Perturbative approach to the dynamics of a linear chain with hierarchical coupling

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The self-similar structure of the dynamical matrix of hierarchical chains of harmonic oscillators is explored to obtain perturbative derivation of the eigensolutions. We focus in particular on the eigenvalue spectrum, determining explicit expressions for the eigenfrequencies, and finding a Cantor set with zero Lebesgue measure. We then bring the calculation to the particular case of a linear chain with equal masses connected by hierarchically distributed springs of power-law type. The spectral dimension, explicitly derived for this system, shows a nonuniversal behavior, being strongly dependent on the details of the coupling. The spectral dimension is then used for computing the lattice specific heat while singularities in the spectrum are analyzed both within the local and global point of view. All results are compared with the numerically determined exact eigenvalues.

I. INTRODUCTION

In the present paper we investigate the properties of a linear chain of harmonic oscillators with identical masses and nearest-neighbor couplings distributed in a hierarchical way. Similar problems of electrons in a hierarchic potential have been discussed by many authors in recent years. In particular, Jona Lasinio, Martinelli, and Scoppola studied the diffusion of electrons through a sequence of potential barriers with hierarchically growing heights, finding a subdiffusive regime.[1] Tight-binding models with diagonal energies following a geometric succession have been found to yield energy spectra that can be continuous or singular, depending on the parameters of the model.[2,3] On the other hand, the same kind of hierarchicity in the hopping terms always gives rise to very rich Cantor-set spectra with a multiplicity of singularities in the density of states.[4,5] The linear chain of oscillators represents a particular system in which these two discrete models are mixed. In fact, in this case the diagonal terms are determined from the off-diagonal terms owing to the translational invariance of the system. Still more important, such a problem in its stationary version is formally equivalent to the problem of diffusion through a hierarchical array of barriers.[6] Moreover, heterostructures made of two media distributed according to a hierarchy of lengths have been recently realized, and modes and frequencies of vibrations have been experimentally measured and compared with theoretical predictions.[7]

Most of the approaches to this kind of problem are based on renormalization or trace map techniques. In the present study, the self-similarity of the system is used to write the dynamical matrix in a recursive way which yields explicit perturbative expressions for the eigensolutions. This approach eventually applies to the case of the infinite chain and allows one, within the limits of the considered approximation, to reveal completely all the properties of the spectrum.

In Sec. II we describe our model and derive the recurrence properties of the dynamical matrix and of its eigensolutions, which are the base of our treatment. Through these properties we obtain first-order perturbation expressions for the solution of the eigenvalue problem. In the remaining sections we discuss in some detail the particular case of a hierarchical system with a geometric succession of coupling constants: $\chi_j=\chi_1\gamma^{j-1}$, $\gamma<1$. In Sec. III we compute the eigenvalues of such a system. It turns out to be a bifurcating treelike spectrum which generates a Cantor set when the system becomes infinite. As a test for our findings, we compare analytical and high-precision numerical values for the eigenfrequencies of finite systems of increasing generation, finding excellent agreement. In Secs. IV and V we discuss the scaling properties of the spectrum. In particular, we derive in Sec. IV the local average density of states and the spectral dimension (both of the finite and semi-infinite system), and use the latter quantity for computing the specific heat in the Debye approximation. Results for large finite systems are compared with those obtained via the eigenfrequency spectrum from direct numerical diagonalization. Results indicate the spectral dimension to be a good parameter for computing integral quantities, in spite of the very fragmented structure of the real density of states. Finally, we focus in Sec. V on the scaling properties of the density of states, determining analytically the local singularities and the global spectrum of exponents. We find that outside the gap the spectrum is characterized everywhere by the same singularity, except at zero frequency, where it is different.

II. MODEL AND SOLUTIONS

We consider a linear harmonic chain of oscillators with nearest-neighbor interactions and scalar displacements. All the particles have the same mass but the force constants are hierarchically distributed along the chain. Be-

ing the first generation made of two masses coupled by a spring of force constant χ_1, the chain at the kth generation is obtained by joining two identical chains of the $(k-1)$th generation by a spring with force constant equal to χ_k:

$$\begin{aligned} &\chi_1\ , \quad k=1\ , \\ &\chi_1\chi_2\chi_1\ , \quad k=2\ , \\ &\chi_1\chi_2\chi_1\chi_3\chi_1\chi_2\chi_1\ , \quad k=3\ , \\ &\cdots, \quad \cdots\ . \end{aligned} \tag{1}$$

At a given generation k, the system is composed of $N=2^k$ particles and it is completely determined once the set of force constants $\{\chi_j\}$, $j=1,2,\ldots,k$, is fixed.

In order to find the force constant of the spring which couples two consecutive sites of index n and $n+1$, the following procedure can be adopted. First of all one must note that for any given integer n ($n=1,\ldots,2^k-1$), the equation

$$n=2^{j-1}(2m-1), \quad j,m=1,2,\ldots, \tag{2}$$

has a unique solution for j and m integer with $1\le j\le k$ and $1\le m\le 2^{k-j}$. Once j and m have been found solving Eq. (2), j individuates the force constant χ_j, while m indicates the number of springs with force constant χ_j which lie on one side of the nth spring.

When imposing free boundary conditions, it can be seen by direct inspection that at a given stage k the dynamical matrix $\hat{D}^{(k)}$ of such a system can be expressed in terms of that at the previous stage $\hat{D}^{(k-1)}$ as

$$\hat{D}^{(k)}=\begin{bmatrix} \hat{D}^{(k-1)} & 0 \\ 0 & \hat{D}^{(k-1)} \end{bmatrix}+\chi_k\hat{\sigma}^{(k)}\ , \tag{3}$$

where the matrix $\hat{\sigma}^{(k)}$ has the same dimension as $\hat{D}^{(k)}$, a 2×2 central block given by

$$\hat{\sigma}=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{4}$$

and all the other elements null.

Equation (3) makes it easy to verify a remarkable property of $\hat{D}^{(k)}$ concerning the eigensolutions of the system. Let us suppose we know the quantities $\omega_\lambda^{(k-1)}$ and $\mathbf{e}_\lambda^{(k-1)}$, the eigenfrequencies and eigenvectors of the dynamical matrix of Eq. (3) for a system of the $(k-1)$th generation ($\lambda=1,2,\ldots,2^{k-1}$). Because of the left-right inversion symmetry of this system, as sketched in (1), the eigenvectors $\mathbf{e}_\lambda^{(k-1)}$ have well-defined parity with respect to the spatial inversion. In particular, at the $(k-1)$th stage, there are $2^{(k-2)}$ odd and $2^{(k-2)}$ even eigenvectors. The parity of $\mathbf{e}_\lambda^{(k-1)}$ being $\eta_\lambda^{(k-1)}$, ($\eta=\pm 1$) then the vectors

$$\mathbf{e}_{\lambda,+}^{(k)}=\frac{1}{\sqrt{2}}(\mathbf{e}_\lambda^{(k-1)},\eta_\lambda^{(k-1)}\mathbf{e}_\lambda^{(k-1)}) \tag{5}$$

are the 2^{k-1} symmetric (+) eigenvectors of the stage k. Because these modes do not stretch the central (χ_k) spring, the corresponding eigenfrequencies $\omega_{\lambda,+}^{(k)}$ are the same as those of the $(k-1)$th stage:

$$\omega_{\lambda,+}^{(k)}=\omega_\lambda^{(k-1)}\ . \tag{6}$$

For the 2^{k-1} antisymmetric modes, the following considerations hold. Let us first observe that in the trivial case $\chi_k=0$, the 2^{k-1} remaining antisymmetric (−) modes have the same eigenvalues as the symmetric ones and eigenvectors given by

$${}^0\mathbf{e}_{\lambda,-}^{(k)}=\frac{1}{\sqrt{2}}(\mathbf{e}_\lambda^{(k-1)},-\eta_\lambda^{(k-1)}\mathbf{e}_\lambda^{(k-1)})\ , \tag{7}$$

where we have indicated by ${}^0\mathbf{e}_{\lambda,-}^{(k)}$ the antisymmetric eigenvectors in the limit $\chi_k=0$, which are the zeroth-order approximation for $\mathbf{e}_{\lambda,-}^{(k)}$, the exact antisymmetric eigenvectors. Moreover, within this approximation,

$${}^0\omega_{\lambda,-}^{(k)}=\omega_\lambda^{(k-1)}\ . \tag{8}$$

When $\chi_k\neq 0$, degeneracy between symmetric and antisymmetric modes is removed and ${}^0\omega_{\lambda,-}^{(k)}$ and ${}^0\mathbf{e}_{\lambda,-}^{(k)}$ are no longer exact solutions of the problem, but can still be considered a good approximation, at least in the $\chi_k\to 0$ limit.

Starting from ${}^0\mathbf{e}_{\lambda,-}^{(k)}$, perturbative solutions can be computed for χ_k small enough. Let us rewrite Eq. (3) as

$$\hat{D}^{(k)}={}^0\hat{D}^{(k)}+\chi_k\hat{\sigma}^{(k)} \tag{9}$$

with implicit definition of the zeroth-order dynamical matrix ${}^0\hat{D}^{(k)}$ and of the "perturbation" $\chi_k\hat{\sigma}^{(k)}$. Let us denote by $x_\lambda^{(k)}$ the eigenvalues of the dynamical matrix, $x_\lambda^{(k)}=(\omega_\lambda^{(k)})^2$. Then the first-order perturbative correction to these eigenvalues gives

$$x_{\lambda,-}^{(k)}\simeq {}^0x_{\lambda,-}^{(k)}+\frac{({}^0\mathbf{e}_{\lambda,-}^{(k)}\chi_k\hat{\sigma}^{(k)}{}^0\mathbf{e}_{\lambda,-}^{(k)})}{({}^0\mathbf{e}_{\lambda,-}^{(k)})^2}\ . \tag{10}$$

At the same time, unperturbed solutions ${}^0\mathbf{e}_{\lambda,-}^{(k)}$ in the above expression can be expressed in terms of the $\mathbf{e}_\lambda^{(k-1)}$ through Eq. (7), yielding

$$x_{\lambda,-}^{(k)}\simeq {}^0x_{\lambda,-}^{(k)}+\chi_k\frac{1}{2}\frac{(e_{\lambda,2^{k-1}}^{(k-1)}+\eta_\lambda^{(k-1)}e_{\lambda,1}^{(k-1)})^2}{({}^0\mathbf{e}_{\lambda,-}^{(k)})^2}\ , \tag{11}$$

where $e_{\lambda,n}^{(k-1)}$ is the nth component of the vector $\mathbf{e}_\lambda^{(k-1)}$.

At this level of approximation, the eigenvectors of the kth stage being obtained by symmetrizing or antisymmetrizing those of the $(k-1)$th stage, it results that all the components of each eigenvector have the same modulus. The modulus of the components can be in turn determined by the normalization condition

$$|{}^0\mathbf{e}_{\lambda,-}^{(k)}|^2=1\ , \tag{12}$$

Therefore $|e_{\lambda,n}^{(k)}|=2^{-k/2}$ and

$$(e_{\lambda,2^{k-1}}^{(k-1)}+\eta_\lambda^{(k-1)}e_{\lambda,1}^{(k-1)})^2=2^{3-k}\ .$$

Thus we can write the expression for the eigenvalues at the kth stage in terms of those at the $(k-1)$th stage

$$\begin{aligned} x_{\lambda,+}^{(k)}&=x_\lambda^{(k-1)}\ , \\ x_{\lambda,-}^{(k)}&=x_\lambda^{(k-1)}+\frac{\chi_k}{2^{k-2}} \end{aligned} \tag{13}$$

for $\lambda=1,2,\ldots,2^{k-1}$. To summarize the previous results: each mode of a given stage gives rise to two modes at the next stage, one symmetric with the same frequency, and one antisymmetric with a squared frequency splitting of $\chi_k/2^{k-2}$.

An explicit expression for the eigenvalues can be written by means of Eq. (13):

$$x_\lambda^{(k)}=\sum_{j=1}^{k}\epsilon_\lambda^{(j)}\frac{\chi_j}{2^{j-2}} \quad (\lambda=1,2,\ldots,2^k) , \tag{14}$$

where k can also be taken infinite. Here $\epsilon_\lambda^{(j)}$ can assume the value 0 or 1. To each possible succession $\{\epsilon_\lambda^{(j)}\}$ of k digits 0,1 there corresponds an eigenvalue of the system, which is therefore in correspondence with a natural number when k is finite. The succession $\{\epsilon_\lambda^{(j)}\}$ can be thought of as the binary representation of the integer λ.

Besides the choice of the $\{\epsilon_\lambda^{(j)}\}$, the actual value of the eigenvalues will depend on the values of the coupling constants $\{\chi_j\}$. Even if the above expression converges for any limited sets of $\{\chi_j\}$, we can in general expect it to give results only for a sufficiently rapid decreasing succession of χ_j's.

An important feature of the approximate eigenvalues, Eq. (14), is that they yield the right value of the trace of the dynamical matrix, i.e., the second spectral moment of the density of states of the system. For a finite stage k one has for the sum of the approximated eigenvalues

$$\sum_{\lambda=1}^{M}x_\lambda^{(k)}=\sum_{\lambda=1}^{M}\sum_{j=1}^{k}\epsilon_\lambda^{(j)}\frac{\chi_j}{2^{j-2}}=\sum_{j=1}^{k}\frac{\chi_j}{2^{j-2}}\left[\sum_{\lambda=1}^{M}\epsilon_\lambda^{(j)}\right] .$$

The sum in the last square bracket is equal to half the total number of eigenvalues $N/2=2^{k-1}$, so that

$$\sum_{\lambda=1}^{M}x_\lambda^{(k)}=\sum_{j=1}^{k}2^{k-j+1}\chi_j$$
$$=2^k\chi_1+2^{k-1}\chi_2+2^{k-2}\chi_3+\cdots+2\chi_k .$$

The same result is obtained from $\mathrm{Tr}[\hat{D}^{(k)}]$ by direct inspection of $\hat{D}^{(k)}$.

III. FREQUENCY SPECTRUM WITH POWER-LAW COUPLINGS

In the following sections we discuss in some detail the case of a hierarchical system with coupling constants given by decreasing geometrical successions. The eigenvalue spectrum of the system is computed, allowing for determination of its spectral dimension and singularities.

Let us take

$$\chi_j=\chi_1\gamma^{j-1}, \quad \gamma\le 1 . \tag{15}$$

Without any loss of generality we can take $\chi_1=1$, obtaining from Eq. (14)

$$x_\lambda^{(k)}=\frac{4}{\gamma}\sum_{j=1}^{k}\epsilon_\lambda^{(j)}\left[\frac{\gamma}{2}\right]^j \quad (\lambda=1,2,\ldots) . \tag{16}$$

The above expression is expected to work reasonably well only for γ small enough.

It is worth noting that, with some straightforward algebra, the λth eigenvalue can be explicitly expressed in terms of the natural number λ, instead of its binary representation $\{\epsilon_\lambda^{(j)}\}$. The result is

$$x_\lambda^{(k)}=4\lambda\left[\frac{\gamma}{2}\right]^k+2(1-\gamma)\sum_{j=0}^{k-1}\left[\frac{\lambda}{2^{k-j-1}}\right]\left[\frac{\gamma}{2}\right]^j , \tag{17}$$

where $[x]$ indicates the integer part of x.

As a first result let us show that the spectrum exhibits a treelike structure and that λ correctly labels the eigenvalues in ascending order. Let us analyze those particular eigenvalues of an infinite system $x_{\lambda^0(m)}$ and $x_{\lambda^*(m)}$ identified respectively by the *patterns*

$$\{\epsilon_{\lambda^0(m)}\}=(\underbrace{0,\ldots,0}_{m},1,0,0,0,\ldots) , \tag{18}$$

with all zero and 1 at the mth position, and

$$\{\epsilon_{\lambda^*(m)}\}=(\underbrace{0,\ldots,0,0}_{m},1,1,1,\ldots) , \tag{19}$$

with zero up to the mth position and 1 at positions $m+1,m+2,\ldots$. From Eq. (16) it follows that

$$x_{\lambda^0(m)}=2\left[\frac{\gamma}{2}\right]^{m-1} ,$$
$$x_{\lambda^*(m)}=\frac{\gamma}{2-\gamma}x_{\lambda^0(m)} . \tag{20}$$

Thus $x_{\lambda^*(m)}<x_{\lambda^0(m)}$, and therefore any pattern with 0 in all the first m positions is associated with an x_λ smaller than $x_{\lambda^0(m)}$. At the same time the eigenvalue related to a pattern with at least one 1 at one of the first m positions

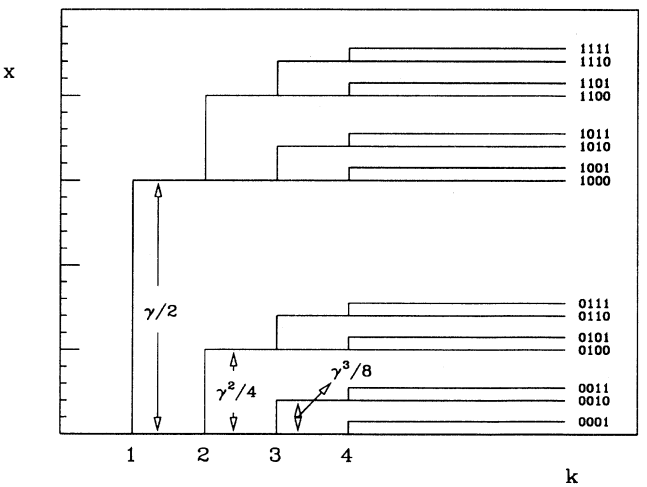

FIG. 1. Scheme of the treelike structure of the eigenvalue spectrum in the case of a power-law distribution of coupling constants. At each generation k the horizontal lines indicate the eigenvalues $x_\lambda^{(k)}$ which can be obtained by considering all the 2^k possible sequences of 0 and 1, $\{\epsilon_\lambda\}$. At the next generation, each eigenvalues λ gives rise to two new eigenvalues $\lambda_\pm$, one with the same value as the parent and the other split off to a higher value by an amount independent of the branch m.

is greater than or equal to $x_{\lambda^0(m)}$.

Since to each pattern can be associated a number in binary representation, the previous considerations provide a means for labeling eigenvalues and show that they can be ordered in increasing value by ordering the corresponding labels. For finite k this simply gives rise to a numerable ordered set of size k (Fig. 1). In the case of infinite systems, to each pattern we can make correspond a real number of the interval $[0,1]$ still in binary representation. This can be obtained by setting a point in front of the pattern. It follows that in this case the spectrum is made of a non-numerable set of zero Lebesgue measure (Cantor set). It can also be noted (Fig. 1) that the distance $x^{(k)}_{\lambda,-}-x^{(k)}_{\lambda,+}$ between two sub-branches growing from the same parent $x^{(k-1)}_{\lambda}$ does not depend on λ, while it does if we consider the distance $\omega^{(k)}_{\lambda,-}-\omega^{(k)}_{\lambda,+}$. We return to this and related points in Sec. V, when discussing the scaling properties of the spectrum.

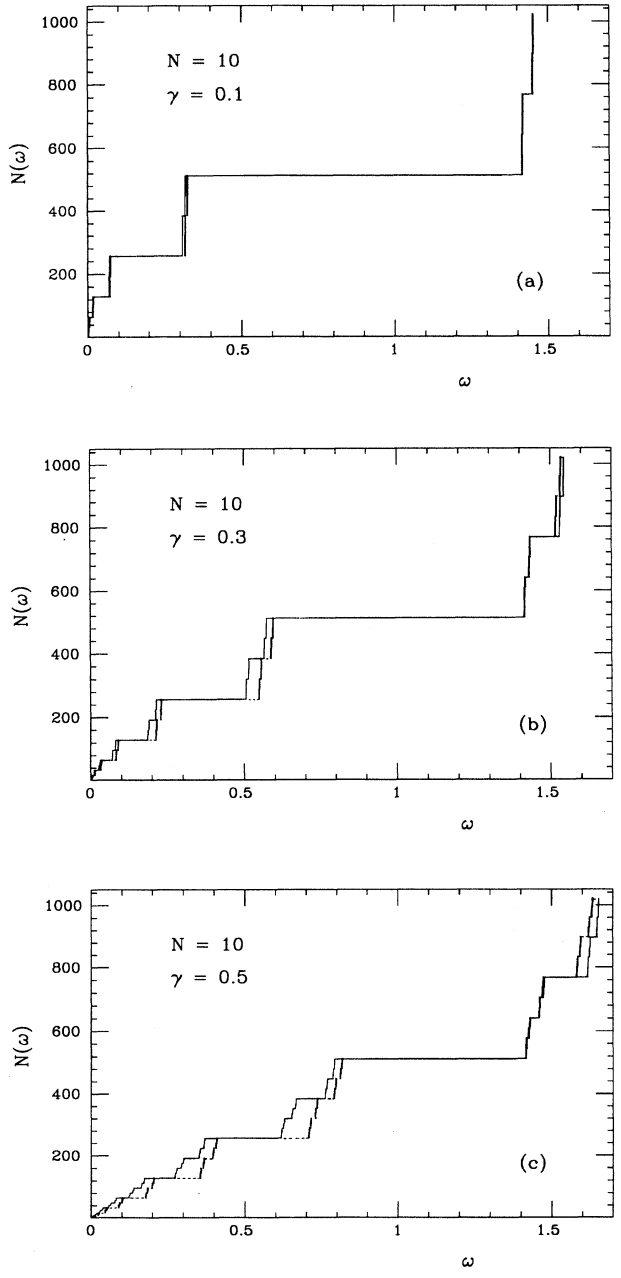

FIG. 2. Comparison between approximated (broken lines) and numerical (full lines) cumulative eigenfrequency spectrum for different coupling constants at generation $k=10$. The cumulative density of states is reported for $\gamma=0.1$ (a), 0.3 (b), and 0.5 (c).

Another interesting property is related to the normalized integrated density of states $\mathcal{N}(\omega)$. Here $\mathcal{N}(\omega)=N(\omega)/2^k$, $N(\omega)$ being the number of eigenmodes with frequency less than ω. It results that for certain frequencies $\mathcal{N}(\omega)$ is independent of the system size. Indeed, if we consider the frequencies $\omega_{\lambda^0(m)}=\sqrt{x_{\lambda^0(m)}}$ defined in Eq. (20), then the number of modes with frequency $\omega<\omega_{\lambda^0(m)}$ in a system with 2^k masses is given by the number of different way in which one can arrange 0 and 1 in the last $k-m$ positions, i.e., $N(\omega_{\lambda^0(m)})=2^{k-m}$, which implies

$$\mathcal{N}(\omega_{\lambda^0(m)})=2^{-m}\ ,$$
$$\omega_{\lambda^0(m)}=\sqrt{2}\left[\frac{\gamma}{2}\right]^{(m-1)/2}\ . \tag{21}$$

The result of Eq. (21) will be used in the next section to derive an explicit expression for the spectral dimension.

In order to verify our results and in order to check the level of approximation of Eq. (16), we computed the eigenfrequency spectrum numerically, by direct diagonalization of the exact dynamical matrix $\hat{D}^k$ using different values of γ. Figures 2(a), 2(b), and 2(c) show the cumulative density of states in the case $k=10$ for $\gamma=0.1, 0.3$, and 0.5 (full line) compared with those calculated from Eq. (16) (dotted line). As can be seen, there is a qualitative agreement between the exact and approximated cumulative density of states, and the agreement becomes quantitative for low values of γ. Moreover, it can be seen that the difference between exact and approximated eigenfrequencies is negative at low frequencies and becomes positive at high frequency, thus preserving the correct second spectral moment of the approximated density of states.

IV. SPECTRAL DIMENSION AND SPECIFIC HEAT

The spectral dimension $\tilde{d}$ describes the smooth behavior of the integrated density of states $\mathcal{N}(\omega)$,

$$\mathcal{N}(\omega)\approx\omega^{\tilde{d}}\ , \tag{22}$$

and can be useful when computing integral quantities, e.g., the specific heat, which should not depend on the fine structure of the spectrum. It was first introduced for describing *fractons,* the localized modes of self-similar structures.[8]

The values of the spectral dimension can be obtained by eliminating m from Eq. (21):

$$\mathcal{N}(\omega_{\lambda^0(m)})=2^{-2\ln(\omega_{\lambda^0(m)})/\ln(\gamma/2)+\text{const}}\ , \tag{23}$$

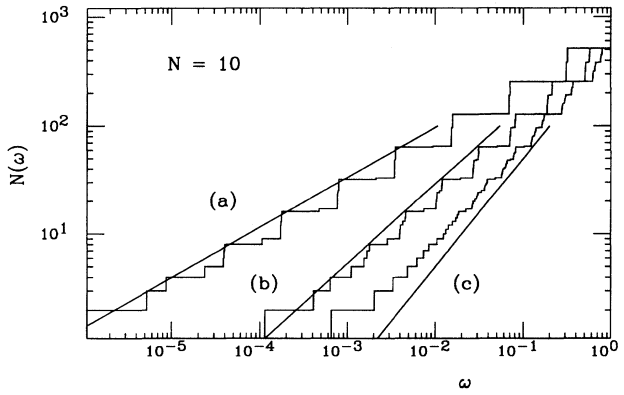

FIG. 3. Numerical computed cumulative density of states as in Fig. 2 but represented on a logarithmic scale for $\gamma=0.1$ (a), 0.3 (b), and 0.5 (c). Straight lines represent the cumulative density of states as described by the spectral dimension, Eq. (24).

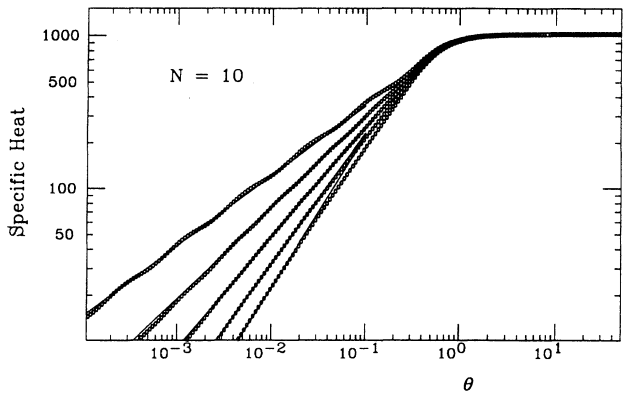

FIG. 5. Specific heat as resulting from using the Debye formula with the spectral dimension derived perturbatively (full line), and the numerically computed eigenvalues (○). The curves represent (from low to high slope) the cases with $\gamma=0.1$, 0.2, 0.3, 0.4, and 0.5.

which then yields

$$\tilde{d}=-\frac{2\ln 2}{\ln\beta}\ , \tag{24}$$

where we have set $\beta=\gamma/2$. Figure 3 shows on a log-log scale the same cumulative density of states as in Fig. 2, together with the one computed from the spectral dimension, Eq. (24). In Fig. 4 the behavior of $\tilde{d}$ as a function of γ is shown. Values from Eq. (24), full line, are compared with values obtained by numerical computations of the density of states (+) and of the specific heat (×, see later). As can be seen, Eq. (24) gives a very good approximation of $\tilde{d}$, at least for $\gamma<0.4$.

The spectrum in the case of low γ looks very singular, so it is interesting to compare physical quantities obtained by means of exact eigenfrequencies with those obtained by using the concept of spectral dimension. For instance, in the Debye approximation the specific heat of the system is expected to behave like

$$C\approx T^{\tilde{d}}\ . \tag{25}$$

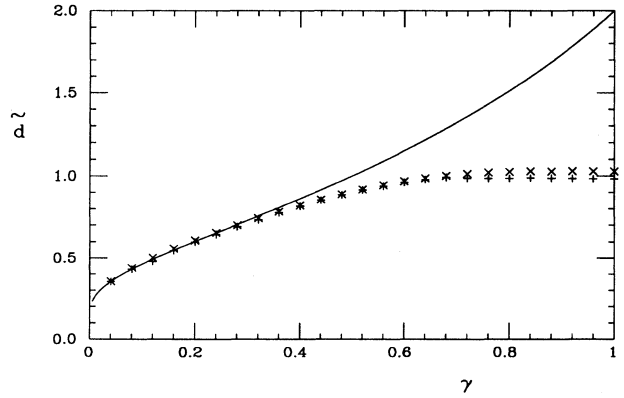

FIG. 4. Behavior of $\tilde{d}$ as function of γ computed through Eq. (24) (full line) compared with those numerically obtained from the density of states (+) and from the specific heat (×).

Results of such a comparison are shown in Fig. 5. In this figure is reported the specific heat of a 1024-mass system ($k=10$), in units of k_B, as a function of the reduced temperature $\theta=k_BT/\hbar$. The open circles indicate the specific heat calculated from the exact eigenfrequencies, i.e.,

$$C=\tfrac{1}{4}k_B\sum_{\lambda}^{2^k}\left[\frac{\omega_\lambda^{(k)}/\theta}{\sinh(\omega_\lambda^{(k)}/2\theta)}\right]^2\ , \tag{26}$$

while the full lines indicate the slope expected from the approximated values of $\tilde{d}$, Eq. (24). It can be seen that agreement is quite good, despite the fact that $\tilde{d}$ yields a very rough approximation of the real, highly singular spectrum.

V. SPECTRUM SINGULARITIES

A. Local scaling

The structure of the eigenvalue distribution reveals the existence of a self-similar spectrum, characterizable through local singularity exponents $\alpha(\omega)$ that can be determined starting from Eq. (14). In order to compute these exponents let us consider, in an infinite system, those branches $x_{k_0}^{(m)}$ specified by patterns like

$$\{\epsilon_{k_0}^{(m)}\}=\overbrace{(\underbrace{\epsilon_1^{(m)},\epsilon_2^{(m)},\ldots,\epsilon_k^{(m)}}_{k},0,\ldots)}^{\infty}, \tag{27}$$

with $\epsilon_1^{(m)},\ldots,\epsilon_k^{(m)}$ chosen arbitrarily. Let us now consider those branches $x_{k^*}^{(m)}$ starting from $x_{k_0}^{(m)}$, and corresponding to sequences

$$\{\epsilon_{k^*}^{(m)}\}=\overbrace{(\underbrace{\epsilon_1^{(m)},\epsilon_2^{(m)},\ldots,\epsilon_k^{(m)}}_{k},1,1,\ldots)}^{\infty}. \tag{28}$$

The distance between the two eigenvalues $x_{k0}^{(m)}$ and $x_{k*}^{(m)}$ is

$$\Delta_k^{(m)}=x_{k*}^{(m)}-x_{k0}^{(m)}=\frac{2\beta^k}{1-\beta},\quad \forall\ m \tag{29}$$

independently of m, i.e., of the choice of the set $\epsilon_1^{(m)},\epsilon_2^{(m)},\ldots,\epsilon_k^{(m)}$.

Also independent of m is the fraction f_k of eigenvalues which fall between $x_{k0}^{(m)}$ and $x_{k*}^{(m)}$:

$$f_k=2^{-k}\ . \tag{30}$$

The average density of states just above $x_{k0}^{(m)}$ is thus

$$\rho_k=\frac{f_k}{\Delta_k^{(m)}}=C(2\beta)^{-k},\quad C=(1-\beta)/2\ , \tag{31}$$

or, by reexpressing k as a function of Δ (we drop the indices),

$$\rho(\Delta)=C^{-\ln 2/\ln\beta}\Delta^{\ln 2\beta/\ln\beta}=C^{\tilde{d}/2}\Delta^{\tilde{d}/2-1}\ . \tag{32}$$

When $k\to\infty$, then $\Delta\to 0$ and one can take $\Delta=2\omega\delta\omega$. From the equality $\rho(\Delta)\Delta=\rho(\delta\omega)\delta\omega$ it follows that

$$\rho(\delta\omega)=(2C\omega)^{\alpha_1}\delta\omega^{\alpha_1-1},\quad \alpha_1=\frac{\tilde{d}}{2}<1\ . \tag{33}$$

As expected for such a critical system, $\rho\to\infty$ as $\delta\omega\to 0$. It results that all points of the spectrum possess the same singularity exponent α_1 except when ω tends to 0. In this case we expect ρ to diverge with an exponent $\alpha_0=2\alpha_1$. It is interesting to note how this result represents an intermediate situation between the electronic case with diagonal hierarchy[2,3] and the one with off-diagonal hierarchy.[4,5] In the first case, a decreasing geometric succession of local energies (which is more clearly related to the present case) produces a continuous spectrum of extended states, plus one single isolated localized state. In contrast, in the second case there is an infinite family of singularities.

B. Global scaling

The presence of two kinds of singularities is reflected in the multifractal spectrum describing the global scaling properties of the eigenvalue spectrum. Let us consider the partition function[9]

$$\Gamma(\tau,q,k)=\sum_{m=1}^{2^k}\frac{\mu^q(\delta\omega_k^{(m)})}{(\delta\omega_k^{(m)})^\tau}\ , \tag{34}$$

where $\tau(q)$ is determined by the condition

$$\Gamma(\tau,q,k)=\begin{cases}0 & \text{for } \tau<\tau(q)\\ \infty & \text{for } \tau>\tau(q)\ .\end{cases} \tag{35}$$

In the present case

$$\mu(\delta\omega_k^{(m)})=\rho(\delta\omega_k^{(m)})\delta\omega_k^{(m)}=f_k=2^{-k}\ ,$$

while from Eq. (29) and from $\omega_k^{(m)}=\sqrt{x_k^{(m)}}$ it follows that

$$\delta\omega_k^{(m)}=\frac{\beta^k}{\sqrt{x_{k0}^{(m)}}\,(1-\beta)}\ . \tag{36}$$

Thus

$$\Gamma(\tau,q,k)=2^{-kq}\beta^{-k\tau}(1-\beta)^\tau\frac{2^{\tau/2}}{\beta}A_k(\tau)\ , \tag{37}$$

having defined

$$A_k(\tau)=\frac{\beta^{\tau/2}}{2}\sum_{m=1}^{2^k}[x_{k0}^{(m)}]^{\tau/2}\ . \tag{38}$$

By taking the logarithm on both sides of Eq. (37) the relation defining $\tau(q)$ can be written

$$\ln[A_k(\tau)]+\text{const}=k[q\ln 2+\tau\ln\beta]\ , \tag{39}$$

which in the limit $k\to\infty$ can be written as

$$L(\tau)=q\ln 2+\tau\ln\beta\ , \tag{40}$$

where

$$L(\tau)=\lim_{k\to\infty}\frac{\ln A_k(\tau)}{k}\ . \tag{41}$$

It is not too hard to evaluate $A_k(\tau)$ for $\tau=0$ and for τ positive even. The result is

$$A_k(2n)=f_n(\beta)2^{k-n}\quad (n\geq 0) \tag{42}$$

with $f_n(\beta)\simeq 2^{n-1}\beta^n$ to the first order in β. Then

$$L(2n)=\lim_{k\to\infty}\frac{(k-n)\ln 2}{k}+\frac{\text{const}}{k}=\ln 2 \tag{43}$$

and Eq. (40) yields, for τ an even natural number,

$$\tau(q)=\tau_1(q)=-(q-1)\frac{\ln 2}{\ln\beta}=\frac{\tilde{d}}{2}(q-1)\ ,$$
$$\tau=0,2,4,\ldots\ . \tag{44}$$

It is also possible to determine the asymptotic behavior of τ for large negative q. In this limit the sum (38) is dominated by the smaller eigenvalue, that is, $A_k(\tau)\approx[x_{\min}]^{\tau/2}$. Since $x_{\min}=2\beta^{k-1}$ the result is

$$L(\tau)\simeq\frac{\tau}{2}\ln\beta \tag{45}$$

and from Eq. (40)

$$\tau(q)=\tau_2(q)\simeq-\frac{2\ln 2}{\ln\beta}=\tilde{d}q,\quad \tau\ll 0\ . \tag{46}$$

These results agree with those of the local scaling. In fact the resulting singularity spectrum $[\alpha,f(\alpha)]$ is

$$\left[\frac{\tilde{d}}{2},\frac{\tilde{d}}{2}\right]\ \text{and}\ (\tilde{d},0)\ . \tag{47}$$

The support of $\alpha_1=\tilde{d}/2$ has the same scaling index as the support of the spectrum, given by $-\tau_1(0)$, whereas the support of $\alpha_0=\tilde{d}$ is just one single point corresponding to $\omega\to 0$ in the spectrum. The $\tau(q)$ curve is determined by α_0 or α_1 according to whether $\alpha_0 q<\alpha_1$ or $\alpha_0 q>\alpha_1$, and shows a crossover at $q=-1$ where $\tau_1(q)=\tau_2(q)$. The

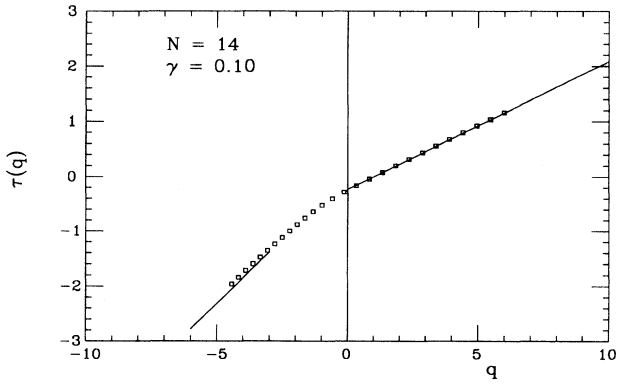

FIG. 6. Main features of the $\tau(q)$ curve, as resulting from analytical (continuous line), and numerical (dotted line) computation at generation $k=14$.

curve $\tau(q)$ is reported in Fig. 6. The squares indicate the points calculated from the exact eigenfrequencies of a $k=14$ system with coupling $\gamma=0.1$ while the full lines are the prediction made using the approximated eigenvalues from Eq. (44) for $q>0$, and from Eq. (46) for $q<0$. The numerical data do not show any sharp crossover at $q=-1$. It is not easy to decide if the sharp crossover exists but is smoothed out by the finite size of the system, or if it does not really exist and it is an artifact of the approximation in the perturbative calculation.

VI. CONCLUSION

We have used a perturbative approach to study a linear chain of oscillators with hierarchical couplings. We have shown that for these systems the recursive structure of the dynamical matrix allows one to obtain many dynamical properties at a good level of approximation.

We have considered in particular the case in which the values of the coupling constants for increasing generations are in geometrical succession. We found in this case an explicit expression for the eigenvalues, which in turn allows for the analytical derivation of many physical properties. Singularities of the eigenvalue spectrum have been determined, using both global and local descriptions. The spectral features are intermediate between those of systems where the hierarchy lies in the diagonal terms, and those where it lies in the off-diagonal ones.[2–5]

An analytical expression for the spectral dimension as a function of the coupling strength has been given, and the resulting behavior of the specific heat with temperature shows that such an exponent describes well the integral quantities, in the presence of very singular spectra also, as in the present case.

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