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Cooperons at the metal-insulator transition revisited: Constraints on the renormalization group and a conjecture

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The effect of Cooperons on the metal-insulator transition in disordered interacting electronic systems is studied. We point out that a proper incorporation of Cooperons into the disordered electron problem must respect a Bethe-Salpeter equation for the effective Cooper interaction amplitude Γ^c. This puts constraints on renormalization-group treatments of the problem. We discuss existing renormalization group approaches, both of the field-theoretic and of the momentum-frequency shell variety, and show that none of them are technically satisfactory. A general analysis of the Bethe-Salpeter equation shows that all possible solutions fall into one of three classes which differ with respect to the scaling behavior of Γ^c. We argue that both of the physically most plausible possibilities lead to logarithmic corrections to scaling, and discuss the experimental implications of this conjecture.

I. INTRODUCTION

The current theoretical description of the metal-insulator transition (MIT) suffers from the fact that there is no consensus about the effects of the particle-particle or Cooper channel on the MIT.[1] The authors who first studied this problem concluded that the presence or absence of the Cooper channel does not qualitatively modify the MIT.[2,3] At first sight this may seem surprising, as the widespread interest in the localization problem was sparked by work on the Cooper channel and the backscattering or weak localization effects it produces.[4] Also, numerous experiments confirmed the presence of these weak localization effects in weakly disordered metallic systems in the absence of magnetic impurities and magnetic fields.[5] However, the assertion appears less surprising if one recalls that electron-electron interaction effects in the presence of disorder lead to many of the same effects as Cooperons.[6] If this is so in the weak disorder regime, and if one acknowledges that electron-electron interactions are in general relevant for the MIT, then it is conceivable that Cooperons do not lead to any additional effects at the MIT over and above those produced by the interplay of interactions and disorder alone. This was in fact the conclusion reached by Finkel'stein[2] and by Castellani *et al.*,[3] who have argued that the Cooper channel is irrelevant, in the sense of the renormalization group (RG), for the MIT, that interaction effects effectively replace Cooperon effects near the MIT, and that MIT with or without Cooperons are qualitatively the same. On the other hand, more recently the present authors have argued that the effective Cooper interaction amplitude is a marginal operator rather than an irrelevant one, and that this marginal operator leads to logarithmic corrections to scaling that are characteristic for those universality classes where Cooperons are present.[7] Since the RG techniques employed in these two respective approaches were quite different it is very hard to see any relations between them and to tell which, if any, of the two results is the correct one. This problem is of more than purely academic interest since it was shown in Ref. 7 that the logarithmic corrections to scaling, if they exist, can reconcile the observed values $s \lesssim 2/3$ of the conductivity exponent s in Si:P (Ref. 8) and some other systems,[9,10] with the rigorous bound[11] that requires $s \geq 2/3$ in three-dimensional (3D) systems.

The present paper is an attempt to clarify this issue. We first show that the Cooper channel problem can be cast in the form of a Bethe-Salpeter equation for the effective Cooper interaction amplitude. This integral equation determines the quantity which the previous treatments disagreed upon, and all perturbative RG calculations must be consistent with it. We then proceed to show that both the field theoretic RG methods[7] and the momentum shell-like methods,[2,3] as previously employed, contain ambiguities, and do not lead to a finite renormalized theory. The origin of these problems is traced back to hidden assumptions in the RG treatments. We conclude that the theory is not renormalizable with the number of renormalization constants assumed for the field-theoretic treatment,[7] and that any momentum shell-type RG produces new terms in the action which are difficult to handle and were ignored in previous treatments.[2,3]

While we are currently unable to remedy the technical problems with the RG treatments, we can classify possible solutions by means of a very general analysis of the Bethe-Salpeter equation. The two most interesting, and physically most plausible, of these solutions predict that in systems without magnetic impurities or external magnetic fields there are logarithmic corrections to scaling at

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the MIT in agreement with the conclusion of Ref. 7.

The plan of this paper is as follows. In Sec. II we consider Finkel'stein's effective field theory for the MIT,[2] and calculate the perturbative corrections to the coupling constants to one-loop order. We also derive the Bethe-Salpeter equation that relates a general, frequency dependent Cooper interaction amplitude γ_c to the Cooper propagator Γ^c. In Sec. III we use field-theoretic renormalization methods to renormalize the parameters that appear in the particle-hole channel. We then discuss both of the previous attempts to renormalize the Cooper channel in order to obtain the scaling behavior of Γ^c, and clarify the origin of the mutual inconsistency of these results. In Sec. IV we classify possible solutions of the Bethe-Salpeter equation with respect to the scaling behavior of Γ^c. This provides us with three possible scaling scenarios for the problem, two of which lead to logarithmic corrections to scaling. Since the structure of the perturbation theory makes the third one unlikely, this leads us to the conjecture that logarithmic corrections to scaling are present in all universality classes that allow for Cooperons. In Sec. V we discuss the experimental implications of this conjecture. In the Appendix we outline the use of the Wilsonian RG for the Cooperon problem.

II. THE FIELD THEORY AND THE LOOP EXPANSION

In the first part of this section we recall the basic field-theoretic description of the disordered electron problem[2,12] and derive the Gaussian propagators of the field theory. Since the technical details of this model have been reviewed in Ref. 1 we will keep this brief. We then explain how to obtain perturbative corrections to the coupling constants by considering the vertex functions to one-loop order. Finally, we derive a Bethe-Salpeter equation for the Cooper propagator.

A. The model

We consider the generalized nonlinear σ model for interacting electrons in the presence of disorder.[2] It is an effective model which is designed to capture the physics determined by the slow modes related to conservation laws, i.e., the diffusion of mass, spin, and energy density. The action can be written[1]

$$S[Q] = -\frac{1}{2G}\int d\mathbf{x}\, \mathrm{tr}\Big(\nabla Q(\mathbf{x})\Big)^2 + 2H\int d\mathbf{x}\, \mathrm{tr}\Big(\Omega Q(\mathbf{x})\Big) - \frac{\pi T}{4}\sum_{u=s,t,c} K_u \int d\mathbf{x}\, \big[Q(\mathbf{x})\circ Q(\mathbf{x})\big]_u. \tag{2.1}$$

Here the field variable Q is a Hermitian, traceless, infinite matrix whose matrix elements, $Q_{nm}^{\alpha\beta}$, are complex 4×4 matrices (spin-quaternions) which comprise the spin and particle-hole degrees of freedom. The labels $\alpha,\beta = 1,2,\ldots,N$ denote replica labels. In deriving Eq. (2.1), quenched disorder has been integrated out by means of the replica trick,[13] and the limit $N\to0$ is implied at the end of all calculations. $n,m=-\infty,\cdots,+\infty$ are Matsubara frequency labels. $\Omega = \mathbb{1}\omega_n$ with $\mathbb{1}$ the identity matrix, $\omega_n = 2\pi T(n+1/2)$, is a fermionic frequency matrix, and tr denotes a trace over all discrete degrees of freedom. The terms $[Q\circ Q]_u$ in Eq. (2.1) are bilinear in Q. They describe the electron-electron interaction, and the explicit forms of the "products" $[Q\circ Q]_u$ can be found in Refs. 1 and 2. $G = 8/\pi\sigma_B$ with σ_B the bare or self-consistent Born conductivity is a measure of the disorder, and $H=\pi N_F/4$ plays the role of a frequency coupling parameter with N_F the bare density of states (DOS) at the Fermi level. K_s and K_t are singlet and triplet particle-hole interaction constants, respectively, and K_c is the singlet particle-particle or Cooper channel interaction constant. At zero frequency the triplet coupling constant in the particle-particle channel vanishes due to the Pauli principle.[14] For simplicity we formulate the theory with a short-range model interaction, i.e., the $K_{s,t,c}$ are simply numbers. For the more realistic case of a Coulomb interaction K_s is x dependent and must be kept under the integral in Eq. (2.1). Most results for this case are easily obtained after all calculations have been performed by essentially putting $K_s = -H$.[2] In Sec. III we will give results for both the short-range and the Coulomb interaction cases.

The matrix Q is subject to the nonlinear constraint,

$$Q^2 = 1. \tag{2.2}$$

This constraint, and the requirements of Hermiticity and zero trace can be eliminated by parametrizing the matrix Q by,[15]

$$Q = \begin{cases} (1-qq^\dagger)^{1/2}-1 & \text{for } n\ge0,\quad m\ge0 \\ q & \text{for } n\ge0,\quad m<0 \\ q^\dagger & \text{for } n<0,\quad m\ge0 \\ -(1-q^\dagger q)^{1/2}-1 & \text{for } n<0,\quad m<0. \end{cases} \tag{2.3a}$$

Here the q are matrices with spin-quaternion valued elements $q_{nm}^{\alpha\beta}$; $n=0,1,\ldots$; $m=-1,-2,\ldots$. It is convenient to expand them in a spin-quaternion basis,

$$q_{nm}^{\alpha\beta} = \sum_{r=0}^{3}\sum_{i=0}^{3} {}_r^i q_{nm}^{\alpha\beta}(\tau_r\otimes s_i). \tag{2.3b}$$

Here $\tau_0 = s_0 = \sigma_0$, and $\tau_j = -s_j = -i\sigma_j$ $(j=1,2,3)$, with σ_j the Pauli matrices. The τ_i are the quaternion basis and span the particle-hole and particle-particle space, while the s_i serve as our basis in spin space.

We have so far given the theory for the so-called generic (G) universality class which is realized by systems without magnetic fields, magnetic impurities, or spin-orbit scattering. Apart from class G, the second universality class with Cooperons is the one with strong spin-orbit scattering (class SO). For class SO the action is shown as above, except that the particle-hole spin triplet channel is absent, i.e., the sum over the spin index i in Eq. (2.3b) is restricted to $i=0$.[16] In what follows we will give results for both class G and class SO.

With the help of Eqs. (2.3) one can expand the action in powers of q,

$$S[Q]=\sum_{n=2}^{\infty}S_n[q], \tag{2.4}$$

where $S_n[q]\sim q^n$. We first concentrate on the Gaussian part of the action,[17]

$$S_2[q]=-4\int_{\mathbf{p}}\sum_{r,i}\sum_{1,2,3,4}{}^i_rq_{12}(\mathbf{p})\,{}^i_rM_{12,34}\,{}^i_rq_{34}(-\mathbf{p}). \tag{2.5a}$$

Here $\int_{\mathbf{p}}\equiv\int d\mathbf{p}/(2\pi)^D$, and $1\equiv(n_1,\alpha_1)$, etc. The matrix M is given by

$${}^i_{0,3}M_{12,34}(p)=\frac{\delta_{1-2,3-4}}{G}\{\delta_{13}[p^2+GH(\omega_{n_1}-\omega_{n_2})]+\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}2\pi TGK_{\nu_i}\}, \tag{2.5b}$$

where $\nu_0=s,\nu_{1,2,3}=t$, and

$${}^i_{1,2}M_{12,34}(p)=-\frac{\delta_{1+2,3+4}}{G}\{\delta_{13}[p^2+GH(\omega_{n_1}-\omega_{n_2})]+\delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\delta_{i0}2\pi TGK_c\}. \tag{2.5c}$$

The Gaussian two-point propagators are given in terms of the inverse of M. They can be put into a standard form by summing over, e.g., n_3 and n_4,

$$\sum_{n_3\geq 0,n_4<0}{}^i_{0,3}M^{-1}_{12,34}(p)/G=(1-\delta_{\alpha_1\alpha_2})\mathcal{D}_{n_1-n_2}(p)+\delta_{\alpha_1\alpha_2}\mathcal{D}^{\nu_i}_{n_1-n_2}(p), \tag{2.6a}$$

$$\sum_{n_3\geq 0,n_4<0}{}^i_{1,2}M^{-1}_{12,34}(p)/G=-(1-\delta_{\alpha_1\alpha_2})\mathcal{D}_{n_1-n_2}(p)-\delta_{\alpha_1\alpha_2}(1-\delta_{i0})\mathcal{D}_{n_1-n_2}(p)-\delta_{i0}\delta_{\alpha_1\alpha_2}\times\frac{\mathcal{D}_{n_1-n_2}(p)}{1+G2\pi TK_cf_{n_1+n_2}(p)},$$

where

$$f_n(p)=\sum_{n_1\geq 0,n_2<0}\delta_{n,n_1+n_2}\mathcal{D}_{n_1-n_2}(p). \tag{2.6b}$$

Here we have introduced the propagators

$$\mathcal{D}_n(p)=[p^2+GH\Omega_n]^{-1}, \tag{2.7a}$$

$$\mathcal{D}^{s,t}_n(p)=[p^2+G(H+K_{s,t})\Omega_n]^{-1}, \tag{2.7b}$$

$$\Delta\mathcal{D}^{s,t}_n(p)=\mathcal{D}^{s,t}_n(p)-\mathcal{D}_n(p), \tag{2.7c}$$

with $\Omega_n=2\pi Tn$ a bosonic Matsubara frequency. Physically, $\mathcal{D}_n$, $\mathcal{D}^s_n$, and $\mathcal{D}^t_n$ are the energy, mass, and spin diffusion propagators.[18] Examining the various terms in Eqs. (2.6) we see that all of them have a standard propagator structure except for the last contribution in Eq. (2.6b). In interpreting these propagators as having a standard structure, the Matsubara frequencies in Eqs. (2.7a)–(2.7c) are taken to be analogous to a magnetic field at a magnetic phase transition, i.e., the MIT occurs at $\Omega_n\to 0$ and Ω_n or the temperature is a relevant perturbation in the RG sense. Using Eqs. (2.6b) and (2.7a), changing the sum to an integral, and placing an ultraviolet cutoff Ω_0 on the resulting frequency integrals shows that $2\pi Tf_{n_1+n_2}(p)$ in Eq. (2.6b) diverges logarithmically in the long wavelength, low-temperature limit. With $K_c>0$, we see that the last term in Eq. (2.6b) is logarithmically small compared to the other terms in Eqs. (2.6).

We conclude this subsection with two remarks. First, the logarithm discussed above that appears in the particle-particle density correlation function is just the usual BCS logarithm. However, since we consider a system with a repulsive Cooper channel interaction, $K_c>0$, which is not superconducting in the clean limit, this does not lead to a Cooper instability. Rather, the last term in Eq. (2.6b) vanishes logarithmically in the limit $p,T\to 0$. If the structure of this term persists for disorder values up to the MIT, and if it couples to the physical quantities such as, e.g., the conductivity, then it will lead to logarithmic corrections to scaling. Second, considering the Gaussian theory one can already anticipate a fundamental problem with any RG treatment of the field theory. To see this, note that at the Gaussian level the two-point vertex functions are given by $S_2[q]$. Examining Eqs. (2.5) we see that at this order no singularities, neither in the ultraviolet nor in the infrared, are present in the vertex functions, and the only cutoff dependence is the restriction that all frequencies must be smaller than some ultraviolet cutoff frequency Ω_0. This should be contrasted with the corresponding two-point Gaussian propagators given by Eqs. (2.6). Because of the last term in Eq. (2.6b), both an infrared singularity and a logarithmic dependence on an ultraviolet cutoff appear. In the usual RG approach such cutoff dependences are eliminated from the field theory by the introduction of suitable renormalization constants. Here, unusual features are that vertex functions and propagators behave differently with respect to their cutoff dependence, and that the cutoff dependent term is logarithmically small rather than large. We will come back to this in Sec. III below.

B. Perturbation theory and the Bethe-Salpeter equation for the Cooper propagator

We now consider the perturbation theory for the vertex functions, which can be generated by standard techniques.[19] For simplicity we restrict our considerations to the two-point vertex function $\Gamma^{(2)}$ and to the one-point vertex function $\Gamma^{(1)}$ which is related to the one-point propagator $P^{(1)}=\langle\mathrm{tr}Q^{\alpha\alpha}_{nn}(\mathbf{x})\rangle$. At one-loop order $\Gamma^{(2)}$ is given by the second derivative with respect to q of the right-hand side of Eq. (2.5a) with the matrix M replaced by

$$ {}^{i}_{0,3}M'_{12,34}(p) = \delta_{1-2,3-4}\Big\{\delta_{13}\Big[\frac{p^2}{G_{n_1 n_2}} + H_{n_1 n_2}(\omega_{n_1} - \omega_{n_2})\Big] + \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3} 2\pi T K^{\nu_i}_{n_1 n_2, n_3 n_4}\Big\}, \tag{2.8a} $$

and,

$$ {}^{i}_{1,2}M'_{12,34}(p) = -\delta_{1+2,3+4}\Big\{\delta_{13}\Big[\frac{p^2}{G_{n_1 n_2}} + H_{n_1 n_2}(\omega_{n_1} - \omega_{n_2})\Big] + \delta_{\alpha_1\alpha_2}\delta_{\alpha_1\alpha_3}\delta_{i0} 2\pi T K^{c}_{n_1 n_2, n_3 n_4}\Big\}, \tag{2.8b} $$

where $G_{n_1 n_2}, H_{n_1 n_2}$, and $K^{s,t,c}_{n_1 n_2, n_3 n_4}$ are given by G, H, and $K_{s,t,c}$, respectively, plus frequency dependent one-loop perturbative corrections. In our notation we have suppressed the fact that these corrections are in general also momentum dependent. In the absence of Cooperons these corrections have been discussed in detail elsewhere.[17] In general the momentum and frequency dependence of the corrections is quite complicated. Here we just give the results to leading order in $1/\epsilon$, with $\epsilon = D - 2$. With Λ an ultraviolet momentum cutoff, and $\bar{G} = GS_D/(2\pi)^D$ with S_D the surface of the D-dimensional unit sphere, we obtain,

$$ G_{n_1 n_2} = G + \frac{G^2}{4}\int_{\mathbf{p}} \mathcal{D}_{n_1 - n_2} - G^2 \int_{\mathbf{p}} [I^s_1(\mathbf{p}, \Omega_{n_1 - n_2}) + 3I^t_1(\mathbf{p}, \Omega_{n_1 - n_2}) + I^c_1(\mathbf{p}, \Omega_{n_1 - n_2})] + G^2 \int_{\mathbf{p}} [I^s_2(\mathbf{p}, \Omega_{n_1 - n_2}) + 3I^t_2(\mathbf{p}, \Omega_{n_1 - n_2})], \tag{2.9a} $$

$$ H_{n_1 n_2} = H + GH \int_{\mathbf{p}} [I^s_1(\mathbf{p}, \Omega_{n_1 - n_2}) + 3I^t_1(\mathbf{p}, \Omega_{n_1 - n_2}) + \frac{1}{2} I^c_1(\mathbf{p}, \Omega_{n_1}) + \frac{1}{2} I^c_1(\mathbf{p}, \Omega_{-n_2})] - GH \int_{\mathbf{p}} [J^s_2(\mathbf{p}, \Omega_{n_1 - n_2}) + 3J^t_2(\mathbf{p}, \Omega_{n_1 - n_2})], \tag{2.9b} $$

$$ K^s_{n_1 n_2, n_3 n_4} = K_s + H - H_{n_1 n_2}, \tag{2.9c} $$

$$ K^t_{n_1 n_2, n_3 n_4} = K_t + G(K_t - K_s)\int_{\mathbf{p}} J_1(\mathbf{p}, \Omega_{n_1 - n_2}) - \frac{GK_t}{2}\int_{\mathbf{p}} J_3(\mathbf{p}, \Omega_{n_1 - n_2}) - G2K_t \int_{\mathbf{p}} I^c_1(\mathbf{p}, \Omega_{n_1 - n_2}) - \frac{G}{2}(H + K_t)^2 \int_{\mathbf{p}} J^c_2(\mathbf{p}, \Omega_{n_1 - n_2}), \tag{2.9d} $$

$$ K^c_{n_1 n_2, n_3 n_4} = K_c - \frac{G}{8}(K_s - 3K_t)\int_{\mathbf{p}} \mathcal{D}_{n_1 - n_2} + G\int_{\mathbf{p}} J^c_1(\mathbf{p}, \Omega_{n_3 - n_2}). \tag{2.9e} $$

With a cutoff frequency $\Omega_0 = O(\Lambda^2/GH)$ the integrands read

$$ I^{s,t}_1(\mathbf{p}, \Omega_n) = \frac{1}{8}\sum_{m=n}^{\Omega_0/2\pi T} \frac{1}{m}\Delta\mathcal{D}^{s,t}_m(\mathbf{p}), \tag{2.10a} $$

$$ I^c_1(\mathbf{p}, \Omega_n) = -\frac{1}{8}G2\pi T \sum_{m=n}^{\Omega_0/2\pi T} \frac{K_c}{1 + G2\pi T K_c f_{-m}(\mathbf{p})}\Big(\mathcal{D}_m(\mathbf{p})\Big)^2, \tag{2.10b} $$

$$ I^{s,t}_2(\mathbf{p}, \Omega_n) = -\frac{G^2}{8}(K_{s,t})^2 2\pi T \sum_{m=n}^{\Omega_0/2\pi T} \Omega_m \mathcal{D}^{s,t}_m(\mathbf{p})\Big(\mathcal{D}_{n+m}(\mathbf{p})\Big)^2\Big[1 - 2p^2\mathcal{D}_{n+m}(\mathbf{p})\Big], \tag{2.10c} $$

$$ J^{s,t}_2(\mathbf{p}, \Omega_n) = \frac{1}{8}(K_{s,t}/H)\frac{2\pi T}{\Omega_n}\sum_{m=n}^{\Omega_0/2\pi T}\Big[\Delta\mathcal{D}^{s,t}_m(\mathbf{p}) + GK_{s,t}\Omega_m \mathcal{D}^{s,t}_m(\mathbf{p})\mathcal{D}_{m+n}(\mathbf{p})\Big], \tag{2.10d} $$

$$ J_1(\mathbf{p}, \Omega_n) = \frac{2\pi T}{8\Omega_n}\sum_{m=1}^{n-1}\Big[\Big(1 - \frac{m}{n}\Big)\mathcal{D}_{m+n}(\mathbf{p}) + \frac{m}{n}\mathcal{D}_m(\mathbf{p})\Big], \tag{2.10e} $$

$$J_3(\mathbf{p},\Omega_n) = -\sum_{m=n}^{\Omega_0/2\pi T} \frac{1}{m}\Delta\mathcal{D}^t_m(\mathbf{p}) - GK_t \sum_{m=n}^{\Omega_0/2\pi T} \mathcal{D}_m(\mathbf{p})\mathcal{D}_{m+n}(\mathbf{p}) + \frac{G}{2}K_s \sum_{m=n}^{\Omega_0/2\pi T} \mathcal{D}^s_m(\mathbf{p})\mathcal{D}_{m+n}(\mathbf{p})$$
$$+\frac{G}{4}K_t \sum_{m=n}^{\Omega_0/2\pi T} \frac{n}{m}\mathcal{D}^s_m(\mathbf{p})\mathcal{D}_{m+n}(\mathbf{p}) + \frac{G}{4}K_t \sum_{m=n}^{\Omega_0/2\pi T} \left(2-\frac{m}{n}\right)\mathcal{D}^t_m(\mathbf{p})\mathcal{D}_{m+n}(\mathbf{p}), \tag{2.10f}$$

$$J^c_2(\mathbf{p},\Omega_n) = 2\pi T \sum_{m=n}^{\Omega_0/2\pi T} \Omega_m \Big(\mathcal{D}_m(\mathbf{p})\Big)^3 \frac{K_c}{1+K_c G 2\pi T f_{-m}(\mathbf{p})}, \tag{2.10g}$$

$$J^c_1(\mathbf{p},\Omega_n) = \frac{1}{2}G2\pi T \sum_{m=n}^{\Omega_0/2\pi T} \frac{1}{m}\Delta\mathcal{D}^s_m(\mathbf{p}) \frac{K_c}{1+G2\pi T K_c f_{-m}(\mathbf{p})}. \tag{2.10h}$$

For the one-point vertex function one finds

$$\Gamma^{(1)}(\Omega_n) = 1 + G\int_{\mathbf{p}} [I^s_1(\mathbf{p},\Omega_n) + 3I^t_1(\mathbf{p},\Omega_n) + I^c_1(\mathbf{p},\Omega_n)]. \tag{2.11}$$

In giving Eqs. (2.9)–(2.11) we have neglected terms that are finite in $D=2$ as $(\Lambda,\Omega_0)\to\infty$, and a δ-function constraint [cf. Eqs. (2.8)] is understood in Eqs. (2.9c), (2.9d), and (2.9e). Some of these terms depend on n_1 and n_2 separately, not just on the difference $n_1 - n_2$. In Eq. (2.9b) we have written a separate dependence on n_1 and n_2 explicitly for later reference. Also, the complete frequency dependence of $K^t_{n_1n_2,n_3n_4}$ and $K^c_{n_1n_2,n_3n_4}$ is more complicated than the one shown. However, for most of our purposes it is sufficient to treat all "external" frequencies as equal, and we do not have to deal with the (substantial) complications that arise from the full perturbation theory.

Finally, let us discuss one important point. To one-loop order the two-point propagators are given by the inverse of the matrix M', Eqs. (2.8) which contains the perturbative corrections to the inverse of the matrix M in Eq. (2.5a). For the particle-hole degrees of freedom one can show in general that, except for irrelevant terms, the matrix M'^{-1} has the same form as the matrix M^{-1}, with the only difference being that the corrected coupling constants appear in M'^{-1}. The underlying reason for this feature is the conservation laws for mass, spin, and energy density. For the particle-particle degrees of freedom the situation is different. There are no conservation laws which guarantee that the form of the last term in Eq. (2.6b) will not change at higher orders in the loop expansion. In general, the Cooper propagator is given by Eq. (2.6b) with the replacement,

$$\frac{K_c}{1+G2\pi T K_c f_{n_1+n_2}(p)} \to \Gamma^c_{n_1n_2,n_3n_4}(p), \tag{2.12a}$$

where Γ^c satisfies a Bethe-Salpeter equation,

$$\Gamma^c_{n_1n_2,n_3n_4} + 2\pi T \sum_{n'_1\geq 0, n'_2<0} \Gamma^c_{n_1n_2,n'_1n'_2} \frac{K^c_{n'_1n'_2,n_3n_4}\delta_{n_3+n_4,n'_1+n'_2}}{p^2/G_{n'_1n'_2} + H_{n'_1n'_2}(\omega_{n'_1}-\omega_{n'_2})} = K^c_{n_1n_2,n_3n_4}. \tag{2.12b}$$

Here we have again suppressed the momentum dependence of Γ^c. Note that if we make the substitutions $K^c_{n'_1n'_2,n_3n_4} \to K_c$, $H_{n'_1n'_2} \to H$, $G_{n'_1n'_2} \to G$, replace the remaining sum by an integral, let $p, T \to 0$, and use an ultraviolet cutoff Ω_0 on the frequency integral, then Γ^c has the standard BCS form,

$$\Gamma^c(p,\omega_{n_1+n_2}) = \frac{K_c}{1+\frac{\gamma_c^{(0)}}{2}\ln\left(\frac{\Omega_0}{D^{(0)}p^2+|\omega_{n_1+n_2}|}\right)}, \tag{2.12c}$$

with $\gamma_c^{(0)} = K_c/H$, and $D^{(0)} = 1/GH$.

The actual Cooper propagator to all orders in perturbation theory would have the simple form given by Eq. (2.12c) only if the coupling constants K^c and H were constant to all orders. In general, the structure of the Cooper propagator will be more complicated and to obtain it one has to solve the integral equation, Eq. (2.12b). This equation expresses a general inversion problem: How to obtain a propagator from a vertex function which has a complicated frequency dependence. In the Cooper channel, in contrast to the particle-hole channel, this problem is not simplified by constraints due to conservation laws. For later reference we symmetrize the Bethe-Salpeter equation by defining

$$\gamma_{n_1n_2,n_3n_4} = \frac{K^c_{n_1n_2,n_3n_4}}{\left[H_{n_1n_2}H_{n_3n_4}\right]^{1/2}}, \tag{2.13a}$$

and

$$\tilde{\Gamma}_{n_1n_2,n_3n_4} = \frac{\Gamma^c_{n_1n_2,n_3n_4}}{\left[H_{n_1n_2}H_{n_3n_4}\right]^{1/2}}. \tag{2.13b}$$

For $T \to 0$ the integral equation for $\tilde{\Gamma}$ then reads

$$\tilde{\Gamma}(\omega,\Omega,\omega'') + \int_0^{\Omega_0} d\omega' \, \frac{\gamma(\omega,\Omega,\omega')\tilde{\Gamma}(\omega',\Omega,\omega'')}{D(\omega',-\Omega-\omega')p^2 + 2\omega' + \Omega} = \gamma(\omega,\Omega,\omega''), \tag{2.14a}$$

with

$$D(\omega',-\Omega-\omega') = [G(\omega',-\Omega-\omega')H(\omega',-\Omega-\omega')]^{-1}. \tag{2.14b}$$

Since we consider the zero-temperature limit we have made the replacements,

$$\begin{aligned} 2\pi T(n_1+n_2) = 2\pi T(n_3+n_4) &\to -\Omega \\ 2\pi T n_3 &\to \omega \\ 2\pi T n_1' &\to \omega' \\ 2\pi T n_1 &\to \omega'' \\ \tilde{\Gamma}_{n_1n_2,n_3n_4} &\to \tilde{\Gamma}(\omega,\Omega,\omega'') \end{aligned} \tag{2.14c}$$

etc., and for definiteness we have assumed $\Omega \geq 0$.

III. THE RENORMALIZATION GROUP FLOW EQUATIONS

In this section we apply a field-theoretic renormalization procedure to the model of Sec. II. We first use normalization point techniques to obtain RG flow equations for all of the coupling constants that appear in the particle-hole channel. These flow equations contain the Cooper propagator $\tilde{\Gamma}$. In order to close the system of flow equations one needs a flow equation for $\tilde{\Gamma}$, and we discuss the reasons why previous attempts to derive one are problematic. In the Appendix we show that the same flow equations are obtained, and the same conclusions are reached, if one uses a Wilson-type momentum-frequency shell RG.

A. Renormalization of the particle-hole channel

The field theory defined by Eqs. (2.1)–(2.2) has the form of a nonlinear σ model with perturbing operators. The pure σ model, i.e., the first term on the right-hand-side (rhs) of Eq. (2.1) with the constraint given by Eq. (2.2), is well known to be renormalizable with two renormalization constants, one for the coupling constant G (the disorder) and one for the renormalization of the Q field.[20] The second term on the rhs of Eq. (2.1) does not require any additional renormalization constants because the coupling constant H in Eq. (2.1) just multiplies the basic Q field, and, therefore, the renormalization of H is determined by the field renormalization constant.[20,21] This model represents the noninteracting localization problem.[12] From a physical point of view the model has a rather restrictive property: The only interaction taken into account is the elastic electron-impurity scattering, and consequently the different Matsubara frequencies in Eq. (2.1) are decoupled.

The situation changes fundamentally with the addition of the last term in Eq. (2.1). Physically, this term describes the electron-electron interaction and hence the exchange of energy between electrons. Technically, this leads to a coupling between the Matsubara frequencies, and an examination of the perturbation theory shows that this term introduces new infrared and ultraviolet singularities. In the absence of interactions, singularities arise only from momentum integrations. With interactions there are singularities due to both momentum and frequency integrations, and the symmetry properties of the model change continuously during the RG procedure. As a consequence, the established results[20] concerning the renormalizability of nonlinear σ models with perturbing operators, which apply to models with a fixed symmetry, are inapplicable. No general results concerning the number of renormalization constants needed are available, and it is *a priori* unclear how to renormalize the model given by the full Eq. (2.1).

All RG treatments of the field theory defined by Eqs. (2.1)–(2.2) so far have ignored this general renormalizability problem. They have assumed, explicitly or implicitly, that the full model is still renormalizable with one extra renormalization constant for each interaction coupling constant which is added. In addition, H acquires a renormalization constant of its own if interactions are present. In the absence of Cooperons, i.e., in a theory which contains only K_s and K_t, there is empirical evidence based on perturbation theory for this assumption being correct.[15,22,23] In the presence of Cooperons things are more complicated as we will see. We will, therefore, first renormalize the particle-hole channel, where the theory is on more solid ground.

It is most convenient to use a normalization point RG. The renormalized disorder, frequency coupling, interaction constants, and q fields are denoted by g, h, k_s, k_t, and q_R, respectively. They are defined by

$$\bar{G} = \mu^{-\epsilon} Z_g g, \tag{3.1a}$$

$$H = Z_h h, \tag{3.1b}$$

$$K_{s,t} = Z_{s,t} k_{s,t}, \tag{3.1c}$$

$${}_{0,3}q = Z^{1/2}\; {}_{0,3}q_R, \tag{3.1d}$$

where μ is an arbitrary momentum scale that is introduced in Eq. (3.1a) to make g dimensionless. The renormalized particle-hole N point vertex functions are related to the bare ones by

$$
{}_{0,3}\Gamma_R^{(N)}(p,\Omega_n;g,h,k_s,k_t,\ldots;\mu,\Lambda)
$$

$$
= Z^{N/2}\,{}_{0,3}\Gamma^{(N)}(p,\Omega_n;G,H,K_s,K_t;\Lambda). \tag{3.2}
$$

Renormalizability implies that all of the $\Gamma_R^{(N)}$ are finite as $\Lambda \to \infty$ for fixed renormalized coupling constants. On the left-hand side of Eq. (3.2) we anticipate the possibility that renormalization may produce new scaling operators which have no counterpart in the bare vertex functions. We will see that this indeed happens, since the particle-hole and particle-particle channels couple, starting at one-loop order.

The Z's in Eqs. (3.1) are not unique. We fix them by the following normalization conditions for the two-point vertex functions,

$$
\frac{1}{8}\frac{\partial}{\partial p^2}\,{}_r^i(\Gamma_R^{(2)})^{\alpha\beta,\alpha\beta}_{n_1n_2,n_1n_2}(p)|_{p=0,\ \omega_{n_1}-\omega_{n_2}=\mu^D/gh,\ \omega_{n_1}+\omega_{n_2}=0} = \frac{\mu^\epsilon}{g}S_D/(2\pi)^D, \tag{3.3a}
$$

$$
\frac{1}{8}\frac{\partial}{\partial(\omega_{n_1}-\omega_{n_2})}\,{}_r^i(\Gamma_R^{(2)})^{\alpha\beta,\alpha\beta}_{n_1n_2,n_1n_2}(p)|_{\alpha\neq\beta,\ p=0,\ \omega_{n_1}-\omega_{n_2}=\mu^D/gh,\ \omega_{n_1}+\omega_{n_2}=0} = h, \tag{3.3b}
$$

$$
\frac{1}{8}\left[{}_0^i(\Gamma_R^{(2)})^{\alpha\alpha,\alpha\alpha}_{n_1n_2,n_1n_2}(p=0) - {}_0^i(\Gamma_R^{(2)})^{\alpha\beta,\alpha\beta}_{n_1n_2,n_1n_2}(p=0)|_{\alpha\neq\beta}\right]_{\omega_{n_1}-\omega_{n_2}=\mu^D/gh,\ \omega_{n_1}+\omega_{n_2}=0} = 2\pi T k_{\nu_i}. \tag{3.3c}
$$

These conditions determine the renormalization constants $Z_{g,h,s,t}$. The wave function or field renormalization constant Z we fix by a normalization condition for the one-point vertex function $\Gamma^{(1)}$, viz.,

$$
{}_0^0(\Gamma_R^{(1)})^{\alpha\alpha}_{nn}|_{\Omega_n=\mu^D/gh} = 1. \tag{3.3d}
$$

The normalization conditions given by Eqs. (3.3) are the conventional ones: At scale μ the renormalized vertex functions are taken to have their tree level structure.

From Eqs. (2.8), (2.9), (3.1), (3.2), and (3.3) the Z's can be determined. We obtain

$$
Z = \left(1/\Gamma^{(1)}(\Omega_n=\mu^D/gh)\right)^2, \tag{3.4a}
$$

$$
Z_g = Z - \frac{\mu^\epsilon}{g}\frac{S_D}{(2\pi)^D}[G_{n_1-n_2}-G]_{\omega_{n_1}-\omega_{n_2}=\mu^D/gh} + O(g^2), \tag{3.4b}
$$

$$
Z_h = Z^{-1} - \frac{1}{h}[H_{n_1-n_2}-H]_{\omega_{n_1}-\omega_{n_2}=\mu^D/gh} + O(g^2). \tag{3.4c}
$$

The fastest way to derive the RG flow equations is to switch from our cutoff regularized theory to dimensional regularization and use a minimal subtraction scheme. We can do so by putting $\epsilon < 0$ in Eqs. (3.4), letting $\Lambda \to \infty$, and then analytically continuing to $\epsilon > 0$. We find,

$$
Z = 1 - \frac{g}{4\epsilon}\left[\frac{-2}{\epsilon} + 3l_t + \tilde\Gamma\right], \tag{3.5a}
$$

$$
Z_g = 1 + \frac{g}{4\epsilon}\left\{5 - 3(1+1/\gamma_t)l_t - \frac{\tilde\Gamma}{2}\right\} + O(g^2), \tag{3.5b}
$$

$$
Z_h = 1 + \frac{g}{8\epsilon}\left[-1 + 3\gamma_t + \tilde\Gamma\right], \tag{3.5c}
$$

$$
Z_s = Z_h, \tag{3.5d}
$$

$$
Z_t = 1 + \frac{g}{2\epsilon}\left[\frac{1}{4\gamma_t} + \frac{1}{4} + \gamma_t + \tilde\Gamma\left(\frac{1}{4\gamma_t} + 1 + \frac{1}{2}\gamma_t\right)\right], \tag{3.5e}
$$

with $l_t = \ln(1+\gamma_t)$, $\gamma_t = k_t/h$. $\tilde\Gamma$ is the Cooper propagator at scale μ,

$$
\tilde\Gamma \equiv \tilde\Gamma(\mu) = \tilde\Gamma(\omega,\Omega,\omega')|_{\omega=\omega'=\mu^D/2gh=\Omega/2}. \tag{3.6}
$$

with $\tilde\Gamma$ from Eq. (2.13b). In giving Eqs. (3.5) and (3.7) below, we have for simplicity neglected terms of order $g\tilde\Gamma^n$ with $n \geq 2$. The omission of these terms does not affect our conclusions.

The one-loop RG flow equations follow from Eqs. (3.1) and (3.5) in the usual way.[19] With $b \sim \mu^{-1}$ and $x = \ln b$ one obtains,

$$
\frac{dg}{dx} = -\epsilon g + \frac{g^2}{4}\left\{5 - 3(1+1/\gamma_t)l_t - \frac{\tilde\Gamma}{2}\right\} + O(g^3), \tag{3.7a}
$$

$$
\frac{dh}{dx} = \frac{gh}{8}[3\gamma_t - 1 + \tilde\Gamma] + O(g^2), \tag{3.7b}
$$

$$\frac{d\gamma_t}{dx} = \frac{g}{8}(1+\gamma_t)^2 + \frac{g}{8}(1+\gamma_t)(1+2\gamma_t)\tilde{\Gamma} + O(g^2), \tag{3.7c}$$

In giving Eqs. (3.5) and (3.7) we have specialized to the case of a Coulomb interaction between the electrons. In this case a compressibility sum rule enforces $\gamma_s = k_s/h = -1$.[2]

Equations (3.7) are valid for the universality class G. For the spin-orbit class SO the analogous results are

$$\frac{dg}{dx} = -\epsilon g + \frac{g^2}{8}[1-\tilde{\Gamma}] + O(g^3), \tag{3.8a}$$

$$\frac{dh}{dx} = -\frac{gh}{8}[1-\tilde{\Gamma}] + O(g^2). \tag{3.8b}$$

Note that Eqs. (3.8) for the universality class SO have, at least in the absence of $\tilde{\Gamma}$, a fixed point describing a MIT,[3] while Eqs. (3.7) for the generic universality class do not. The runaway problem posed by Eqs. (3.7) for the class G has been solved by a perturbation theory and resummation to all orders.[24] However, this problem is decoupled from the Cooper channel problem discussed here, and for our present purposes, Eqs. (3.7) are sufficient.

Equations (3.7) and (3.8) are identical to the flow equations obtained in Refs. 2 and 3, which supplemented them by a flow equation for $\tilde{\Gamma}$. They are also identical to the flow equations obtained in Ref. 7. In that paper, however, an explicit expression for $\tilde{\Gamma}$ was used, and an additional flow equation for the Cooper interaction amplitude $\gamma_c = k_c/h$ was derived. These two approaches yielded qualitatively different results. In the next subsection we will show that these different results are due to different assumptions concerning the renormalization of the Cooper channel, and we will discuss the validity of these assumptions.

B. Attempts to renormalize the particle-particle channel

The RG flow equations given by Eqs. (3.7) and Eqs. (3.8) are not closed because they contain the Cooper propagator $\tilde{\Gamma}$. More generally, we have not yet considered the renormalization of the particle-particle channel, which is necessary in order to complete the theory. What has been done in the literature to close the system of flow equations amounts to various implicit assumptions. Here we investigate this point in some detail. We will state the assumptions explicitly, and show how the mutually contradicting results in the literature follow from them. We will then analyze the resulting descriptions with respect to internal consistency. To what extent they are consistent with the Bethe-Salpeter equation will be discussed in Sec. IV.

1. Attempt 1: A flow equation for $\tilde{\Gamma}$

It is clear from the results of the previous subsection that the renormalization procedure used so far needs to be expanded: The appearance of $\tilde{\Gamma}$ in Eqs. (3.7) – (3.8) indicates that the Cooper channel couples to the particle-hole channel at one-loop order. One can now proceed based on the following.

Assumption 1. One can consider $\tilde{\Gamma}$ an effective renormalized coupling constant which should be treated on equal footing with g, h, k_s, and k_t. No other renormalization of Cooper channel quantities is necessary.

This is the assumption made implicitly in Refs. 2 and 3. It may be motivated by the fact that to zeroth order in the disorder there is a logarithmic singularity in, e.g., Eq. (2.6b) that can be viewed as a scale dependent renormalization of K_c. It implies that Eq. (3.2) should be completed to read

$$_{0,3}\Gamma_R^{(N)}(p,\Omega_n; g,h,k_s,k_t,\tilde{\Gamma};\mu,\Lambda)$$

$$= Z^{N/2}\,{}_{0,3}\Gamma^{(N)}(p,\Omega_n;G,H,K_s,K_t;\Lambda). \tag{3.2$'$}$$

Equation (3.2′) acknowledges the fact that the renormalization procedure has generated an additional operator whose scaling properties have to be determined in order to complete the description of the renormalized theory.

One can define a renormalization constant $\bar{Z}_c$ by

$$\gamma_c^{(0)} = \bar{Z}_c\tilde{\Gamma}. \tag{3.9}$$

To zeroth order in the disorder Eqs. (3.6) and (3.9) give

$$\bar{Z}_c = (1-\tilde{\Gamma}x)^{-1}, \tag{3.10a}$$

$$\frac{d\tilde{\Gamma}}{dx} = -\tilde{\Gamma}^2. \tag{3.10b}$$

A few remarks are in order in the context of Eq. (3.10b): (1) It has the standard form of a RG flow equation for a marginal operator. (2) In this approach the RG is used to obtain the BCS logarithm. It correctly reproduces the result for $\tilde{\Gamma}$ to lowest order in the disorder, viz.

$$\tilde{\Gamma}(\mu) = \frac{\gamma_c^{(0)}}{1+\gamma_c^{(0)}x} + O(g), \tag{3.11}$$

which is easily obtained from Eqs. (2.14a) and (3.6). (3) A crucial question is what the structure of the higher loop corrections to Eq. (3.10b) will be. If, as has been asserted in Refs. 2 and 3, $\tilde{\Gamma}$ flows to a nonzero fixed point value, $\tilde{\Gamma} \to \tilde{\Gamma}^*$, then Eq. (3.10a) seems to imply that $\bar{Z}_c$ has a Cooper-type singularity at a finite scale $x = \ln b = 1/\tilde{\Gamma}^*$ which does not correspond to any physical phase transition. This conclusion is not necessarily correct, since terms of higher order in the disorder could change the behavior of $\bar{Z}_c$. Nevertheless, we will see that the structure of Eq. (3.10a) represents a serious problem and leads to an internal inconsistency.[25] We will discuss this point in connection with Eq. (3.15) below.

Within this approach the one-loop RG flow equation can be obtained by using Eqs. (2.9) and (2.13a) in Eq. (2.14a), and iterating to first order in the disorder. From Eqs. (3.1) and (3.9) one can then obtain a flow equation for $\tilde{\Gamma}$. The resulting equation is quite complicated, and we will illustrate only a few points. First, let us make contact with Refs. 2 and 3 by retaining only the first two terms in Eq. (2.9e), neglecting the corrections to H, putting $p = 0$, and working in $D = 2$. In this approximation, γ in Eq. (2.14a) is given to leading logarithmic accuracy by,

$$\gamma(\omega,\Omega,\omega') = \gamma_c^{(0)} - a_1 \ln\left(\frac{\omega+\omega'+\Omega}{\Omega_0}\right), \tag{3.12a}$$

with,

$$a_1 = \begin{cases} (g/16)(-\gamma_s + 3\gamma_t) & \text{for class G} \\ (g/16) & \text{for class SO}\ , \end{cases} \tag{3.12b}$$

for the generic and spin-orbit universality classes, respectively. Here we consider the short-range model for simplicity. Using Eq. (3.12a) in the procedure discussed above, we obtain the following one-loop flow equation for $\tilde{\Gamma}$,

$$\frac{d\tilde{\Gamma}}{dx} = -\tilde{\Gamma}^2[1-a_1 c] - 4a_1\tilde{\Gamma}\ln 2 + 2a_1 + O(g^2, x^2 e^{-2x})\ , \tag{3.13a}$$

with

$$c = \int_0^\infty \frac{dz}{z+1/2}\ln\left(\frac{z+3/2}{z+1/2}\right) - \frac{\pi^2}{12}. \tag{3.13b}$$

Two things should be noted: (1) the coefficients in Eq. (3.13a) are universal and, consequently, this equation has the form of a standard RG flow equation; (2) at the MIT in, e.g., the SO universality class, a_1, which is of $O(g)$, is a constant of order $g^* = O(\epsilon)$. If Eq. (3.13a) is valid, then this implies that $\tilde{\Gamma}$ goes to a fixed point value at the MIT which is of order $\epsilon^{1/2}$. In a strict ϵ expansion the terms of $O(g\tilde{\Gamma})$ and higher on the rhs of Eq. (3.13a) can be neglected, and to leading order the equation takes the form

$$\frac{d\tilde{\Gamma}}{dx} = -\tilde{\Gamma}^2 + 2a_1. \tag{3.13a$'$}$$

This result is identical to those in Refs. 2 and 3, except for the prefactor of the a_1, which is four in these references. This difference is due to the fact that for a Coulomb interaction, which was considered in these references, an additional term appears in Eq. (2.9e). This leads to a more complicated kernel than the one in Eq. (3.12a), and to an additional factor of 2 in Eq. (3.13a$'$). Since this difference is not relevant for our purposes we restrict ourselves to the simpler kernel for the short-range case, Eq. (3.12a).

Above we argued that the functional form of the renormalization constant $\tilde{Z}_c$, Eq. (3.10a), potentially leads to problems. We, therefore, check the approach for internal inconsistencies. To illustrate that a problem does indeed exist, we add to Eq. (3.12a), via Eq. (2.13a), the Cooper propagator contribution to $H_{n_1 n_2}$ and $H_{n_3 n_4}$, i.e., the integral over I_1^c in Eq. (2.9b). The result is a kernel in Eq. (2.14a) which is given by

$$\begin{aligned}\gamma(\omega,\Omega,\omega') =& \gamma_c^{(0)} - a_1\ln\left(\frac{\omega+\omega'+\Omega}{\Omega_0}\right) \\ &+ a_2\gamma_c^{(0)}\Big(\ln[1-\tfrac{1}{2}\gamma_c^{(0)}\ln(\omega/\Omega_0)] \\ &+ \ln[1-\tfrac{1}{2}\gamma_c^{(0)}\ln(\omega'/\Omega_0)]\Big),\end{aligned} \tag{3.14a}$$

with

$$a_2 = g/16. \tag{3.14b}$$

The last two terms in Eq. (3.14a) occur because of the dependence of γ on the Cooper propagator. In the last term in Eq. (3.14a) we have neglected a dependence on Ω which is irrelevant for our purposes. Using Eq. (3.14a) in the RG procedure yields the flow equation,

$$\begin{aligned}\frac{d\tilde{\Gamma}}{dx} =& -\tilde{\Gamma}^2[1-a_1c-2a_2] - 4a_1\tilde{\Gamma}\ln 2 + 2a_1 \\ &- a_2\tilde{\Gamma}^2\int_0^{b^2}\frac{dz}{(z+1/2)^2}\ln[1-\frac{\tilde{\Gamma}}{2}\ln(2z)] + O(g^2).\end{aligned} \tag{3.15}$$

The crucial point is that as $x = \ln b \to \infty$ the last term in Eq. (3.15) does not exist because of a singularity at $b \sim \exp(1/\tilde{\Gamma}) \sim \exp(1/\epsilon^{1/2})$. Notice that this breakdown of the RG flow equation for the Cooper propagator is nonperturbative in nature. Previous derivations of Eq. (3.13a$'$),[2,3] were based on perturbative expansions in both g and $\tilde{\Gamma}$ and cannot be used to discuss the singularity in Eq. (3.15). Furthermore, if one expands the last term in Eq. (3.15) in powers of $\tilde{\Gamma}$ then a non-Borel summable divergent series is obtained. The structure of this singularity resembles what is known as the renormalon problem in quantum field theory.[19] We also note that this singularity is obviously related to the one discussed in point (3) below Eq. (3.10b).

2. *Attempt 2: Renormalizing the Cooper vertex function*

A quite different attempt to renormalize the Cooper channel has been made in Ref. 7. It is based on the following.

Assumption 2a. The Cooper channel is renormalizable with the same wave function renormalization as the particle-hole channel and an additional renormalization constant for the Cooper interaction constant K_c.

This means that Eqs. (3.1) are supplemented by

$$K_c = Z_c k_c, \tag{3.1c'}$$

$$_{1,2}q = Z^{1/2}\ _{1,2}q_R, \tag{3.1d'}$$

and that the renormalization statement for *all* N-point vertex functions is generalized from Eq. (3.2) to

$$\Gamma_R^{(N)}(p,\Omega_n;g,h,k_s,k_t,k_c;\mu,\Lambda) = Z^{N/2}\,\Gamma^{(N)}(p,\Omega_n;G,H,K_s,K_t,K_c;\Lambda). \tag{3.2''}$$

The new renormalization constant Z_c we determine from a normalization condition for the two-point Cooper vertex function,

$$\frac{1}{8}\Big[{}^0_1(\Gamma_R^{(2)})^{\alpha\alpha,\alpha\alpha}_{n_1n_2,n_1n_2}(p=0) - {}^0_1(\Gamma_R^{(2)})^{\alpha\beta,\alpha\beta}_{n_1n_2,n_1n_2}(p=0)|_{\alpha\neq\beta}\Big]_{\omega_{n_1}+\omega_{n_2}=-\mu^D/gh,\ \omega_{n_1}-\omega_{n_2}=0} = -2\pi T k_c\ . \tag{3.16}$$

From this we obtain,

$$Z_{s,t} = Z^{-1} - \frac{1}{k_{s,t}}[K^{s,t}_{n_1n_2,n_3n_4} - K_{s,t}]_{\omega_{n_1}-\omega_{n_2}=\mu^D/gh} + O(g^2), \tag{3.17}$$

and minimal subtraction yields

$$Z_c = 1 + \frac{g}{4\epsilon}\left[\frac{-2}{\epsilon} + 3l_t + \tilde{\Gamma}\right] + \frac{g}{4\epsilon}(1+3\gamma_t)/\gamma_c + \frac{g}{\epsilon^2}\tilde{\Gamma}/\gamma_c, \tag{3.18}$$

where $\gamma_c = k_c/h$. We then obtain flow equations for γ_c,

$$\frac{d\gamma_c}{dx} = \frac{g}{4}(3\gamma_t+1) + \frac{g\gamma_c}{8}\left\{-\frac{4}{\epsilon} + 6l_t + 1 - 3\gamma_t + \tilde{\Gamma} + \frac{8}{\epsilon}\frac{\tilde{\Gamma}}{\gamma_c}\right\} + O(g^2), \tag{3.19a}$$

$$\frac{d\gamma_c}{dx} = \frac{g}{4} + \frac{g\gamma_c}{8}\left\{-\frac{4}{\epsilon} + 1 + \tilde{\Gamma} + \frac{8}{\epsilon}\frac{\tilde{\Gamma}}{\gamma_c}\right\} + O(g^2) \tag{3.19b}$$

for the universality classes G and SO, respectively. Notice the terms $\sim 1/\epsilon$ in Eqs. (3.19), which are characteristic for the case Coulomb interactions [for a short-range interaction, the equivalent terms are $\sim \ln(1+\gamma_s)$]. The presence of these terms in the RG flow equation for γ_c reflects the well-known $(\ln)^2$ singularity that exists in the disorder expansion of the single-particle density of states (DOS) in $D = 2$.[6] The appearance of this type of term in the Cooper channel has been discussed before in the context of superconductivity ($\gamma_c < 0$) by Fukuyama and co-workers.[26] It has sometimes been claimed that these DOS effects are absent in all flow equations for the interaction constants.[27] We stress that this statement depends on exactly what quantity is considered. We find that it is true for γ_s and γ_t, but not for the Cooperon amplitude γ_c. It is also important to note that the presence of these $1/\epsilon$ terms in the flow equations *per se* does not create any problems. They do appear, e.g., in the renormalization of the single-particle DOS, for which a careful application of the above method[28] leads to results that are consistent both internally and with those obtained by other methods.[2]

One still needs to determine the scaling behavior of $\tilde{\Gamma}$, since it enters the flow equations, Eqs. (3.19). According to Assumption 2a, $\tilde{\Gamma}$ does not acquire a renormalization constant of its own, but rather is interpreted as the Cooper propagator, whose scaling behavior follows from that of the Cooper vertex function. Reference 7 did not solve the inversion problem given by the full Bethe-Salpeter equation, but notice that, with an irrelevant $\tilde{\Gamma}$, Eqs. (3.19) yield a finite fixed point (FP) value for γ_c, which in turn yields a logarithmically vanishing $\tilde{\Gamma}$ from Eq. (2.14a). This together with the zero-loop result for $\tilde{\Gamma}$, i.e., Eq. (3.11), leads to the following.

Assumption 2b. Near the MIT, $\tilde{\Gamma}$ is adequately represented by

$$\tilde{\Gamma}(\mu) = \frac{\gamma_c}{1+\gamma_c x}. \tag{3.20}$$

With these assumptions, γ_c approaches a finite FP value, and $\tilde{\Gamma}$ is a logarithmically irrelevant operator. This is in sharp contrast to Eq. (3.13a′), which predicts a finite FP value of $\tilde{\Gamma}$.

3. Discussion and mutual inconsistency of the attempts

The two assumptions made in Refs. 2, 3, and 7, respectively, which we have spelled out in the last two subsections, taken together with the Bethe-Salpeter equation, Eq. (2.14a), are mutually inconsistent. This can be seen as follows. Assumption 1 leads to a finite FP value of $\tilde{\Gamma}$. The structure of Eq. (2.14a) is such that the only way this can happen is if γ diverges in the critical, or zero-frequency, limit. However, $\tilde{\Gamma}$ = const. inserted into Eqs. (3.19) leads to a finite FP value of γ_c or γ. Hence Assumptions 1 and 2 are contradictory, and this explains why, as we have seen, the two approaches lead to qualitatively different results. Apart from the renormalon problem, which we have discussed in connection with Eq. (3.15), both approaches, taken by themselves, appear reasonable and logically consistent. In order to see that neither one of them really is, we need to consider what has *not* been achieved by either renormalization procedure.

Attempt 1 considers $\tilde{\Gamma}$ the only relevant quantity in the Cooper channel. At this point it is important to remember that $\tilde{\Gamma}$ really is a part of the Cooper propagator, even though Attempt 1 treated it as an effective coupling constant. Equation (3.9) can, therefore, be interpreted as postulating a wave function renormalization for the Cooper channel, viz., $\tilde{Z}_c$, which is different from the one for the particle-hole channel. This raises a number of questions. For instance, it suggests that the propagator $\mathcal{D}_n(p)$, which is common to both the particle-hole and the Cooper channels [cf. the first two terms on the rhs of Eqs. (2.6)], is renormalized differently in the two channels. It is unclear how to interpret this. More importantly, however, this approach never considers how to renormalize the Cooper vertex function. It is clear that the Eqs. (3.9) and (3.10a) do not lead to a finite Cooper vertex function. One concludes that the theory is not properly renormalized by this approach. The problem is essentially that a finite propagator does not automatically imply a finite vertex function, as it does in simpler theories, because of the nontrivial inversion problem posed by the Bethe-Salpeter equation. We stress that one cannot dismiss the Cooper vertex function as unphysical, since with a suitable (albeit not realizable) external field it determines the free energy.

Attempt 2 suffers from the opposite problem. It renormalizes the Cooper vertex function and succeeds in making it finite, but it never considers the renormalization of the corresponding propagator. Indeed, in this approach it is the Cooper propagator that is not finite. We further note that if one tries to renormalize the Cooper propagator using Assumption 2a, then one runs into the renormalon problem again.

We conclude at this point that neither attempt is technically satisfactory, and that they both suffer from essentially the same shortcoming, which is not at all obvious from the previous publications. We further show in the Appendix that the problems we encounter are not due to the particular renormalization method used, and that the same conclusions are reached if one works with a Wilson-type momentum-shell RG. Since it is at present unclear how to overcome these technical problems concerning the renormalization of the Cooper channel, we next turn to a general discussion of the inversion problem. Our goal is to formulate the restrictions that are put on any renormalization attempt by the Bethe-Salpeter equation.

IV. A DISCUSSION OF THE BETHE-SALPETER EQUATION

The preceding section has shown that the results of the two attempts to obtain the scaling behavior of $\tilde{\Gamma}$ from a renormalization of the Cooper channel are not reliable, since neither one succeeds in producing a consistently renormalized theory. We now turn directly to the Bethe-Salpeter equation in order to determine the restrictions put on the scaling of γ_c and $\tilde{\Gamma}$ by the inversion problem.

In the first part of this section we discuss some general features of the Bethe-Salpeter equation, Eq. (2.14a). We conclude that the results of either one of the attempts discussed in Sec. III B are consistent with the Bethe-Salpeter equation, although the technical details were not. We will also see that the Bethe-Salpeter equation allows for a third scaling scenario which, however, seems unlikely given the structure of the perturbation theory. We then argue that both of the first two scaling scenarios, despite their rather different asymptotic scaling behaviors, lead to logarithmic corrections to scaling. The consequences of this are discussed in Sec. IV B.

A. Solutions of the Bethe-Salpeter equation

As was pointed out in the preceding subsection, to complete the RG description started in Sec. III A the scaling properties of the Cooper propagator given by Eqs. (2.13), (2.14a), and (3.6) need to be determined. In Sec. III B we showed why previous attempts to do so are not satisfactory. Here we pursue a different approach. We acknowledge that the Bethe-Salpeter equation puts constraints on any renormalization procedure, since it couples the scaling behavior of γ_c and $\tilde{\Gamma}$.

In principle one should solve the Bethe-Salpeter equation numerically together with the RG flow equations. However, at present this is precluded by the unsolved problems concerning the renormalization of the Cooper channel which were discussed in Sec. III. Since the kernel of the Bethe-Salpeter equation is determined by the scaling behavior of γ_c, this program must await the construction of a finite renormalized theory including the Cooper channel. We, therefore, set ourselves a more modest goal, viz., to determine what kind of scaling behaviors are consistent with the constraints imposed by Eqs. (2.13) and (2.14a). We will find that the Bethe-Salpeter equation allows for three different scaling scenarios, two of which lead to logarithmic corrections to scaling.

To simplify our considerations we work at zero momentum. We further specialize to a model with a separable kernel. The main points can be illustrated using a kernel that is a sum of two separable parts,

$$\gamma(\omega,\Omega,\omega'') = f_1(\omega,\Omega)\, f_1(\omega'',\Omega) + f_2(\omega,\Omega)\, f_2(\omega'',\Omega). \tag{4.1a}$$

Note that Eq. (4.1a) satisfies the symmetry requirement,

$$\gamma(\omega,\Omega,\omega'') = \gamma(\omega'',\Omega,\omega). \tag{4.1b}$$

Equation (4.1b) is a consequence of the symmetry property $K^c_{n_1n_2,n_3n_4} = K^c_{n_3n_4,n_1n_2}$. Inserting Eq. (4.1a) into Eq. (2.14a) leads to a separable integral equation that can be easily solved. We obtain

$$\tilde{\Gamma}(\omega,\Omega,\omega'') = \left[(1+J_1)(1+J_2) - J_3^2\right]^{-1}\Big\{ f_1(\omega,\Omega)f_1(\omega'',\Omega)\,(1+J_2) + f_2(\omega,\Omega)f_2(\omega'',\Omega)\,(1+J_1) - [f_1(\omega,\Omega)f_2(\omega'',\Omega) + f_1(\omega'',\Omega)f_2(\omega,\Omega)]J_3 \Big\}, \tag{4.2a}$$

with

$$J_{1,2} = \int_0^{\Omega_0} d\omega' \, \frac{[f_{1,2}(\omega',\Omega)]^2}{2\omega'+\Omega}, \tag{4.2b}$$

and

$$J_3 = \int_0^{\Omega_0} d\omega' \, \frac{f_1(\omega',\Omega) f_2(\omega',\Omega)}{2\omega'+\Omega}. \tag{4.2c}$$

Attempt 1 in Sec. III B suggested that $\tilde{\Gamma}(\mu)$ given by Eq. (3.6) goes to a finite fixed point value at the MIT, and that the approach to criticality is characterized by a conventional power law. To see how this type of behavior can be realized from Eqs. (4.1) and (4.2), we specialize to criticality, and put $f_2 = 0$. If f_1 diverges like, e.g.,

$$f_1(\omega,\Omega) \sim (\omega+\Omega)^{-\alpha}, \quad (\alpha > 0), \tag{4.3}$$

then the FP value of $\tilde{\Gamma}$ is,

$$\tilde{\Gamma}^* = 2^{1-2\alpha} 3^{-2\alpha} \left[\int_0^\infty \frac{dx}{(x+1/2)(x+1)^{2\alpha}} \right]^{-1}, \tag{4.4a}$$

and near the FP $\tilde{\Gamma}$ satisfies the flow equation,

$$\frac{d\tilde{\Gamma}}{d\ln\mu^2} = \tilde{\Gamma}^2 2\alpha/\tilde{\Gamma}^* - 2\alpha\tilde{\Gamma}, \tag{4.4b}$$

that is, $\tilde{\Gamma} - \tilde{\Gamma}^* \sim \mu^{4\alpha}$. Similarly, if $f_2 = 0$ and f_1 vanishes at the MIT [Eq. (4.3) with $\alpha < 0$], then $\tilde{\Gamma}$ also vanishes and satisfies a flow equation with universal coefficients. The marginal, logarithmic approach to zero occurs if f_2 vanishes and f_1 either approaches a constant, or vanishes logarithmically at the MIT.

If we assume that these asymptotic results are not tied to the separable kernel, but are generic properties of the general Bethe-Salpeter equation, then we have the following scaling scenarios.

Scenario 1: γ diverges at the MIT, and $\tilde{\Gamma}$ has a finite FP value. The approach to the FP is power-law-like. *Scenario 2*: γ approaches a finite constant (or vanishes logarithmically), and $\tilde{\Gamma}$ vanishes logarithmically. *Scenario 3*: Both γ and $\tilde{\Gamma}$ vanish like a power law.

Scenarios 1 and 2 are realized by the results of Attempts 1 and 2 in Sec. III B, respectively. We conclude that neither one of these results, taken by itself, violates the constraints that follow from the Bethe-Salpeter equation. The third scenario, though we cannot exclude it, appears unlikely, given the structure of perturbation theory: The second term on the rhs of Eq. (2.9e) tends to drive K_c and hence γ towards larger values.

In all three of the scaling scenarios presented above $\tilde{\Gamma}$ satisfies an autonomous differential equation with universal coefficients. This feature is unlikely to be generic, since it clearly follows from our *ansatz* for the kernel of the Bethe-Salpeter equation, which we assumed above to consist of only one scaling part. From a more general point of view, going beyond the asymptotic behavior, one expects a more complex result. For instance, even if $\gamma \to \gamma^*$ at the MIT one expects a correction that vanishes either as a power law or as a logarithm. In fact, Eqs. (3.19) predict this kind of behavior. In our model calculation this happens if $f_2 \neq 0$. With Eqs. (4.2) it is easy to see that $\tilde{\Gamma}(\mu)$ does not satisfy an autonomous differential equation if $f_2 \neq 0$. Of course, this just reflects the obvious fact that in general an integral equation cannot be reduced to a single differential equation. An important conclusion we draw from this is that in general $\tilde{\Gamma}$ will not be a simple scaling quantity, but rather will consist of multiple scaling parts. This violates the assumption made in Attempt 1 in Sec. III B 1. Furthermore, even if γ diverges at the MIT, then one generically still expects a finite subleading contribution. Using this in either Eq. (4.2a) or in the actual Bethe-Salpeter equation, Eq. (2.14a), one finds that (1) $\tilde{\Gamma}(\mu)$ does not satisfy a single differential equation, and (2) $\tilde{\Gamma}$ approaches a finite FP value $\tilde{\Gamma}^*$, but logarithmically slowly so. We are thus led to the conclusion that in general Scenario 1 above should be replaced by a more general scenario as follows. *Scenario 1'*: γ diverges at the FP, and $\tilde{\Gamma}$ has a finite FP value, which is approached logarithmically.

We conclude that for both the case where γ diverges and where it approaches a constant at the MIT one expects a logarithmically slow approach to the FP value of $\tilde{\Gamma}$. The only other possibility is that γ vanishes as a power law at the MIT. For this case $\tilde{\Gamma}$ also vanishes as a power law, which does not seem likely considering the structure of perturbation theory.

B. Logarithmic corrections to scaling

In the preceding subsection we have argued that in general one expects $\tilde{\Gamma}$ to approach its fixed point value logarithmically slowly. This result is consistent with Eqs. (3.19) which give $\gamma_c \to \gamma_c^*$ at the MIT, which in turn implies that $\tilde{\Gamma}$ vanishes logarithmically slowly at the MIT. Because $\tilde{\Gamma}$ couples to all physical quantities, cf. Eqs. (3.7), we conclude that in all universality classes where Cooperons are present, logarithmic corrections to scaling will appear. Note that this conclusion is independent of the spatial dimensionality and depends only on whether or not the kernel γ has a constant contribution at the MIT. We also note that this is a zero-temperature, quantum mechanical effect that might be relevant for other quantum phase transitions.

For a specific example of an observable quantity, let us consider the electrical conductivity σ, which is related to the disorder by

$$\sigma = 8 S_D b^{-\epsilon}/(2\pi)^D \pi g(b). \tag{4.5}$$

Note that in giving Eq. (4.5) we have used units such that $e^2/\hbar$ is unity, and we have ignored the possibility of charge renormalization. For a discussion of the latter point we refer the reader elsewhere.[28] With t the dimensionless distance from the critical point at zero temperature, and $\delta\tilde{\Gamma} = \tilde{\Gamma} - \tilde{\Gamma}^*$, the conductivity satisfies the scaling equation,

$$\sigma(t,T) = b^{-\epsilon}\,\mathcal{F}[b^{1/\nu}t, b^{z}T, \delta\tilde{\Gamma}(b)]. \quad (4.6)$$

Here ν is the correlation length exponent, z is the dynamical scaling exponent, and of the irrelevant variables in the scaling function $\mathcal{F}$ we have kept only the one that decays most slowly at the MIT, i.e., $\delta\tilde{\Gamma}$.

At zero temperature we let $b = t^{-\nu}$, and assume that $\mathcal{F}[1,0,x]$ is an analytic function of x since it is evaluated far from the MIT. We obtain

$$\sigma(t\to 0, T=0) \approx \sigma_0 t^{s}\left\{1 + \frac{a_1}{\ln(1/t)} + \frac{a_2}{[\ln(1/t)]^2} + \dots\right\}, \quad (4.7)$$

with $s = \nu(D-2)$. Here σ_0 is an unknown amplitude and the a_i are unknown expansion coefficients. Depending on what the subleading behavior of $\delta\tilde{\Gamma}(b)$ is, the a_i with $i \geq 2$ could carry a very weak t dependence (e.g. $a_2 \sim \ln\ln t$).

An interesting consequence of the logarithmically marginal operator $\tilde{\Gamma}$ is that the dynamical scaling exponent in Eq. (4.6) is ill defined. To see this we use that z is normally defined by

$$z = d + \kappa, \quad (4.8)$$

with κ the exponent that determines the anomalous dimension of h. Equation (3.7b) suggests that in general the linearized RG flow equation for h will have the form

$$\frac{d\ln h}{dx} = \tilde{\kappa} + \alpha\delta\tilde{\Gamma}, \quad (4.9)$$

with α a universal constant and $\tilde{\kappa} = \kappa$ if $\delta\tilde{\Gamma}$ vanishes as a power law. However, if $\delta\tilde{\Gamma}$ vanishes logarithmically at the MIT, then

$$h(b) = b^{\tilde{\kappa}}(\ln b)^{\alpha}, \quad (4.10)$$

i.e., h does not scale as $b^{\tilde{\kappa}}$. This in turn implies that Eq. (4.6) should be replaced by

$$\sigma(t,T) = b^{-\epsilon}\,\mathcal{F}[b^{1/\nu}t, b^{D}h(b)T, \delta\tilde{\Gamma}(b)]. \quad (4.11)$$

At $t = 0$ and as $T \to 0$, Eqs. (4.10) and (4.11) give

$$\sigma(t=0, T\to 0) \approx \sigma_0 T^{\epsilon/z}[\ln(1/T)]^{\epsilon\alpha/z} \times\left\{1 - \frac{\epsilon\alpha^2}{z}\frac{\ln\ln(1/T^{\epsilon/z})}{\ln(1/T)} + \dots\right\}. \quad (4.12)$$

We conclude that for $\sigma(t=0,T)$ the asymptotic scaling is in general determined by logarithms.

V. DISCUSSION

In this paper we have discussed two previous attempts to renormalize the Cooper channel in the disordered electron problem.[2,3,7] We have shown that both of these approaches relied heavily on implicit assumptions concerning the renormalizability of the theory, the number of renormalization constants needed, and which quantities are simple scaling operators. We have clarified the origin of the mutually contradictory results obtained in these papers, and have shown that they result from inconsistent assumptions. We have further come to the conclusion that the Cooper propagator $\tilde{\Gamma}$ is in general not a simple scaling operator, and that none of these approaches is technically satisfactory.

One of the problems with the approaches of Refs. 2, 3, and 7 was found to be the fact that neither appreciated the seriousness of the inversion problem that is posed by the nontrivial relation between the propagator and the vertex function in the Cooper channel. We have formulated this problem in the form of a Bethe-Salpeter equation. A general discussion based on a separable kernel model suggested the three scaling scenarios presented in Sec. IV. The physically most important result of this discussion is that one expects the Cooper propagator $\tilde{\Gamma}$ to approach its fixed point value logarithmically slowly. This conclusion is essentially just a consequence of the BCS-type logarithm that appears in the Cooper channel at the Gaussian level. This was the physical content of Ref. 7. The current analysis leads us to the conjecture that the logarithmic corrections to scaling do indeed exist, even though their technical derivation (but not the physical arguments) in Ref. 7 was questionable. The point is that the structure of the Bethe-Salpeter equation ensures that the BCS logarithm is not an artifact of the Gaussian theory, but survives at higher orders for all kernels that are physically plausible given the structure of perturbation theory.

Some consequences of the logarithmic corrections to scaling for the analysis of existing experiments have been discussed in Ref. 7. In particular, that paper showed that Eq. (4.7) gives rise to an apparent or effective conductivity exponent $s_{\rm eff}$ which can be substantially smaller than the true asymptotic exponent s. It was shown that the difference between s and $s_{\rm eff}$ can be large enough to account for the observation $s_{\rm eff} \sim 0.5$ in Refs. 8 and 29 without violating the rigorous bound $s \geq 2/3$.[11] Since this was explained in Ref. 7 in some detail, there is no need to repeat it here. Also, since no theoretical value for the asymptotic exponent s in 3 D is known, no quantitative statements can be made. It is, therefore, important to see if the model can predict any qualitative features that are sensitive to the presence of the logarithmic corrections to scaling and whether or not such features are observed.

There are two obvious features predicted by our model which follow from the qualitative magnetic field dependence of the Cooperon. The first one is a strong magnetic field dependence of the effective exponent $s_{\rm eff}$. If the logarithmic corrections to scaling are caused by the Cooper propagator, which acquires a mass in a magnetic field, then any nonzero field must act to destroy the logarithms. It thus follows from the model that Si:P, or any material, in a magnetic field, must show a value of $s_{\rm eff}$ that is larger than 2/3. Any observation of an $s_{\rm eff}$ smaller than 2/3 in a magnetic field would rule out the logarithmic corrections to scaling as the source of the anomalously small value of $s_{\rm eff}$.

The second feature concerns the temperature dependence of the conductivity. It is well known that those materials which show an $s_{\rm eff} < 0$ also show a change of sign of the temperature derivative $d\sigma/dT$ of the conductivity as the transition is approached.[9,30] Within the RG description of the MIT this phenomenon can be explained by a change of sign of the g^2 term in the g-flow equation, Eq. (3.7a) or (3.8a). Since it is observed in Si:B, for which Eq. (3.8a) is the relevant flow equation, as well as in Si:P, the change of sign cannot be associated with the scaling behavior of the triplet interaction constant, but must be due to $\tilde{\Gamma}$. Since a magnetic field suppresses the Cooperon channel, this feature should therefore disappear in a sufficiently strong magnetic field.

Let us make these predictions somewhat more quantitative. The relevant magnetic field scale B_x is given by the magnetic length $l_B = \hbar c/eB$ being equal to the correlation length $\xi \simeq k_F t^{-\nu}$. Let us assume that the boundary of the critical region is given by $t \simeq 0.1$. With $k_F \simeq 4 \times 10^6$ cm^{-1} for Si:B or Si:P near the MIT and with $\nu \simeq 1$ we then obtain $B_x \simeq 1$ T. The model thus predicts that for magnetic fields exceeding about 1 T the observed effective conductivity exponent should be larger than 2/3 and the change of sign of $d\sigma/dT$ observed in zero field should disappear.

Both of these features seem to have been observed in the recent experiments by Sarachik and co-workers.[9,30] The Cooperon-induced logarithmic corrections to scaling thus provide a consistent explanation for several observed features of doped silicon which otherwise would be mysterious.

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APPENDIX: THE WILSONIAN RENORMALIZATION GROUP APPROACH

The basic philosophy behind the field theoretic renormalization approach employed in Sec. III is to eliminate all singular ultraviolet cutoff dependences, i.e., singularities of the theory as $(\Lambda, \Omega_0) \to \infty$, by introducing renormalization constants Z_i $(i = 1, 2, \ldots)$. RG flow equations are then obtained by examining how the Z_i depend on the cutoff.[19] This approach is believed to be equivalent to the Wilsonian RG, which examines how the theory changes when the ultraviolet cutoff is changed from, e.g., Λ to Λ/b with b a RG rescaling factor.[31] However, only a limited amount of work has been done on the formal relationship between these two formulations of the RG.[32] It is, therefore, not entirely obvious that the problems encountered in Sec. III are not due to our using the field-theoretic renormalization method (which failed, as we have seen, to achieve the desired cutoff independence). We, therefore, show in this Appendix that the same results as in Sec. III are obtained, and the same problems are encountered, if a Wilsonian approach is used.

We first remark that *a priori* it is not clear whether one should use a momentum-shell or a momentum-frequency shell RG procedure. Usually, to describe dynamical critical phenomena a momentum-shell approach is used.[33] In general one would argue that, since the RG is just a formal technique, either approach, correctly applied, should give the correct result. With this in mind, Eqs. (2.9) – (2.11) can be immediately used within a Wilsonian RG approach. Within a *bona fide* momentum-shell RG we use the replacement,

$$\int_{\mathbf{p}} \to \int_{\Lambda/b<p<\Lambda}, \tag{A1}$$

which leads to coupling constants that are dependent on the scale factor b. As a conventional example of using this technique we write the second term on the rhs of Eq. (2.9a) for $\epsilon \to 0$ as,

$$\frac{G^2}{4}\int_{\Lambda/b<p<\Lambda} \mathcal{D}_{n_1-n_2} \approx \frac{G\bar{G}}{8}\ln\left(\frac{\Lambda^2 + GH\Omega_{n_1-n_2}}{\Lambda^2/b^2 + GH\Omega_{n_1-n_2}}\right)$$
$$\approx \frac{G\bar{G}}{4}\ln b. \tag{A2}$$

Here we have used that the critical limit is at zero frequency, $\Omega_{n_1-n_2} = 0$.

We next note that there is a problem with applying this approach to some of the terms in Eqs. (2.9). For instance, the integral over I_c in Eq. (2.9a) leads to a term proportional to

$$\bar{G}\int_{\Lambda/b}^{\Lambda}\frac{dp}{p}\frac{\gamma_c^{(0)}}{\left[1+\gamma_c^{(0)}\ln(\Lambda/p)\right]} = \bar{G}\ln\left[1+\gamma_c^{(0)}\ln b\right]. \tag{A3}$$

Within the spirit of a differential momentum-shell RG one would expand the rhs of Eq. (A3), replacing it by $\bar{G}\gamma_c^{(0)}\ln b$. Next one would take a derivative with respect to $\ln b$ to obtain a RG flow equation. An implicit assumption in this procedure is that the RG equations will automatically generate all of the higher order logarithmic terms in Eq. (A3). However, in the present case this will not occur since the momentum-shell RG can only produce logarithms that are due to momentum integrals, while the higher order logarithms in Eq. (A3) are due to a frequency integral in I_c in Eq. (2.9a). Consequently, one must not expand the rhs of Eq. (A3), but rather consider the derivative of Eq. (A3) with respect to $\ln b$ as the definition of the zeroth order approximation for the Cooper propagator,

$$\tilde{\Gamma} = \frac{\gamma_c^{(0)}}{1+\gamma_c^{(0)}\ln b}. \tag{A4}$$

Using this reasoning one reproduces the flow equations given by Eqs. (3.7) and (3.8).

Note that by giving the definition, Eq. (A4), of $\tilde{\Gamma}$, we effectively acknowledge that the RG generates new terms in the renormalized action. That is, in order to renormalize the action, or the vertex functions, one must enlarge the parameter space. As in the case of the field-theoretic renormalization method, a RG flow equation for $\tilde{\Gamma}$ is needed to complete the description. Because the logarithmic structure in Eq. (A3) arises from a frequency integral, a momentum-frequency RG procedure is necessary to obtain this flow equation. To illustrate the procedure we restrict ourselves to zeroth order in the disorder, and show how to obtain a flow equation for $\tilde{\Gamma}$ whose zeroth order approximation is given by Eq. (A4). To proceed we split $q_{n_1 n_2}$ into high- and low-frequency components,

$$q_{n_1 n_2} = q^{>}_{n_1 n_2}\Theta(\Omega_0 - |n_1 - n_2|)\Theta(|n_1 - n_2| - \Omega_0/b^z) + q^{<}_{n_1 n_2}\Theta(\Omega_0/b^z - |n_1 - n_2|), \tag{A5}$$

with z the dynamical scaling exponent, $z = 2 + O(\epsilon)$. Examining the Gaussian action given by Eqs. (2.5) we see that the last term in Eq. (2.5c) couples $q^>$ and $q^<$. Therefore, even at zeroth order in the disorder the frequency shell RG will cause a nontrivial renormalization of the action. This is not true if one considers only the particle-hole channel, where $q^>$ and $q^<$ do not couple at that order. The renormalization can be carried out in the usual way by integrating out the $q^>$ degrees of freedom. Here we follow a slightly different, but equivalent, route. At zero momentum, and for $m = -n_1 - n_2 \geq 0$, the last term in the Cooper propagator given by Eq. (2.6b) is proportional to

$$\Gamma = \frac{\gamma_c^{(0)}}{1 + \gamma_c^{(0)} \int_0^{\Omega_0} \frac{d\omega}{2\omega + \Omega_m}}. \tag{A6}$$

Now, after the renormalization Ω_0 in Eq. (A6) is replaced by Ω_0/b^z. To compensate for this, $\gamma_c^{(0)}$ must be replaced by a scale dependent function $\tilde{\gamma}_c$ given by

$$\frac{1}{\tilde{\gamma}_c} = \frac{1}{\gamma_c^{(0)}} + \int_{\Omega_0/b^z}^{\Omega_0} \frac{d\omega}{2\omega + \Omega_m} \approx \frac{1}{\gamma_c^{(0)}} + \int_{\Omega_0/b^z}^{\Omega_0} d\omega/2\omega = \frac{1}{\gamma_c^{(0)}} + \frac{1}{2}\ln b^z. \tag{A7}$$

Using $z = 2 + \epsilon$ one obtains

$$\tilde{\gamma}_c = \tilde{\Gamma} = \frac{\gamma_c^{(0)}}{1 + \gamma_c^{(0)} \ln b}, \tag{A8}$$

or the RG flow equation

$$\frac{d\tilde{\Gamma}}{d\ln b} = -\tilde{\Gamma}^2. \tag{A9}$$

Note that it is not quite clear in this approach whether the Cooper interaction amplitude γ_c or the Cooper propagator $\tilde{\Gamma}$ is the renormalized quantity. However, because $\Omega_0/b^z \to 0$ as $b \to \infty$ these quantities become identical in the critical limit.

The most important conclusion that we draw from these considerations is that a Wilson-type RG approach leads to the same flow equations as the field-theoretic renormalization procedure. It therefore follows that the inconsistency found within the field-theoretic approach is common to both procedures. To put it another way, the inversion problem posed in terms of the Bethe-Salpeter equation must be solved, irrespective of which variant of the RG one is using. If one worked strictly within the RG framework, we would expect that calculations to higher order in the disorder will show that additional nontrivial terms in the renormalized action are generated by the RG.

[1] D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).

[2] A. M. Finkel'stein, Z. Phys. B **56**, 189 (1984); Zh. Eksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)].

[3] C. Castellani, C. DiCastro, G. Forgacs, and S. Sorella, Solid State Commun. **52**, 261 (1984).

[4] E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979); for a review, see, P. A. Lee and T. V. Ramakrishnan, Rev. Mod. Phys. **57**, 287 (1985).

[5] G. Bergmann, Phys. Rep. **101**, 1 (1984).

[6] B. L. Altshuler and A. G. Aronov, Solid State Commun. **30**, 115 (1979); B. L. Altshuler, A. G. Aronov, and P. A. Lee, Phys. Rev. Lett. **44**, 1288 (1980).

[7] T. R. Kirkpatrick and D. Belitz, Phys. Rev. Lett. **70**, 974 (1993).

[8] T. F. Rosenbaum, R. F. Milligan, M. A. Paalanen, G. A. Thomas, R. N. Bhatt, and W. Lin, Phys. Rev. B **27**, 7509 (1983).

[9] P. Dai, Y. Zhang, and M. P. Sarachik, Phys. Rev. B **45**, 3984 (1992).

[10] W. N. Shafarman, D. W. Koon, and T. G. Castner, Phys. Rev. B **40**, 1216 (1989).

[11] J. Chayes, L. Chayes, D. S. Fisher, and T. Spencer, Phys. Rev. Lett. **57**, 2999 (1986). This paper proved $\nu \geq 2/D$ with ν the correlation length exponent. For a heuristic argument to this effect, see A. B. Harris, J. Phys. C **7**, 1671 (1974). In order to conclude that $s \geq 2/D$ one needs in addition Wegner's [Z. Phys. B **25**, 327 (1976)] scaling law $s = \nu(D-2)$ which holds in all existing theories of the MIT for $2 < D < 4$.

[12] F. Wegner, Z. Phys. B **35**, 207 (1979).

[13] See, e.g., G. Grinstein, in *Fundamental Problems in Statistical Mechanics VI*, edited by E. G. D. Cohen (North Holland, Amsterdam, 1985), p.147.

[14] A disorder generated, frequency dependent triplet particle-particle interaction constant has been discussed in D. Belitz and T. R. Kirkpatrick, Phys. Rev. B **46**, 8393 (1992).

[15] M. Grilli and S. Sorella, Nucl. Phys. B **295** [**FS 21**], 422 (1988); D. Belitz and T. R. Kirkpatrick, Nucl. Phys. B **316**, 509 (1989).

[16] K. B. Efetov, A. I. Larkin, and D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. **79**, 1120 (1980) [Sov. Phys. JETP **52**, 568 (1980)].

[17] T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **41**, 11 082 (1990).

[18] C. Castellani, C. DiCastro, P. A. Lee, M. Ma, S. Sorella, and E. Tabet, Phys. Rev. B **30**, 1596 (1986); C. Castellani and C. DiCastro, *ibid.* **34**, 5935 (1986); C. Castellani, C. DiCastro, G. Kotliar, P.A. Lee, and G. Strinati, Phys. Rev. Lett. **59**, 477 (1987); Phys. Rev. B **37**, 9046 (1988).

[19] See, e.g., J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon, Oxford, 1989); M. Le Bellac, *Quantum and Statistical Field Theory* (Clarendon, Oxford, 1991).

[20] E. Brézin, J. Zinn-Justin, and J. C. Le Guillou, Phys. Rev. D **14**, 2615 (1976).

[21] D. R. Nelson and R. A. Pelcovits, Phys. Rev. B **16**, 2191 (1977).

[22] C. Castellani, C. DiCastro, and G. Forgacs, Phys. Rev. B **30**, 1593 (1984); C. Castellani, C. DiCastro, and S. Sorella, *ibid.* **34**, 1349 (1986).

[23] Even in that case, however, there is no proof of the assumption. It is entirely conceivable that higher order calculations would reveal the need for, e.g., different wave-function renormalizations in the spin-singlet and spin-triplet channels. We will proceed under the assumption that this is not the case.

[24] T. R. Kirkpatrick and D. Belitz, Phys. Rev. B **45**, 3187 (1992).

[25] Often the RG is thought of as strictly a technique to resum logarithmic divergencies. If one adopts this interpretation one is tempted to expand Eq. (3.10a) in powers of $\gamma_c x$ and ignore the singularity implied by the infinite series. We will show below Eq. (3.15) that such an expansion is invalid.

[26] H. Fukuyama, H. Ebisawa, and S. Maekawa, J. Phys. Soc. Jpn. **53**, 1919 (1984); see also, H. Fukuyama, Physica **126B**, 306 (1984); S. Maekawa, H. Ebisawa, and H. Fukuyama, Progr. Theor. Phys. Suppl. **84**, 154 (1985), and references therein.

[27] A. M. Finkel'stein, in *Anderson Localization*, edited by T. Ando and H. Fukuyama (Springer, New York, 1988).

[28] D. Belitz and T. R. Kirkpatrick, Phys. Rev. B. **48**, 14 072 (1993).

[29] Very recently, the interpretation of the data in Ref. 8 has been questioned [H. Stupp, M. Hornung, M. Lakner, O. Madel, and H.v. Löhneysen, Phys. Rev. Lett. **71**, 2634 (1994). These authors assert that the rounding of the data ascribed to sample inhomogeneities in Ref. 8 instead signals a crossover and that the effective exponent is 1.3. In that case, logarithmic corrections to scaling may still be present, but in the absence of a theoretical value for s in $D=3$ the experiment would not be a suitable test of their presence. In Ref. 7 we have shown that even if the value of Ref. 8 is the correct one, the presence of logarithmic corrections to scaling can solve the exponent puzzle.

[30] P. Dai, Y. Zhang, S. Bogdanovich, and M. P. Sarachik, Phys. Rev. Lett. **48**, 4941 (1993).

[31] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).

[32] J. Polchinski, Nucl. Phys. B **231**, 269 (1984).

[33] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).