# Lifshitz tail in a model of interacting particles

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The density of states of noninteracting disordered particle systems shows a characteristic behavior deep in the band, known as the Lifshitz tail. In the present work the Lifshitz problem is reconsidered in a model one-dimensional disordered system with "minimal" interactions. The interaction has a form which allows the calculations to be performed asymptotically exactly in the tail, using nonperturbative techniques. Correlation between the particles leads to a considerable decrease of the density of states. These findings indicate that interactions may favor delocalization, a result obtained earlier, using phenomenological approaches, perturbative analysis, and the renormalization-group method.

Disorder may drastically modify the behavior of a physical system. A celebrated example of this effect is the localization transition,[1] which takes place when the concentration of impurities in a conductor reaches a critical value. Originally localization was studied in noninteracting electronic systems. Using a scaling theory it has been established that whenever a transition takes place at a finite impurity concentration (i.e. in dimensions higher than two) it is of second order; at zero temperature the static conductivity $\sigma$ vanishes according to a power law.[2] Shortly after it was pointed out that interaction between electrons may influence localization.[3] In the presence of interactions, however, the theory became so complicated, that despite great efforts, it is still not fully understood.[1] Meanwhile, the study of transport properties in many other systems revealed that localization is a ubiquitous phenomenon; it may manifest itself whenever a wave propagates in a random medium. Thus, in the presence of impurities sound wave attenuation is anomalous,[4] light propagation is strongly modified[5] and bosonic systems in general behave in many respect similarly to their electronic counterparts.[6]

The difficulties associated with interacting disordered particle systems stem from the fact that randomness renormalizes the interaction, typically increasing its strength. As a consequence, a naive perturbative approach in terms of the interaction is not possible. To deal with this problem one either attempts to sum up infinite perturbation series, or resorts to nonperturbative methods. The former approach was used by Finkelstein[7] and later extended by Belitz and Kirkpatrick,[1] who constructed the renormalization-group equations for both the disorder parameter and the relevant interaction constants. The starting point in the renormalization-group method is the pure, noninteracting system, and the transition is approached from the conducting side. To get more insight on the effect of interactions on localization, the renormalization-group equations have been explicitly studied in one dimension by Giamarchi and Schultz,[8] who showed, that interactions may lead to delocalization. In electronic systems an insulator-perfect metal ($\sigma=$ infinity) transition takes place at zero temperature, and the delocalized phase is characterized by superconducting fluctuations. In bosonic systems in the analogous delocalized state the superfluid correlation functions exhibit power-law behavior. Since in the absence of interactions, in one dimension all one-particle states, both electronic and bosonic are localized for arbitrarily weak disorder, an approach based on perturbation theory around the extended states may be questionable, even if the resulting series is summed to infinity.

The present work was motivated by the fact that the problem of interacting particles in the presence of disorder is still not fully resolved.[1] Here we investigate a disordered many-body model in one dimension, with "minimal interactions" (see below). We concentrate on the Lifshitz problem[9] by studying the one-particle density of states for large negative energies, $E \ll 0$ (in the tail of the band) using a nonperturbative, instanton approach.[10] Applying this technique in the absence of interactions between particles the asymptotic behavior of the density of states was obtained by Cardy[10] (and later reproduced by others[11]) in the form of a stretched exponential

$$n_0(E) \propto \frac{|E|}{g} \exp\left[-\text{const}\frac{|E|^{3/2}}{g}\right] . \tag{1}$$

Here $g$ characterizes the disorder (see below). The above result is derived by evaluating the averaged one-particle Green's function

$$\overline{G(x,x';E)} = \overline{\langle x|(H_0-E)^{-1}|x'\rangle} \quad \text{with } E \to E+i0 . \tag{2}$$

In (2) $\langle x|$ and $|x'\rangle$ are one-particle states and the Hamiltonian is defined as $H_0 = -\nabla^2/2 + V(x)$ (using units $h/2\pi = m = 1$). The bar denotes averaging over the distribution of the static random potential $V(x)$, which is assumed to be Gaussian,

$$\langle V(x)V(x')\rangle = g\delta(x-x') . \tag{3}$$

For negative energies the matrix element in (2) can be represented by a functional integral over some scalar (Bose) field $\varphi(x)$, as

 

$$G(x,x';E)=\frac{1}{Z}\int D\varphi[\varphi(x)\varphi(x')] \times\exp\{-\int dy[\varphi(y)(|E|+H_0)\varphi(y)]\}\ . \tag{4}$$

Here $Z$ is the "partition function," defined as $Z=\exp(S_0)$, with $S_0$ being the action given by the exponent in (4). The above representation for the Green's function allows the averaging to be performed conveniently using the replica trick.[10] The density of states is obtained by $n_0(E)=(1/\pi)\mathrm{Im}G(x,x;E)$, leading to (1) (for details, see Refs. 10). The same result can be obtained without the replica trick, using supersymmetric field theory.[12]

In order to study the effect of interactions, we introduce an additional term in the action, by $S=S_0-\lambda\int dy\,\varphi^4(y)$, with $\lambda>0$. The following remarks are due in connection with the model defined by $S$.

The resulting action resembles that of a $\varphi^4$ theory with a repulsive contact interaction between the particles described by the field. In order to clarify the meaning of $S$, let us recall the proper definition of a Green's function in a zero-temperature field theory defined by the Hamiltonian $H_0$ (Ref. 13)

$$G(x,x';E)=\frac{1}{Z}\int D\phi[\int dt\, e^{iEt}\phi(x;t)\phi(x';0)]\exp\left\{\int d\bar{t}\int dy\left[\phi(y;\bar{t})\left[i\frac{\partial}{\partial\bar{t}}-H_0\right]\phi(y;\bar{t})\right]\right\}\ . \tag{5}$$

[Here we explicitly used that $G(x,x';t,t')\equiv G(x,x';t-t')$.] In (5) the Green's function has been written for a real field. Expressions (4) and (5) are equivalent. This is a consequence of the quadratic nature of the exponent in (5). After Fourier transformation, the action becomes diagonal in the energy variable and with the notation $\varphi(x)=\phi(x;E)$,[14] (5) reduces to (4), irrespective of the properties of the random potential in the Hamiltonian. When a $\varphi^4(y)$ or $\phi^4(y;\bar{t})$ term is added, respectively, to the exponents in (4) and (5), it mixes different energies in the latter, whereas it does not lead to such an effect in the former. Expression (4), with the action $S$ now corresponds to forcing the exponent in (5) [with $\phi^4(y;\bar{t})$] to a "diagonal form" in terms of the energy variable and then again using the notation $\varphi(x)=\phi(x;E)$. Our additional term in the action $S$ does correspond to a (repulsive) interaction, but in terms of standard field theory its precise nature is not evident.

Our motivation to introduce the $\varphi^4$ term in (4) is that it represents the simplest complication of the noninteracting theory, which still allows an exact evaluation of the density of states in the limit of large negative energies. In this sense it describes a "minimal interaction."

To proceed, we replicate the field and perform the average over the distribution of the random potential. After trivial rescalings of the original variables, the averaged Green's function can be represented as

$$G(x,x';E)=\lim_{n\to 0}\left[\frac{|E|}{g-\lambda}\right]^{n/2+1}\int\prod_{i=1}^{n}D\varphi_i\left[\frac{1}{n}\prod_{k=1}^{n}\varphi_k(x)\varphi_k(x')\right]\exp S[\varphi] \tag{6}$$

with

$$S[\varphi]=-\frac{|E|^{3/2}}{2(g-\lambda)}\int dy\left\{\sum_{i=1}^{n}\left[\varphi_i^2-\varphi_i\frac{d^2}{dy^2}\varphi_i+\frac{\lambda}{2(g-\lambda)}\varphi_i^4\right]-\frac{g}{2(g-\lambda)}\left[\sum_{i=1}^{n}\varphi_i^2\right]^2\right\}\ . \tag{7}$$

Here $n$ is the replica index and $\varphi_i(i=1,2,\ldots,n)$ are now the components of a vector in replica space. In arriving at the above action, we assumed that $\lambda<g$, and our subsequent analysis is valid only in this case. The functional integral in (6) is well defined if $g\le 0$, whereas in the present case, with $g>0$, the best we can hope for is an analytic continuation from $\mathrm{Re}g<0$ to $\mathrm{Re}g>0$ (for more details on the analytic continuation, see Refs. 15 and 16). With $\lambda=0$, the action in (7) has rotational symmetry in replica space. For any finite $\lambda$ this symmetry is broken. The two fourth-order terms compete and this may lead to interesting consequences even if $\lambda<g$.

As mentioned, we use the instanton technique to calculate the averaged Green's function. Therefore, we consider the saddle point of (7). This leads to the following "classical equation of motion"

$$\left[1-\frac{d^2}{dx^2}-\frac{g}{(g-\lambda)}\sum_{j=1}^{n}\varphi_j^2+\frac{\lambda}{(g-\lambda)}\varphi_i^2\right]\varphi_i=0\ . \tag{8}$$

Apart from the trivial solution $\varphi_i=0$, this equation has spacially uniform solutions. The $k$th-type uniform solution ($k=1,2,\ldots,n$) is a replica vector, which has $k$ nonzero components, each being equal to $N(k)=\pm[(kg-\lambda)/(g-\lambda)]^{-1/2}$. The multiplicity of the $k$th-type solution is $\binom{n}{k}$ [any combination of the $k$ nonzero components is a solution to Eq. (8)]. These solutions correspond to $n$-component vectors which point to the vertices of a $n$-dimensional hypercube in replica space, reflecting the cubic symmetry of the model. Calculation of the Gaussian fluctuations around the trivial and uniform solutions, leads to a vanishing imaginary part of the Green's function. This results in zero contribution to the density of states.

Equation (8) possesses also nonuniform $k$th-type instanton solutions, with nonzero components $\chi(x)N(k)$,

$$\chi(x)=\pm\sqrt{2}\frac{1}{\cosh(x-x_0)}\ . \tag{9}$$

Here $x_0$ is the center of the instanton, which due to the translational symmetry of the action $S$ is arbitrary. The solutions with only $k=1$,

$$\varphi_1^{cl,(1)}=\begin{bmatrix}\chi N(1)\\0\\\\.\\0\end{bmatrix},\quad \varphi_2^{cl,(1)}=\begin{bmatrix}0\\\chi N(1)\\0\\.\\0\end{bmatrix},\ldots$$

$$\varphi_n^{cl,(1)}=\begin{bmatrix}0\\0\\.\\.\\\chi N(1)\end{bmatrix} \tag{10}$$

were used by Bulajevskii, Panyakov, and Sadovski[17] to study the properties of dirty superconductors.

The "classical" action on these instanton solutions is finite, and can easily be evaluated. For each of the $\binom{n}{k}$ $k$th-type instanton solutions this results in $S_k^{cl}=-4|E|^{3/2}/3(g-\lambda/k)$. Next, we evaluate the contribution of Gaussian fluctuations. Writing $\varphi_i^{(k)}=\varphi_i^{cl,(k)}+\eta_i^{(k)}$ for the $k$th-type solution, with $\varphi_i^{cl,(k)}$ being the instanton solutions discussed above, the corresponding action can be expressed as $S_k=S_k^{cl}-[|E|^{3/2}/2(g-\lambda)]\delta S_k$, $\delta S_k=\int dx \sum_{i,j}\eta_i^{(k)}(x)\ \mathbf{M}_{i,j}^{(k)}\eta_j^{(k)}(x)$. Here $\mathbf{M}^{(k)}$ is a $n\times n$ symmetric matrix in replica space. After diagonalization with a similarity transformation $\mathbf{M}^{(k)}$ becomes (we use the same notation for the diagonal form)

$$M^{(k)}=\begin{bmatrix}M_1 & & 0 & & \\ & \begin{matrix}M_2^{(k)} & 0 & 0\\ 0 & M_2^{(k)} & 0\\ 0 & 0 & M_2^{(k)}\end{matrix} & & 0 & \\ & & \begin{matrix}M_3^{(k)} & 0 & 0 & 0\\ 0 & M_3^{(k)} & 0 & 0\\ 0 & 0 & M_3^{(k)} & 0\\ 0 & 0 & 0 & M_3^{(k)}\end{matrix} & & \\ 0 & & & & \ddots\end{bmatrix} \tag{11}$$

Here $M_1$, $M_2^{(k)}$, and $M_3^{(k)}$ in the $1\times1$, $(k-1)\times(k-1)$ and $(n-k)\times(n-k)$ blocks, respectively, are operators given by

$$M_1=1-\frac{d^2}{dx^2}-3\chi^2\ , \tag{12a}$$

$$M_2^{(k)}=1-\frac{d^2}{dx^2}-\frac{kg-3\lambda}{kg-\lambda}\chi^2\ , \tag{12b}$$

$$M_3^{(k)}=1-\frac{d^2}{dx^2}-\frac{kg}{kg-\lambda}\chi^2\ . \tag{12c}$$

Some remarks on the eigenvalues of these operators can be made without any calculation. Due to the continuous translational symmetry of $S$, which is broken by the instanton solutions, Goldstone's theorem[13] requires that one of these operators have a zero eigenvalue. [In the $\lambda=0$ case, $S$ is $O(n)$ symmetric in replica space. The corresponding instanton solutions reduce this symmetry to $O(n-1)$, and there must exist additional $(n-1)$ zero eigenvalues.] Indeed, the explicit calculation of the spectra of the operators in (12) reveals[18] that $M_1$ possesses a zero eigenvalue, a single negative eigenvalue ($=-3$) and a continuous spectrum of positive eigenvalues. The operator $M_2^{(k)}$ has only positive eigenvalues, irrespective of the allowed values of $k$, $g$, and $\lambda$. The structure of the spectrum of $M_3^{(k)}$ depends crucially on the parameters. For $\lambda=0$, $M_3^{(k)}$ is independent of $k$, has a continuous positive spectrum, a single zero mode (reflecting the rotational symmetry of the action) and no discrete negative modes. When $\lambda\neq0$, the zero mode disappears, and discrete negative modes appear, whose number depends on the values of $k$, $g$, and $\lambda$.

The zero and negative modes pose a real problem when the Gaussian fluctuations are evaluated, and must be dealt with separately.[13,16] The apparent divergence caused by the zero mode can be eliminated, when translational symmetry is restored by summing the contribution of instantons with different $x_0$ (for details, see the discussion in Ref. 12). The presence of the negative eigenvalues requires the analytic continuation of the corresponding Gaussian integrals,[15,16] which may result in imaginary contributions to the Green's function. With all these difficulties out of the way, the expression for the Green's function can be written as

$$G(x,x;E)=|E|D_1^{-1/2}\lim_{n\to0}\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}\frac{k}{kg-\lambda}[D_2^{(k)}]^{(1-k)/2}\times[D_3^{(k)}]^{(k-n)/2}\exp\left[-\frac{4}{3}|E|^{3/2}\frac{k}{kg-\lambda}\right]\ . \tag{13}$$

Here $D_1$, $D_2^{(k)}$, and $D_3^{(k)}$ denote, respectively, the determinants of $M_1$, $M_2^{(k)}$, and $M_3^{(k)}$ (without zero modes).

In the $\lambda=0$ case, $D_2^{(k)}=D_3^{(k)}\equiv D_2$ in Eq. (13), independently of $k$, and the limit $n\to 0$ can easily be performed by summing the binomial coefficients. This sum leads to a factor ln2. In the noninteracting theory, defined by the action $S_0$, the final result for the Green's function is the same as (13) (with $\lambda=0$), except that ln2 is replaced by 1. This discrepancy is the consequence of the breaking of the $O(n)$ replica symmetry in the model with interaction. For finite $\lambda$ the vectors in replica space corresponding to the saddle-point solutions point towards the vertices of the unit $n$-dimensional cube. If $\lambda$ is not too small (see below), these solutions represent a good approximation to the functional integral. However, when $\lambda$ is small, other contributions, not represented by (13) become equally important. In the noninteracting model, instead of the sum in (13) over discrete directions of the saddle-point replica vectors, one integrates over all directions.[10] This leads to the factor 1.

For general $\lambda$, the evaluation of the sum in (13) in the $n\to 0$ limit is highly nontrivial. In order to proceed, we employ the result that for any power series of the form $\sum_{k=1}^{n}\binom{n}{k}a^k$, with arbitrary $a$ one has the identity[19]

$$\lim_{n\to 0}\frac{1}{n}\sum_{k=1}^{n}\binom{n}{k}a^k=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}a^k . \tag{14}$$

[The factors in (13) which are not powers of $k$, can be formally written as power series in $\lambda/k$. One can then show that an identity similar to (14) holds for each term of this series.] Next, we make use of the fact that $|E|$ is large, consequently, the major contribution in (13) comes from large values of $k$. In the large $k$ limit $M_2^{(k)}$ and $M_3^{(k)}$ become identical and assume their interaction free form, which is also independent of $k$ [see Eqs. (12b) and (12c)]. The expression for the Green's function takes the form (with $D_2$ defined above)

$$G(x,x;E)=\frac{|E|}{g}\sqrt{D_2/D_1}\times\sum_{k=1}^{\infty}\frac{(-1)^{(k+1)}}{k}\exp\left[-\frac{4|E|^{3/2}}{3(g-\lambda/k)}\right] . \tag{15}$$

By replacing the summation with integration, one can apply again the saddle-point method [with separating the even and odd $k$ terms in (15)]. The resulting integrand has a stationary point at $k^*=|E|^{3/2}\lambda/3g^2$. Since, for consistency, the value of $k^*$ must be large, it implies that our discussion is valid only for $\lambda\gg g^2/|E|^{3/2}$. In the opposite limit the Green's function can be calculated using a perturbation expansion in terms of $\lambda$. Finally, evaluating (15) at the saddle point leads to

$$G(x,x;E)\propto\frac{g^3}{\lambda^2|E|^2}\sqrt{D_2/D_1}\exp\left[-\frac{4|E|^{3/2}}{3g}\right] . \tag{16}$$

As mentioned earlier, $M_1$ has a single discrete negative eigenvalue ($D_1<0$), which leads to a purely imaginary expression for the Green's function as given by (16). (The ratio of the determinants $D_2/D_1$ can be evaluated explicitly,[20] leading to $-3$.) Clearly, the true Green's function does have a nonvanishing real part, which, however, cannot be determined with the present method.

Comparing the density of states $n_0$ and $n$, respectively, for the interaction free case [expression (1)] and in the presence of interaction [derived from (16)], in the limit of large negative energies, we obtain

$$\frac{n}{n_0}\propto\frac{g^4}{\lambda^2|E|^3} , \tag{17}$$

which is the main result of this work. Expression (17) is valid for $g>\lambda\gg g^2/|E|^{3/2}$.

In conclusion, we studied the problem of the Lifshitz tail in a disordered model system of bosons with a "minimal" repulsive interaction. The interaction term was obtained by retaining in the action of the standard field-theoretic formulation of the general problem only a subset of Fourier components of the bosonic fields in the energy variable. As a consequence, the application of the nonperturbative instanton method allows an asymptotically exact calculation of the one-particle Green's function. The result for the density of states, when compared with the corresponding expression for the free case [see (17)], shows that the interaction (at least within the present model) tends to diminish the Lifshitz tail. This conclusion is reminiscent to that of Efros and Shklovsky, who argued that in the localized phase of disordered electronic systems repulsive interactions lead to the vanishing of the density of states at the Fermi level, i.e., to the so-called Coulomb gap.[21] An additional interesting feature of (17) is that for a given value of $\lambda$ the relative magnitude of the interacting density of states decreases with diminishing disorder.

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