Incompressible quantum Hall states in Josephson-junction arrays

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(Received 19 May 1993; revised manuscript received 23 August 1994)

We have shown that the charges in two-dimensional arrays of small Josephson junctions form Laughlin incompressible states in a magnetic field, so that the charge density is fixed in finite intervals of external parameters. Thus the system resembles a two-dimensional electron gas in the quantum Hall regime. In contrast to the latter one, the system is shown to be equivalent to a Bose gas with a weak contact repulsive interaction. We present numerical results for the main plateau in the charge density and also a toy model that enables us to develop the picture of all plateaus.

A two-dimensional Josephson junction array consists of a set of small superconducting islands connected by tunnel junctions. Such systems are now a subject of intensive theoretical[1–3] and experimental[4] investigations (see Ref. 5 for review). Their properties are primarily governed by *macroscopic* quantum effects. Indeed, the state of the array may be completely described either in terms of phase differences between the islands or in terms of their quantum conjugated variables, the numbers of Cooper pairs on each island. The latter description corresponds to a picture of discrete charges hopping coherently from island to island and interacting with one another.

This picture resembles the *microscopic* description of a two-dimensional gas of interacting electrons which exhibits two remarkable phenomena: the integer and fractional quantum Hall effects (FQHE).[6] It was shown[7] that the fractional quantum Hall effect has its origin in *incompressible* states. In these states the particle density is fixed in a finite range of chemical potential.

A question of interest for us is how deep the similarity between these two physical systems is. We will show that under certain conditions the extra Cooper pairs (or simply charges) on the islands of the array form a low density phase. In this phase the Coulomb interaction between the charges is strong, but its range is short in comparison with the average interparticle distance. Hence, at low energies the charges avoid one another within the interaction range, which effectively reduces the otherwise strong interaction. For this reason we establish that the system in a low density phase is equivalent to a Bose gas with *weak* contact repulsion. This equivalence allows us to show explicitly that the charges form Laughlin-like incompressible states in a magnetic field. As a result, the charge density in the array changes stepwise as a function of external parameters. Some results of this work have been published in a short version.[8]

Let us start with a quantitative description of the system. A regular infinite Josephson array (Fig. 1) is characterized by the Josephson energy $E_J = \hbar I_c/2e$ of the junctions, the capacitance C of the junctions, and the capacitance C_0 of the islands to the ground and to the gate electrode. The Hamiltonian of the system[5,9] can be written in the charge representation,

$$H = \sum_{(ij)} \frac{E_J}{2}(|n_i+1, n_j-1\rangle\langle n_i, n_j|e^{iA_{ij}} + \text{H.c.}) + \frac{(2e)^2}{2}\sum_{i,j} C_{ij}^{-1} n_j n_i - 2eV\sum_i n_i, \quad (1)$$

where n_i is the number of Cooper pairs on the island i. The first term describes Josephson tunneling and may look more familiar in the phase representation: $E_J\sum_{(ij)}\cos(\phi_{ij} - A_{ij})$ where the sum is taken over the neighboring islands. A transverse magnetic field is described by the vector potential $A_{ij} = (2e/\hbar c)\vec{A}\cdot\vec{r}_{ij}$ (we measure distances in units of lattice spacing a). The Coulomb interaction is determined by the inverse capacitance matrix C_{ij}^{-1}. By changing the gate voltage V one can alter the concentration of charges in the array.

The properties of the array depend crucially on the relation between the Josephson energy E_J and charging energy $E_C = (2e)^2/\max(C, C_0)$. We concentrate on the limit $E_C \gg E_J$ when the charge description is more appropriate and no stable vortices can occur. To be concrete, we assume that $C \gg C_0$. For $E_J \to 0$ the state of the system is determined by the electrostatics. The charges enter the array if the voltage V exceeds the Coulomb blockade threshold $V_t = eC_{00}^{-1}$.[10] It is crucial that the repulsive interaction between the charges in the

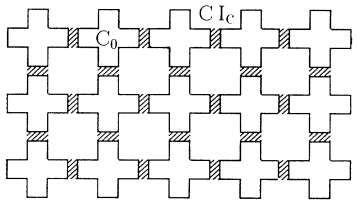

FIG. 1. The layout of a two-dimensional Josephson junction array.

array has a finite range $\Lambda = \sqrt{C/C_0} \gg 1$.[5] For this reason, the concentration n of charges jumps sharply from zero to values of the order of Λ^{-2} slightly above V_t.

This does not happen for small but finite E_J, since the charges gain kinetic energy which increases with increasing density. As a result, the concentration increases linearly with gate voltage $n \propto V - V_c$ above the threshold $V_c \equiv V_t - E_J/e$, and the phase with low density $n \ll \Lambda^{-2}$ exists in a relatively broad interval $0 < V - V_c \lesssim E_J C_0/(eC)$ (see below). At higher voltages the charges become localized, forming a sort of Wigner lattice. Let us note, that a long range interaction would result in the formation of a Wigner lattice at arbitrary low charge density.

In the low density phase ($n \ll \Lambda^{-2}$) two charges cannot be on the same island since it costs a large amount of energy $\simeq E_C$. Configurations with lower energies have either one or no charges on each island, enabling a description in terms of the *coordinates* of charges.[2] The configurations obtained by permutation of coordinates are identical, which implies *Bose* statistics of charges. Moreover, at still lower energies, $E \ll E_J$, an effective mass approximation is valid, and the original Hamiltonian (1) can be mapped onto the Hamiltonian of interacting bosons in the continuous limit,

$$H = \sum_i (p_i - 2eA/c)^2/2M + \sum_{i<j} U(r_i - r_j) - \mu N, \quad (2)$$

where $M \equiv \hbar^2/E_J$ and $\mu \equiv 2e(V - V_c)$. The interaction $U(r) = (E_C/2\pi)K_0(r/\Lambda)$ is logarithmic at small distances $r \ll \Lambda$ and decreases exponentially $U(r) \simeq (E_C/2\pi)\exp(-r/\Lambda)$ at $r \gg \Lambda$.

We consider first the ground state of the system in zero magnetic field. For a naive picture of uncorrelated charges one gets an unphysical result: the potential energy per particle, $U_p \simeq n\Lambda^2 E_C$, is much larger than the kinetic energy, $K_p \simeq \hbar^2 n/M = nE_J$. The potential energy U_p can be decreased substantially if one introduces the correlations preventing the particles to be close (to distances $\sim \Lambda$) to each other. To do this, we present a ground state as a product of Jastrow functions $\Psi(r_1, \ldots, r_N) = \prod_{i<j} \Phi(r_i - r_j)$ which were successfully used in the study of strongly interacting Bose systems like ^{4}He.[11] We will normalize the wave function in such a way that $\Phi(r \to \infty) = 1$.

Since the range of the interaction is small, $\Lambda \ll n^{-1/2}$, one can assume that the function $\Phi(r)$ changes at small distances, $r \ll n^{-1/2}$. In this case the pair distribution function $g(r)$ is simply related to the Jastrow functions, $g(r) = |\Phi(r)|^2$ and the standard expression[11] for the energy of the system reads $E = N(N-1)W_0/S$, where

$$W_0 = \int d^2r \{|\hat{p}\Phi(r)|^2/2M + U(r)|\Phi(r)|^2/2\} \quad (3)$$

and S is the area of the array. Since the pair distribution function tends to unity for $r > n^{-1/2}$, we impose the boundary condition $\Phi(r) = 1$ for $r > l$, where $l \sim n^{-1/2}$ is a variational parameter.

The function $\Phi(r)$ corresponding to the minimum of energy (3) is almost zero within the interaction range and increases logarithmically $\Phi(r) = \ln(r/\Lambda)/\ln(l/\Lambda)$ for $\Lambda \lesssim r < l$. In what follows we will concentrate on the limit of very small radius of the interaction, $\ln(n^{-1/2}/\Lambda) \gg 1$. In this case, the main contribution to the energy comes from the integration (3) over the range $\Lambda \ll r \ll l$. For this reason, the energy $W_0 = \pi/M \ln(l/\Lambda)$ does not depend much on details of the hard-core interaction potential (characterized only by its range Λ) and on the parameter l. Choosing $l = n^{-1/2}$ (Ref. 12) we obtain the energy $E_0 \equiv E(N+1) - E(N)$ needed to add a particle to the system,

$$E_0 = 2\pi n E_J/\mathcal{L}, \ \mathcal{L} = \ln(n^{-1/2}/\Lambda) + O(1). \quad (4)$$

Therefore, the concentration of charges increases linearly with the chemical potential $n = \mu\mathcal{L}/(2\pi E_J)$ in the low density phase $n\Lambda^2 \ll 1$. The latter exists in the range $0 < \mu \lesssim E_J C_0/C$, as we mentioned above.

Treating the problem in the presence of a magnetic field, we use standard notation for the quantum Hall effect: the Landau level separation $\hbar\omega_c \equiv 2eH\hbar/Mc$, the magnetic length $l_0 \equiv (\hbar c/2eH)^{1/2}/a$, and the filling factor $\nu \equiv 2\pi l_0^2 n$ (here a is a lattice spacing of the array). On the other hand, in Josephson arrays the natural units are E_J and frustration $f = Ha^2/\Phi_0$. The following relations connect these two systems of notation: $\hbar\omega_c \equiv 2\pi f E_J$, $l_0 = 1/\sqrt{2\pi f}$, $\nu = n/f$.

We consider now strong magnetic field $\nu \sim 1$ (or $Ha^2/n \sim \Phi_0$). Let us note that the kinetic energy per particle $K_p \geq \hbar\omega_c/2 = \pi n E_J/\nu$ is much larger than the energy E_0 (4) in the interacting case without the field. It is natural to assume that the increase of the energy due to the interaction is of the order of E_0 also in the magnetic field. This means that the wave function cannot be substantially modified by the interaction *globally*. In particular, it still varies slowly $(d\Psi/dr)/\Psi \simeq l_0^{-1}$ at large interparticle distances $|r_i - r_j| \sim l_0 \simeq n^{-1/2}$. We, therefore, present the trial wave function in a form of a product

$$\Psi(r_1, \ldots, r_N) = \tilde{\Psi}(r_1, \ldots, r_N) \prod_{i<j} \Phi(r_i - r_j). \quad (5)$$

between a part $\tilde{\Psi}(r_1, \ldots, r_N)$ which varies at distances $\sim l_0$ and Jastrow functions $\Phi(r)$ which change on a much shorter scale $r < l \ll l_0$ [$\Phi(r) = 1$ for $r > l$].

The energy (2) of the state (5) consists of two contributions, $E = K + U_{eff}$. The first term comes from the integration over a part of the configuration space $\vec{r} = (r_1, \ldots, r_N)$, where no particles are close to each other ($|r_i - r_j| > l$). It simply describes the kinetic energy for a slowly varying part of the wave function, $K = N \int d^{2N}\vec{r} |(p_1 - 2eA(r_1)/c)\tilde{\Psi}|^2/2M$. The integration over the rest of the configuration space gives rise to the energy $U_{eff} = N(N-1)W_0 \int d^{2N}\vec{r}\,\delta(r_1 - r_2)|\tilde{\Psi}|^2$ due to local modification of the wave function. Note that the vector potential still does not enter in the formula (3) for W_0, since $\Phi(r)$ changes fast on the scale l_0. Therefore, W_0 is given by its zero field expression, $W_0 = \pi/M \ln(l/\Lambda)$. Although we formally have considered $l \ll l_0 \simeq n^{-1/2}$,

one is allowed to change l with l_0 (or with $n^{-1/2}$) under the large logarithm, $\ln(l/\Lambda) \to \mathcal{L}$, see Eq. (4).

Estimating the energies we have assumed that $\ln(n^{-1/2}/\Lambda) \gg 1$. In order to check the quality of our approximations for not too small values of $\Lambda/n^{-1/2}$ we have evaluated the energy E of two particles in an eigenstate with zero angular momentum. We assumed a hard wall interaction potential $U(r) \to \infty$ for $|r| < \Lambda$ and $U(r) = 0$ otherwise. Due to the interaction the energy increases by an amount $\Delta E \equiv E - \hbar\omega_c = 0.1115, 0.2263, 0.7745$ (in units $\hbar\omega_c/2$) for $\Lambda/2l_0 = 10^{-4}, 10^{-2}, 0.3$. The results of our approximation $U_{eff} = 0.1086, 0.2171, 0.8306$ [for $\mathcal{L} = \ln(2l_0/\Lambda)$] deviate from exact values only by a few percent (2.7%, 4%, and 7.2%, respectively).

The kinetic and potential energies obtained above correspond to the following effective Hamiltonian for the slowly varying part $\tilde{\Psi}(\vec{r})$ of the wave function,

$$H_{eff} = \frac{1}{2M}\sum_i (p_i - 2eA/c)^2 + \frac{2\pi\hbar^2}{M\mathcal{L}}\sum_{i<j}\delta(r_i - r_j) - \mu N. \quad (6)$$

It is important that the new contact interaction (or pseudopotential) is weak $U_{eff}/K \lesssim \mathcal{L}^{-1} \ll 1$. For this reason the system is in the fractional quantum Hall regime. From now on we will work with the effective Hamiltonian (6).

For the filling factors $\nu = 1/2m$ we apply the Laughlin trial wave function[7] for bosons,

$$\tilde{\Psi}_m = \prod_{i<j}(z_i - z_j)^{2m} \exp\left(-\sum_i |z_i|^2/4l_0^2\right), \quad (7)$$

where $z_j \equiv x_j + iy_j$. The potential energy (6) is exactly *zero* since in the state $\tilde{\Psi}_m$ two charges cannot both be at the same point. For this reason at $\mu = \hbar\omega_c/2$ the charges will enter the array without increasing the energy as long as they can form Laughlin states. Their density will be as high as possible, which corresponds to $m = 1$, $\nu = 1/2$. A further increase in density will be blocked by the incompressibility of the Laughlin state. Indeed, in order to increase the density one should create a "quasielectron" excitation.[7] The excitation energy Δ must be of the order of the potential energy (6) for an uncorrelated particle, $\Delta \simeq E_0$, $E_0 = \hbar\omega_c/2\mathcal{L}$ (4). In order to put one charge in the array two quasielectrons should be created.

Now we are able to formulate the main result of this work. Under the conditions we have specified the charge density changes abruptly from zero to $\nu = 1/2$ and remains constant in the interval $0 < \tilde{\mu} < 2\Delta$, where $\tilde{\mu} \equiv \mu - \hbar\omega_c/2$. Thus, an incompressible Laughlin state may occur in the Josephson junction array.

Numerical studies of the systems with a small number of particles are proven to be very useful in analyzing incompressible states and their excitations.[7] In order to evaluate the energy Δ we have performed symbolic computations[13] for systems of several $N = 4, 5$ particles. We tried quasielectron wave functions proposed by Laughlin[7] and Jane;[14] these were modified according to Bose statistics. For the first choice we obtained $\Delta/E_0 = 0.686, 0.672$ for $N = 4$ and 5, respectively. The second choice gives very similar results: $\Delta/E_0 = 0.669, 0.637$. Therefore, we can conclude that $\Delta/E_0 = 0.64 \pm 0.03$. The width of the main plateau at $\nu = 1/2$ is marked by an arrow in Fig. 2.

The scheme we used to derive Eq. (6) may be extended to calculate the effective potential energy in higher orders and thus resolve the fine structure of density plateaus at $\nu < 1/2$. In this way we estimate the energy per particle $E_m \simeq E_0(\Lambda/l_0)^{4m}$ in the Laughlin state $\tilde{\Psi}_m$. This state occurs in the interval $E_m < \tilde{\mu} < E_{m-1}$. Since $l_0 \gg \Lambda$, the widths of the intervals rapidly decrease with m. Therefore, the filling factor as a function of μ rapidly changes from $\nu = 0$ to $\nu = 1/2$ in a narrow region $0 < \tilde{\mu} < E_1$ as shown in the inset of Fig. 2.

The question remains, how the charge density changes with further increase of the chemical potential. Several ways were proposed to construct the hierarchy of incompressible states in a two-dimensional electron gas.[15,7,14] We have modified the generation rules for bosons and computed the energies of the corresponding states for few particles. The results were close to the values (4) for uncorrelated particles and we were unable to resolve new incompressible states. One of the possible reasons is that the standard schemes may not adequately describe the correlations in systems of bosons, especially at large filling factors $\nu > 1$.

To grasp the correlations at arbitrary ν we develop a simple "toy" model based on the following trial wave function:

$$\tilde{\Psi}(z_1, \ldots, z_N) = \prod_{k=1}^{l} \Psi^{(k)}(z_{n_{k-1}+1}, \ldots, z_{n_k}), \quad (8)$$

with $0 = n_0 < n_1 < \cdots < n_l = N$. Here $\Psi^{(k)}$ are the Laughlin wave functions (7) for $m = 1$, possibly with

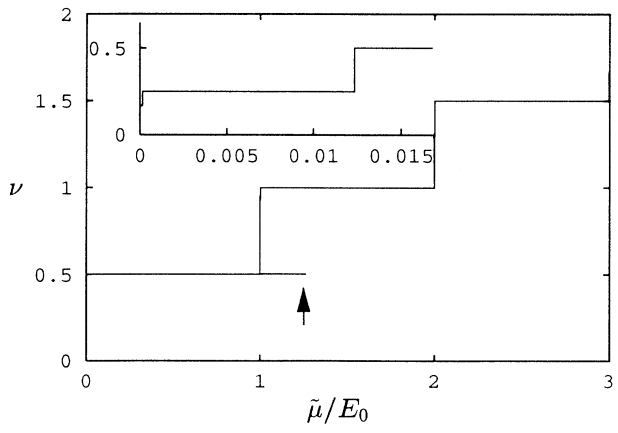

FIG. 2. Filling factor of charges versus chemical potential. The arrow shows the width of the main plateau ($\nu = 1/2$), computed numerically. The ladder corresponds to a toy model. Inset: A set of exponentially narrowing plateaus for higher Laughlin fractions, $\nu = 1/2, 1/4, 1/6$. We have chosen $\Lambda/l_0 = 1/3$.

holes, thus corresponding to $\nu_k \leq 1/2$. Different functions Ψ_k relate to different groups of particles, and the total filling factor is $\nu = \sum_{k=1}^{l} \nu_k$. The interaction occurs only between particles of different groups, and the energy density is given by $E/S = fE_0(\nu^2 - \sum_k \nu_k^2)$. This should be minimized with respect to n_k and l at a given ν. As a result we obtain that all Laughlin states are completely filled, $\nu_k = 1/2$ for $k = 1, \ldots, l-1$, except one state with a smaller filling $\nu_l = \nu - (l-1)/2 < 1/2$. The energy density E/S is a linear function of ν with cusps at $\nu = l/2$. Hence, the state with $\nu = l/2$ is incompressible in the range $(l-1)E_0 < \tilde{\mu} < l$ of a chemical potential. The dependence of a filling factor on the chemical potential (Fig. 2) looks similar to the one for the *integer* quantum Hall effect. This may be surprising since the latter has its origin in Fermi statistics, whereas in our case the incompressibility arises from the interaction of particles.

We wish the toy wave function (8) were symmetric with respect to particle permutations. Unfortunately, we did not manage to estimate rigorously the energy for the symmetrized wave function. Why do we think that we can use the toy model at least as a guideline? First, it reproduces on average the linear increase of the concentration n with the chemical potential [see Eq. (4)]. Second, the width of the first plateau predicted by the toy model is in good agreement with our numerical results (see Fig. 2).

Making use of the analogy to the FQHE, we conjecture that the incompressible states manifest themselves in a quantization of the Hall conductance of the array. Indeed, in the absence of dissipation the Hall conductance $\sigma_{xy} = (2e)^2\nu/h$ is fixed if ν is fixed. According to our results, we expect at least two sets of Hall plateaus. One occurs near the threshold voltage and corresponds to the *fractional* quantum Hall effect, $\sigma_{xy} = 2e^2/hm$. Another is predicted by the toy model and corresponds to the *integer* quantum Hall effect, $\sigma_{xy} = 2e^2 l/h$.

The effect can be observed if the charges in the array form a low density phase. For uniform arrays with $E_J \ll E_C$ and $C_0 \ll C$, this phase exists in the range $0 < V - V_c \lesssim E_J C_0/(eC)$ of gate voltage. The irregularities in the array (nonuniform parameters of the cells and random offset charges at the islands) may suppress the effect, just as disorder extinguishes the FQHE in two-dimensional electron gas. An analysis of incompressible states in arrays with such irregularities seems to be an interesting problem for future investigations.

We wish to thank C. Bruder, D. E. Khmelnitsky, K. K. Likharev, A. van Otterlo, M. Tinkham, G. Zimanyi, and especially G. Schön and J. E. Mooij for many valuable discussions. We thank D. V. Averin, R. Chen, and L. Fonseco for critical reading of the manuscript. This work was supported in part by SFB 195 of the DFG. One of us (A.O.) appreciates the hospitality of SUNY at Stony Brook where part of this work was completed.

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[1] U. Eckern and A. Schmid, Phys. Rev. B **39**, 6441 (1989); B. J. van Wees, Phys. Rev. Lett. **65**, 255 (1990).

[2] R. Fazio and G. Schön, Phys. Rev. B **43**, 5307 (1991).

[3] M. P. A. Fisher, Phys. Rev. Lett. **65**, 923 (1990); C. Bruder, R. Fazio, and G. Schön, Phys. Rev. B **47**, 342 (1993).

[4] H. S. J. van der Zant, F. C. Fritschy, W. J. Elion, L. J. Geerligs, and J. E. Mooij, Phys. Rev. Lett. **69**, 2971 (1992); T. S. Tighe, M. T. Tuominen, J. M. Hergenrother, and M. Tinkham, Phys. Rev. B **47**, 1145 (1993).

[5] J. E. Mooij and G. Schön, in *Single Charge Tunneling,* Vol. 294 of NATO *Advanced Study Institute, Series B: Physics,* edited by H. Grabert and M. H. Devoret (Plenum Press, New York, 1992).

[6] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980); D. C. Tsui, H. L. Störmer, and A. C. Gossard, *ibid.* **48**, 1559 (1982).

[7] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983); T. Chakraborty and P. Pietiläinen, *The Fractional Quantum Hall Effect* (Springer-Verlag, Berlin, 1988).

[8] Yu. V. Nazarov and A. A. Odintsov, Physica B **194-196**, 1737 (1994).

[9] One can neglect the quasiparticle tunneling if the charging energy and the temperature are small compared to the superconducting gap; see P. Lafarge, P. Joyez, D. Esteve, C. Urbina, and M. H. Devoret, Nature (London) **365**, 422 (1993).

[10] N. S. Bakhvalov, G. S. Kazacha, K. K. Likharev, and S. I. Serdyukova, Physica B **173**, 319 (1991).

[11] G. D. Mahan, *Many-Particle Physics* (Plenum Press, New York, 1990), Chap. 10.

[12] Formally the energy decreases with the increase of l. On the other hand, our approximation $g(r) = |\Phi(r)|^2$ is valid if $l \ll n^{-1/2}\ln(l/\Lambda)$. Our choice $l = n^{-1/2}$ is near the upper boundary of this range. A rigorously estimated contribution to the energy from the integration of the energy density[11] over the range $r > n^{-1/2}$ is small [of the order of $O(1)$ in Eq. (4)] because of a fast decay of $1 - g(r)$.

[13] We appreciate the MAPLE software package.

[14] J. K. Jane, J. Phys. Chem. Solids **51**, 889 (1990).

[15] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984); **52**, 2390(E) (1984).