribution to the small difference in these numbers is believed to be a real difference in the ground-state energies of the two cases. The $Z_6$ case has a residual entropy at $T=0$, because about 1.7% of the spins see a zero effective field in a ground state. Splitting the degeneracy of these minimum-energy states should result in a net lowering of the ground-state energy. The loose spins probably aid in the equilibration process at low temperatures. Not surprisingly, the fraction of loose spins (and therefore the entropy at $T=0$) becomes smaller as $D$ is reduced. No

loose spins were seen for $D=4$, which has a ferromagnetic ground state for the $Z_6$ case.

## IV. NUMERICAL RESULTS

The specific heat, $c_H$, of an $L=64$ lattice (sample $S2$) with $D=0$ is displayed in Fig. 1(a). The qualitative agreement with the experimental data[10] for dilute YEr alloys is quite satisfactory, which shows that Eq. (8) retains the essential features of the randomly pinned SDW, as claimed. The same data are replotted on a log-log scale in Fig. 1(b). It should be clear to the reader from these specific-heat data that we are looking at a true phase transition, with $T_c=1.940\pm0.002$ for this sample. The data for sample $S1$ are similar. There is a small smearing of the specific heat of $S1$ near $T_c$, which will be discussed later. The points shown were obtained by numerically differentiating the energy. The specific heat was also computed by calculating the fluctuations in the energy at fixed temperature, yielding similar but noisier results. Near $T_c$ the sample was run for about $5\times10^5$ Monte Carlo steps per spin (MCS) at each $T$, with sampling after each 20 MCS. This was reduced further away from $T_c$, and in the range $1.5\le T\le1.7$ averaging was conducted over 61 440 MCS, after discarding the first 20 480 MCS for equilibration.

From Fig. 1(b) we obtain the critical exponent $\alpha$ and the universal amplitude $A/A'$ which characterize[42] the specific-heat cusp at $T_c$. The values are

$$\alpha=-0.76\pm0.03\ ,\quad A/A'=1.00\pm0.02\ . \tag{9}$$

The value of $A/A'$ is similar to that of the pure $XY$ model,[43] but the value of $\alpha$ is very different. A reduction in the value of $\alpha$ is a typical effect of randomness.

To see that the model is not really ferromagnetic for $T<T_c$ when $D$ is large, we look at the way $|M|$ behaves as a function of $L$ and $T$. In Fig. 2(a) we show $M(T)$ for

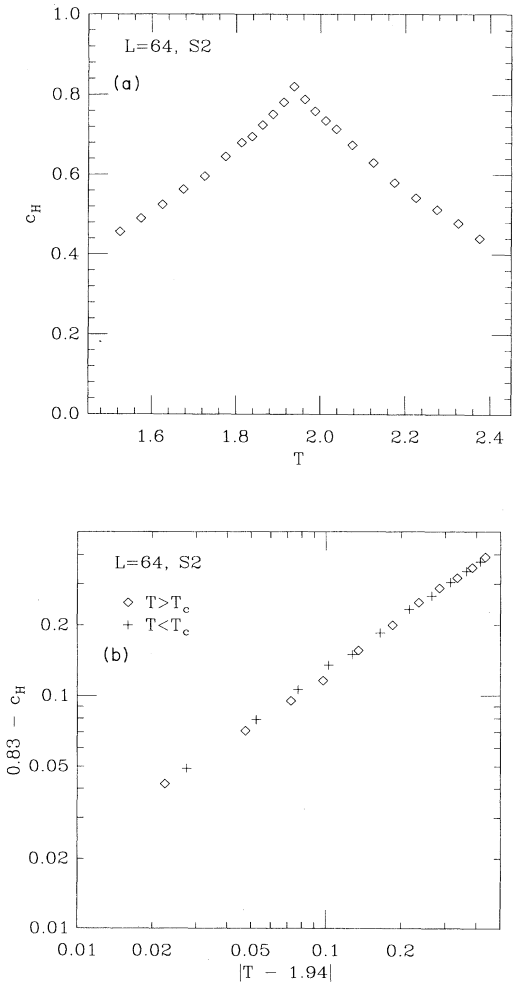


FIG. 1. Specific heat of the random anisotropy $Z_6$ model with $D=\infty$ on an $L=64$ simple cubic lattice (sample $S2$) near $T_c$: (a) linear plot; (b) log-log plot.

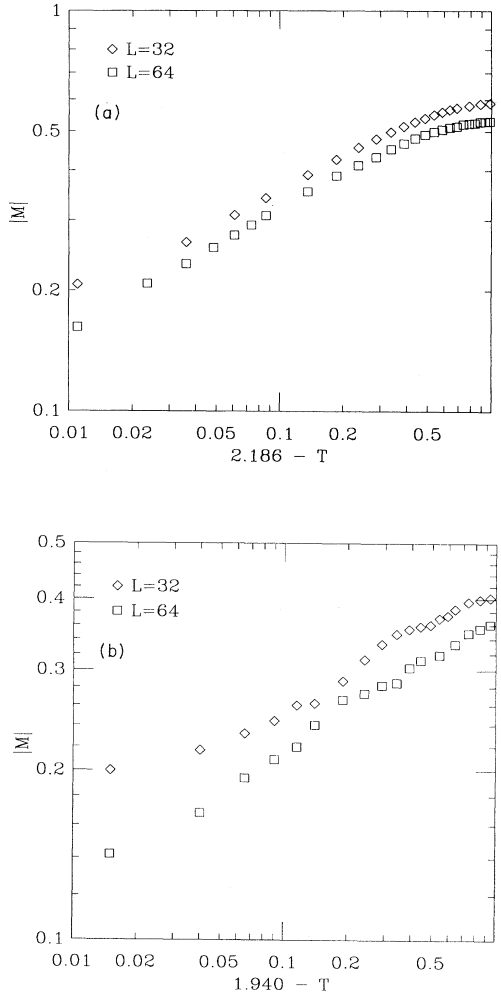


FIG. 2. Magnetization of the random anisotropy $Z_6$ model on simple cubic lattices with $L=32$ and 64, log-log plot: (a) $D=6$; (b) $D=\infty$.

an $L=32$ lattice and an $L=64$ lattice with $D=6$, on a log-log scale. In Fig. 2(b) we show the same thing for $D=\infty$. In both cases it appears that

$$M(T,D,L)=M(0,D,L)g(T,D) \tag{10}$$

provides a good representation of the data. It is interesting to note that $g(T,6)$ is not monotonic, but has a smooth maximum near $T=1.00$, and then decreases by about 5% as $T$ is lowered to zero.

If we fit the behavior of $g(T,D)$ near $T_c$ by a power law, we find an effective exponent $\tilde{\beta}\approx 0.31$ for $D=6$, and $\tilde{\beta}\approx 0.26$ for $D=\infty$. It seems clear that $\tilde{\beta}$ is not a critical exponent in the usual sense, because there is no true long-range ferromagnetic order here. Nevertheless, we are left with a puzzle, because the quoted value of $\beta$ from the neutron-scattering experiment of Caudron *et al.*[12] is $0.5\pm 0.1$.

The magnetic susceptibility for $T>T_c$ and $D=\infty$ is shown in Fig. 3, for the $L=64$ sample $S2$. Both

$$\chi_M=\frac{1}{2T}\left[1+\frac{1}{L^3}\sum_{i\neq j}\langle\cos(\theta_i-\theta_j)\rangle\right] \tag{11a}$$

and its "longitudinal" part

$$\tilde{\chi}_M=\frac{1}{T}[\langle M^2\rangle-\langle|M|\rangle^2] \tag{11b}$$

are shown. The angle brackets indicate a thermal average. The existence of a nonzero value of $\langle|M|\rangle$, even though $T>T_c$, is a finite-size effect. As expected, both $\chi_M$ and $\tilde{\chi}_M$ diverge with the same exponent, $\gamma_M$. Assuming a simple power-law divergence, $\gamma_M$ seems to have a value of $1.70\pm 0.05$. This is in somewhat better agreement with the experimental measurement,[12,14] $\gamma_M=1.9\pm 0.2$, than was our value for $\tilde{\beta}$. We will return to this point shortly.

To measure the critical exponent $\eta$, we look at the behavior of the two-spin correlation function $C_2(\mathbf{k},T_c)$. This is shown, averaged over angles, for both $L=64$ lattices at $T=1.95$, in Fig. 4. The slope of the log-log plot as $|\mathbf{k}|\to 0$ is $\eta-2$, so we find

$$\eta=0.01\pm 0.03\ . \tag{12}$$

We now observe that the value of the correlation length exponent, $\nu$, obtained from the hyperscaling relation $d\nu=2-\alpha$ is $0.92\pm 0.01$, while the value from the relation $\gamma=\nu(2-\eta)$ is $0.85\pm 0.04$. The consistency is not very good. The author believes that the estimates of $\alpha$ and $\eta$ are more reliable than the estimate of $\gamma$, which is probably too low. A value of $\gamma_M=1.83$ satisfies both of the scaling relations, and also improves the agreement with experiment. Given the unusual size dependence of the magnetization, Eq. (10), the existence of large corrections to scaling for $\chi_M$ is not surprising. The same problem was also noted in the analysis of the high-temperature series[25] for $\chi_M$.

When $T$ is near $T_c$ it is interesting to look at the probability distribution of the magnetization. This is shown in Fig. 5 for sample $S1$, and in Fig. 6 for sample $S2$. The data for each of these contour plots is taken from a run of about $5\times 10^5$ MCS, after equilibrating. We see that in both cases there is significant asymmetry about the origin at $T=1.95$. This shows that the phase space is starting to bifurcate, and that the probability for hopping between the two wells in the time of our run is small. At $T=1.925$ the two pictures differ qualitatively. For sample $S1$ a second bifurcation has occurred, along an axis which is perpendicular to the first bifurcation. Only one transition between the two wells in Fig. 5(b) was observed. For sample $S2$ a second bifurcation does not occur, although in Fig. 6(b) we can see a tendency for the contours to become elliptical. A search of the phase space of $S2$ was conducted using various cold-start initial conditions, and the author is confident that there is not a second pair of wells for this sample.

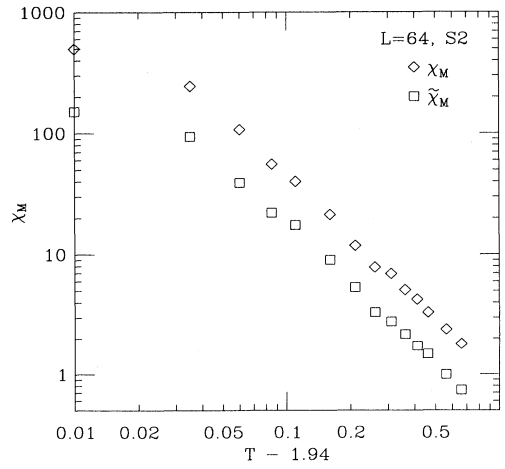


FIG. 3. Magnetic susceptibility of the random anisotropy $Z_6$ model with $D=\infty$ on an $L=64$ simple cubic lattice (sample $S2$) for $T>T_c$, log-log plot. $\chi_M$ is defined in Eq. (11a), and $\tilde{\chi}_M$ is defined in Eq. (11b).

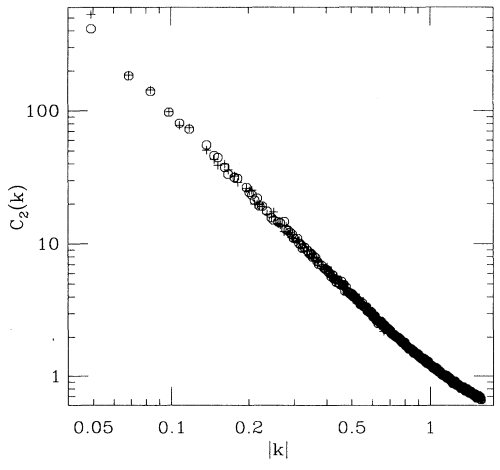


FIG. 4. Angle-averaged two-spin correlation function at $T=1.95$, for the random anisotropy $Z_6$ model with $D=\infty$ on $L=64$ simple cubic lattices, log-log plot. The pluses show averaged data from 24 states of $S1$ sampled at 20 480 MCS intervals, and the octagons show equivalent data from 24 states of $S2$.

A state from each of the two wells of Fig. 5(b) was slowly cooled down to $T=0$. The *total* energies (not per spin) of the minimum-energy states found for these two Gibbs states differ by only 3, while the height of the energy barrier between the two wells can be estimated to be about 200. (Recall that the multiplicity of the minimum-energy states in each well is large, due to the loose spins.) Minimum-energy states from the two different wells are almost orthogonal, having a dot product of about $-0.068$. Thus the phase space of $S1$ has an almost perfect fourfold symmetry. It is clear that this fourfold symmetry cannot be caused by the sixfold anisotropy imposed on Eq. (8), because $Z_4$ is not a subgroup of $Z_6$. Therefore the fourfold symmetry also should occur for Eq. (4) when $L$ is large, and for the experimental systems.

The author believes that when $L$ is increased beyond 64 the probability that a sample will exhibit this fourfold symmetry of the phase space will increase rapidly toward 1. As $L\rightarrow\infty$ the two bifurcations ought to merge into a single fourfold splitting. There seems to be no reason to expect any additional bifurcations for a three-dimensional system. Two orthogonal states are sufficient to provide a basis for a two-component spin model. It would be interesting to test these extrapolations by studying an $L=128$ system. This could be done with a supercomputer.

Because the two bifurcations in sample $S1$ occur sequentially, rather than simultaneously, the critical properties are slightly smeared out. This smearing should disappear as $L$ is increased, and a single, well-defined critical point is expected. We also expect that Eq. (7) will become accurate for large $L$, as required by scaling theory. For $L=64$ the magnetizations of our three minimum-energy states are 0.3565 and 0.3635 for $S1$, and 0.3728 for $S2$. This is already significantly below the extrapolation of Eq. (6), 0.394, although perhaps some allowance should be made for the difference between Eq. (4) and Eq. (8).

Fourier transforming a minimum-energy state from

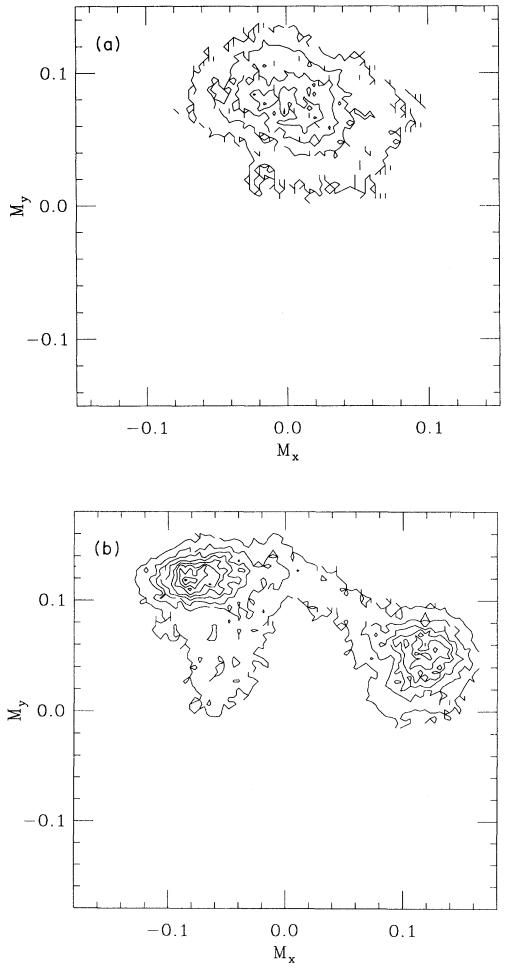


FIG. 5. Contour plots of the probability density of the magnetization for sample $S1$, taken from a run of about $5\times10^5$ MCS at each $T$: (a) $T=1.95$; (b) $T=1.925$. Note that the scales are different in (a) and (b).

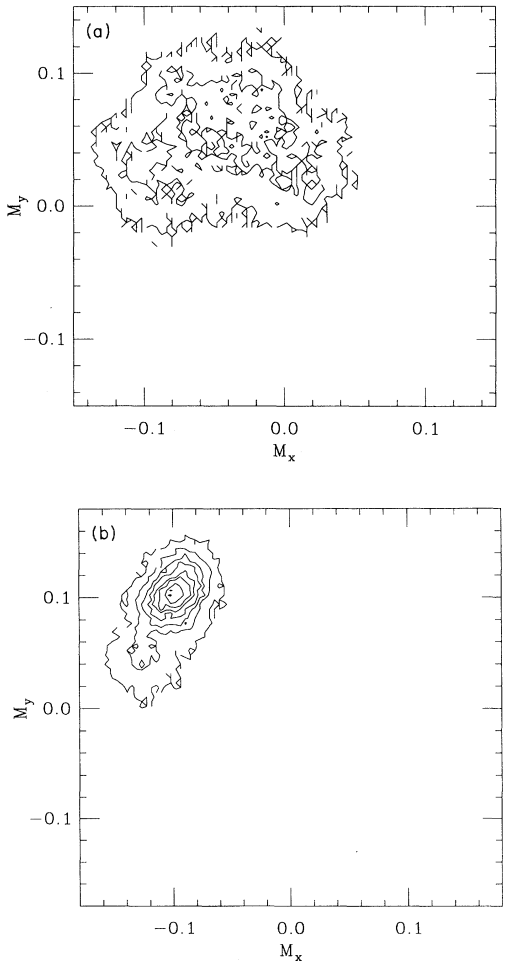


FIG. 6. Contour plots of the probability density of the magnetization for sample $S2$, taken from a run of about $5\times10^5$ MCS at each $T$: (a) $T=1.95$; (b) $T=1.925$. The scales in (a) and (b) are the same ones used in Figs. 5(a) and 5(b), respectively.

each of the two orthogonal wells of $S1$ and a minimum-energy state from $S2$, and averaging over angles, we get an estimate of the $T=0$ two-spin correlation function, $C_2(|\mathbf{k}|, T=0)$, which is displayed in Fig. 7. The slope at small $|\mathbf{k}|$ agrees with Eq. (5), and defining the exponent $\eta_1$ in the usual fashion gives us

$$\eta_1 = -0.4 \pm 0.1 \ . \tag{13}$$

This behavior has been observed in small-angle neutron-scattering experiments on several amorphous magnetic alloys.[27–29] It would be very interesting to examine carefully the shape of the SDW peak in the neutron scattering from YEr, to see if it shows the $|\mathbf{k}|^{-2.4}$ behavior also.

## V. DISCUSSION

For the two lattices with $L=64$ and $D=\infty$ the energy $E(T_c)$ is $-1.014 \pm 0.002$, compared to[38] $E(T_c) = -0.994$ for the pure $XY$ model. What may be more intriguing is that the probability, $P(|\mathbf{v}|)$, of a vortex loop[44] (of either sign) threading through a plaquette at $T_c$ is also lower than for the pure $XY$ model, having the value $0.1490 \pm 0.0004$. (At $T=\infty$, $P(|\mathbf{v}|)$ is 0.3241 for a $Z_6$ model, as compared to 1/3 for an $XY$ model.[44])

The above estimates ignore the downward shift[26] of the effective value of $T_c$ as $L$ increases. Making an allowance for this effect, we estimate that the true, infinite lattice $T_c$ for $D=\infty$ is

$$T_c = 1.935 \pm 0.010 \ . \tag{14}$$

This suggests that the value of $P(|\mathbf{v}|)$ at the true $T_c$ is slightly less that 0.149. The conjecture that the exact value is

$$P(|\mathbf{v}|) = 4/27 = 0.\overline{148} \tag{15}$$

at $T_c$ for $D=\infty$ is highly attractive. This number is exactly twice the vortex density of the random $h_i$ field, which is easily calculated to be 8/27. The implication is that at $T_c$ there may be a symmetry of the transfer matrix which relates the static vortices in the $h_i$ field to the dynamic vortices in the $l_i$ field.

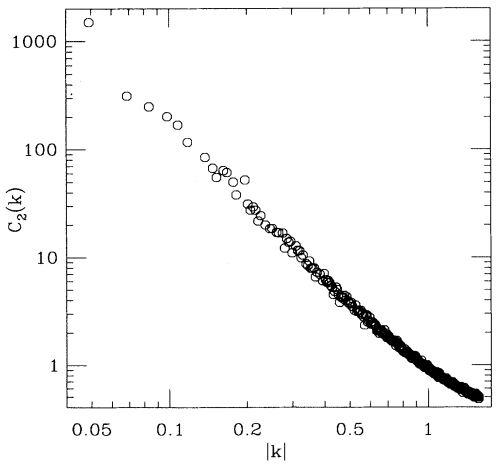


FIG. 7. Angle-averaged two-spin correlation function at $T=0.0$, for the random anisotropy $Z_6$ model with $D=\infty$ on $L=64$ simple cubic lattices, log-log plot. The data are an average over three states, two from $S1$ and one from $S2$, as described in the text.

The conjecture of Eq. (15) gives a special status to the $Z_6$ model with $D=\infty$, as compared to Eq. (4) or any other model in the same universality class. This is not unreasonable. The $Z_6$ model with $D=\infty$ is the simplest, and thus the most constrained, model in the universality class.[45] Therefore it is the one most likely to obey some special symmetry relation.[46] Duality relations for ensembles of random Hamiltonians are known for some two-dimensional Potts models.[47] In this case, however, we are suggesting a relation which is only expected to hold at $T_c$. This is somewhat reminiscent of the relation found by Stephen and Mittag[48] between the transfer matrix of a two-dimensional Potts model at $T_c$ and the Hamiltonian of a one-dimensional spin-1/2 Heisenberg model.

The fluctuations of the magnetization in $Z_6$ models obey a kind of "triangle equality." Define the partial magnetizations

$$M_n(X) = P(n|X) - P(n+3|X) \ , \tag{16}$$

where $n=0$, 1, or 2, and $P(n|X)$ is the probability that a spin, $l_i$, in the set $X$ has the value $n$. Suppose that we have two sets of spins, $X$ and $X'$. These could be two different sets of spins, or the same set of spins at two different times. Now order the $M_n$ according to $|M_a(X)| \le |M_b(X)| \le |M_c(X)|$, and similarly for the primed variables. Then if

$$|M_a(X)| + |M_b(X)| = |M_c(X)| \ , \tag{17a}$$

and

$$|M_{a'}(X')| + |M_{b'}(X')| = |M_{c'}(X')| \ , \tag{17b}$$

it follows that the $M_n(X+X')$ will also satisfy a triangle equality, *even if the direction of the maximum $M_n$ changes.* This means that the triangle equality acts as a fixed-point relation among the three $M_n$. If we average over large sets of spins and long times, deviations from the triangle equality should become small, unless the system is in a ferromagnetic phase.

In a mean-field theory with random anisotropy, deviations from the triangle equality in the ferromagnetic phase may be of either sign. If $D$ is large, the mean-field theory[23] predicts a ferromagnetic phase with $|M_c| < |M_a| + |M_b|$. In three dimensions, however, this does not happen. In the Monte Carlo results for $D=6$ and even $D=\infty$, the triangle equality is obeyed all the way down to $T=0$. This may be viewed as merely a reflection of the fact that these cases do not have true long-range order. It is probably the explanation for the almost complete lack of "squaring up" of the YEr SDW, as seen in the small intensity of the third harmonic peak in the neutron-scattering experiments.[12,13] For the $D=4$ case, $|M_c| > |M_a| + |M_b|$ when $T < T_c$, because the six-fold anisotropy is able to stabilize a ferromagnetic phase.

## VI. CONCLUSION

In this work we have used Monte Carlo simulations to study a version of the two-component random anisotropy magnet, which is a model of randomly pinned spin-density waves in YEr, and also of certain amorphous magnetic alloys. We have found an unusual second-order phase transition, from the paramagnetic phase into a phase with power-law decay of spin correlations but no true long-range order. In many respects our results agree with the existing experimental data. It would be worthwhile to improve the precision of the experiments, and to extend the computer simulations to larger samples. Due to the remarkable nature of the numerical results presented here, the author believes that it may be possible to find some exact analytical results for this problem.

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