Generalized quantum Mattis spin glasses with p-spin interactions

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A generalization of the p-spin-interaction quantum multicomponent Mattis model in a transverse field is investigated by use of the Suzuki-Trotter approach with the thermodynamic perturbation theory. The phase diagram is obtained in the limit $p\rightarrow\infty$, and the effects of quantum fluctuations on the first-order phase transitions separating the disordered phase from the ordered phases are discussed. We find that paramagnetic phase is characterized by transverse ordering, whereas the ordered phases are not affected by the quantum fluctuations.

The study of the effect of quantum fluctuations on phase transitions in the Ising and Heisenberg spin-glass models has been a subject of much interest.[1–29] In particular, the quantum infinite-range transverse Ising spin-glass model has been studied very extensively[2–15,22–25] by means of various approaches. This model is a simple and, as recently demonstrated, an experimentally accessible example[23,24,28–31] of a classical disordered magnetic system which can be converted into its quantum analog. In quantum transverse spin-glass models with infinite-range interactions much attention has been paid to the description of the phase transition between the paramagnetic and spin-glass phases. It has been found that the transverse field has the effects of destroying the spin-glass phase, and when strong enough, destroys the spin-glass phase even at $T=0$. Lately, the p-spin-interaction spin-glass model in a transverse field with $p\rightarrow\infty$ is solved by using the static approximation.[32,33] For the case of the classical Ising spin glasses, such an exactly solvable model which is referred to as "the simplest spin glass," is the so-called Derrida random-energy model[34,35] which consists of a collection of independent random-energy levels. In addition, we should also note that the study of this model is very closely related to a straightforward generalization[36–38] of the Hopfield neural-network models[39,40] which considered higher-order interactions. Correspondingly, the natural extension of the classical Hebbian learning rule to order p is defined by

$$J_{i1\cdots ip}=\sum_{\mu=1}^{n}\xi_{i1}^{\mu}\xi_{i2}^{\mu}\cdots\xi_{ip}^{\mu}\ , \tag{1}$$

i.e., a set of n uncorrelated patterns $\{\xi_i^{\mu}\}$ $\{i=1,2,\ldots,N,\mu=1,2,\ldots,n\}$ in which ξ_i^{μ} is either $+1$ or -1 with equal probability, is encoded in the interaction matrix.

On the other hand, a generalization[41,42] of the classical Mattis spin-glass model[43] with the competing interactions

$$J_{ij}=\frac{1}{N}\left[J_0+\sum_{\mu=1}^{n}\xi_i^{\mu}\xi_j^{\mu}\right] \tag{2}$$

has been studied by various authors. The study of this generalized Mattis model is very interesting not only in understanding the behavior of those relatively realistic spin-glass models but also in the context of models of neural networks. For $J_0=0$ this reduces to the Hopfield neural-network model with a finite number n of memorized patterns. The purpose of this paper is to study a generalization of the p-spin-interaction quantum multicomponent Mattis model with the interactions given by a generalized Hebb rule

$$J_{i1\cdots ip}=\frac{p!}{N^{p-1}}\left[J_0+\sum_{\mu=1}^{n}\xi_{i1}^{\mu}\xi_{i2}^{\mu}\cdots\xi_{ip}^{\mu}\right]\ , \tag{3}$$

like in the infinite-range spin-glass model, the presence of $P!/N^{p-1}$ in the interaction is necessary to ensure an extensive free energy.[45] We will apply the Suzuki-Trotter formula[44] to reduce the model to an equivalent classical one and then use the perturbation theory for this effective classical model. We are interested in particular in the $p\rightarrow\infty$ limit of quantum transverse Mattis model, where the thermodynamics of the model can be solved exactly. The effect of quantum fluctuations on phase diagrams is examined.

We consider the model with the Hamiltonian

$$H=-\sum_{(i1\cdots ip)}J_{i1\cdots ip}\sigma_{i1}^{z}\sigma_{i2}^{z}\cdots\sigma_{ip}^{z}-\Gamma\sum_i\sigma_i^{x}\ , \tag{4}$$

where the σ are Pauli spin matrices and Γ is the transverse field.

Applying the Suzuki-Trotter formula[44] to the partition function, the effective classical Hamiltonian with an extra "Trotter" dimension is obtained in the Mth approximation as

$$H_{\rm eff}=-\frac{1}{M}\sum_{k=1}^{M}\sum_{(i1\cdots ip)}J_{i1\cdots ip}\sigma^z_{i1,k}\sigma^z_{i2,k}\cdots\sigma^z_{ip,k}-\frac{a}{\beta}\sum_{k=1}^{M}\sum_{i=1}^{N}\sigma_{i,k}\sigma_{i,k+1}-\frac{MNC}{\beta}\ , \tag{5}$$

where we define

$$a\equiv\tfrac{1}{2}\ln[\coth(\beta\Gamma/M)]\ , \tag{6}$$

$$c\equiv\tfrac{1}{2}\ln[\tfrac{1}{2}\sinh(2\beta\Gamma/M)]\ , \tag{7}$$

$\sigma_{i,k}=\pm1$ is the classical Ising spin on the lattice (i,k), i is the position in the original system, and k is the index for the Trotter dimension introduced due to the quantum dynamics. Ultimately the limit $M\to\infty$ must be taken. In Eq. (5), β is the inverse temperature and the exchange interaction $J_{i1\cdots ip}$ is as given in Eq. (3).

The partition function of the system can be written

$$Z=\mathrm{Tr}\exp(-\beta H_{\rm eff})\ . \tag{8}$$

We can approach the statistical mechanics of the effective Hamiltonian, Eq. (5), by using thermodynamic perturbation theory. The starting point is to consider the spin-spin interaction terms,[46]

$$-\beta V=\frac{\beta}{M}\sum_{k=1}^{M}\sum_{(i1\cdots ip)}J_{i1\cdots ip}\sigma_{i1,k}\sigma_{i2,k}\cdots\sigma_{ip,k}\ , \tag{9}$$

as a perturbation of the single-site Hamiltonian

$$-\beta H_0=a\sum_{k=1}^{M}\sum_{i=1}^{N}\sigma_{i,k}\sigma_{i,k+1}+MNC\ . \tag{10}$$

The free energy of the full system can be expressed in terms of the reference system free energy F_0 and the interaction averaged in the reference system, $\langle V\rangle_0$:

$$-\beta F=-\beta F_0-\beta\langle V\rangle_0\ , \tag{11}$$

with

$$-\beta F_0=\ln\mathrm{Tr}\exp(-\beta H_0)\ . \tag{12}$$

We introduce the average magnetization, which distinguishes the ferromagnetic from the paramagnetic phase,

$$m=\frac{1}{MN}\sum_{k=1}^{M}\sum_{i=1}^{N}\langle\sigma_{i,k}\rangle\ , \tag{13}$$

and define an "optimum" reference system by using a Lagrange multiplier λ to enforce the constraint that the order parameter m takes on its self-consistently determined values in the reference system itself. Then the reference system Hamiltonian can be revised as

$$-\beta H_0=a\sum_{k=1}^{M}\sum_{i=1}^{N}\sigma_{i,k}\sigma_{i,k+1}-\lambda\left[Nm-M^{-1}\sum_{k=1}^{M}\sum_{i=1}^{N}\sigma_{i,k}\right]+MNC\ , \tag{14}$$

and the reference system partition function becomes

$$Z_0=\mathrm{Tr}\exp(-\beta H_0)=\prod_{i=1}^{N}(e^{-m\lambda}Z_0^{(i)})\ , \tag{15}$$

where

$$Z_0^{(i)}=\sum_{\{\sigma_k\}}\exp\left[a\sum_{k=1}^{M}\sigma_k\sigma_{k+1}+\frac{\lambda}{M}\sum_{k=1}^{M}\sigma_k+Mc\right]\ . \tag{16}$$

Equation (15) is nothing other than the partition function of the Ising chain in a longitudinal field with the periodic boundary condition and can be solved by the transfer-matrix method.[47] The resulting reference free energy F_0 can be obtained by taking $M\to\infty$ limit,

$$-\beta F_0/N=-m\lambda+\ln 2\cosh[\lambda^2+(\beta\Gamma)^2]^{1/2}\ . \tag{17}$$

The remaining term in the full free energy, Eq. (11), is obtained by

$$\langle V\rangle_0=\mathrm{Tr}\left\{\left[-\frac{J_0p!}{MN^{p-1}}\sum_{k=1}^{M}\sum_{(i1\cdots ip)}\sigma_{i1,k}\sigma_{i2,k}\cdots\sigma_{ip,k}-\frac{p!}{MN^{p-1}}\sum_{k=1}^{M}\sum_{(i1\cdots ip)}\sum_{\mu=1}^{n}(\xi^\mu_{i1}\sigma_{i1,k})(\xi^\mu_{i2}\sigma_{i2,k})\cdots(\xi^\mu_{ip}\sigma_{ip,k})\right]\exp(-\beta H_0)\right\}\times[\mathrm{Tr}\exp(-\beta H_0)]^{-1}=-J_0Nm^p-N\sum_{\mu=1}^{n}(q^\mu)^p\ , \tag{18}$$

where the overlap q^u which describes a spin-glass phase,[42–44] is defined by

$$q^\mu=\frac{1}{MN}\sum_{k=1}^{M}\sum_{i=1}^{N}\xi_i^\mu\langle\sigma_{i,k}\rangle\ . \tag{19}$$

Accordingly, the free energy per spin is given by

$$-\beta f=-m\lambda+\ln 2\cosh[\lambda^2+(\beta\Gamma)^2]^{1/2}+\beta J_0m^p+\beta\sum_{\mu=1}^{n}(q^\mu)^p\ . \tag{20}$$

Here the order parameter m may be determined by minimizing the free energy with respect to m:

$$\lambda=\beta J_0pm^{p-1}+\beta p\sum_{\mu=1}^{n}(q^\mu)^{p-1}\xi^\mu\ . \tag{21}$$

From Eq. (21) we see that the Lagrange multiplier λ is a random variable, and the final expression of the free energy has to be averaged over the random variables ξ^μ.

Finally, the free energy averaged over disorder is

$$f = J_0(p-1)m^p + (p-1)\sum_{\mu=1}^{n}(q^\mu)^p - \frac{1}{\beta}\left\langle \ln 2\cosh\beta\left[\Gamma^2 + \left[J_0 p m^{p-1} + p\sum_{\mu=1}^{n}(q^\mu)^{p-1}\xi^\mu\right]^2\right]^{1/2}\right\rangle , \tag{22}$$

where $\langle \cdots \rangle$ denotes the average over the distribution of ξ^μ.

For convenience, we consider, as usual, only one of patterns (the first, say) to have a macroscopic overlap with the configuration $\{\sigma_i\}$ and the remaining $(n-1)$ patterns to have an overlap at most of order $O(1/\sqrt{N})$ without loss of generality. This Mattis spin-glass solution is described by the order parameter of the form $q^\mu = q\delta_{\mu 1}$. In this case,

$$f = J_0(p-1)m^p + (p-1)q^p - T\left\langle \ln 2\cosh\frac{1}{T}[\Gamma^2 + (J_0 p m^{p-1} + p q^{p-1}\xi)^2]^{1/2}\right\rangle . \tag{23}$$

In the limit of large p, Eq. (23) has four possible solutions: (1) A paramagnetic phase (P) with $m=0$ and $q=0$,

$$f_P = -T\ln 2\cosh(\Gamma/T) . \tag{24}$$

(2) A spin-glass (SG) solution with $m=0$ and $q=1$,

$$f_{SG} = -1 . \tag{25}$$

(3) Ferromagnetic (FM) solutions with $m=1$ and $q=0$,

$$f_{FM} = -J_0 , \tag{26}$$

i.e., the free energy in the FM phase depends only on J_0 and not on T and Γ, and (4) A mixed phase (M) with $m=1$ and $q=1$,

$$f_M = -J_0 + p - 1 . \tag{27}$$

Unlike the case in the $p=2$ model, the mixed phase is suppressed in the existence of the p-spin interaction with $p \to \infty$.

The phase transition between the P and SG phases is determined by equating the free energies of two phases which gives the equation

$$T\ln 2\cosh(\Gamma/T) = 1 . \tag{28}$$

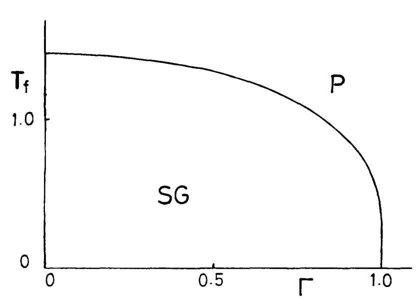

FIG. 1. Dependence of the spin-glass freezing temperature T_f on the transverse field Γ (in units of J).

Figure 1 shows the dependence of the freezing temperature T_f on the transverse field Γ. We see that the SG freezing temperature T_f as a function of Γ decreases continuously from its maximum value $T_f = 1/\ln 2$, as Γ grows, and vanishes for the critical value of the transverse field $\Gamma(T_f=0)=1$. Similarly, the P-to-FM and SG-to-FM phase boundaries are determined by the equations $f_P = f_{FM}$ and $f_{SG} = f_{FM}$, respectively. The resulting phase diagram of the model is presented in Fig. 2, where all transitions are first-order ones. For $J_0 = 0$ we expect no FM ordering, hence in the high-temperature phase $T > T_f$, the system is paramagnetic which is characterized by transverse ordering, while for $T < T_f$ the system is in the frozen spin-glass phase and the free energy is independent of T and Γ. When $J_0 \neq 0$, a ferromagnetic phase exists in addition to the paramagnetic phase and SG phase. There is a transition from the paramagnetic to the ferromagnetic phase when $T\ln 2\cosh(\Gamma/T) = J_0$. This transition line stops when it meets the separation line between the paramagnetic and the spin-glass phase, $T\ln 2\cosh(\Gamma/T)=1$. This happens for $J_0 = 1$. In the present large-p case, quantum fluctuations have the effect of destroying the ordered phases, but the nature of the ordered phases is similar to the classical model.

In conclusion, we have studied the generalized Mattis spin-glass model with p-spin interaction in the presence of transverse fields. The general formulation for the free energy of the models is derived, and the phase diagrams are obtained in the limit of large p. Unlike the $p=2$ case, there is no second-order phase transitions at all temperatures. In the frozen phases, all quantum fluctuations are suppressed, and the problem reduces to the classical limit. Quantum fluctuations make the SG and FM phases unstable, and even the frozen-ordered phases, which depend strongly on Γ, can be destroyed.

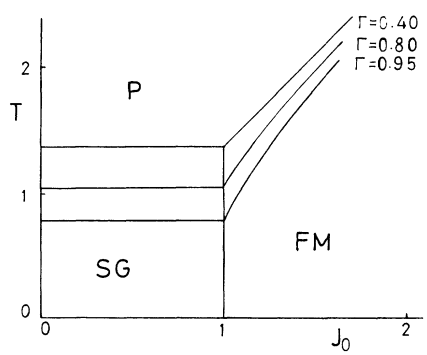

FIG. 2. The phase diagram of the systems in the large p limit for three values of Γ (in units of J).

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[1]A. J. Bray and M. A. Moore, J. Phys. C **13**, L665 (1980).
[2]H. Ishii and T. Yamamoto, J. Phys. C **18**, 6225 (1985).
[3]K. D. Usadel, Solid State Commun. **58**, 629 (1986).
[4]K. Walasek and K. Lukierska-Walasek, Phys. Rev. B **38**, 725 (1988).
[5]D. Thirumalai, Q. Li, and T. R. Kirkpatrick, J. Phys. A **22**, 3339 (1989).
[6]P. Ray, B. K. Chakrabarti, and A. Chakrabarti, Phys. Rev. B **39**, 11 828 (1989).
[7]Y. Q. Ma and Z. Y. Li, Phys. Lett. A **145**, 19 (1990); **148**, 134 (1990).
[8]G. Buttner and K. D. Usadel, Phys. Rev. B **41**, 428 (1990); **42**, 6385 (1990).
[9]Y. Y. Goldschmidt and P. Y. Lai, Phys. Rev. Lett. **64**, 2467 (1990).
[10]L. De Cesare, K. Lukierska-Walasek, and K. Walasek, Phys. Rev. B **45**, 8127 (1992).
[11]Y. Q. Ma, Z. Y. Li, and C. D. Gong, J. Phys. Condens. Matter **3**, 4687 (1991).
[12]Y. Q. Ma and C. D. Gong, Phys. Rev. B **45**, 793 (1992).
[13]Y. Q. Ma and C. D. Gong, Phys. Rev. B **48**, 12 778 (1993).
[14]S. Dattagupta, B. Tadic, R. Pirc, and R. Blinc, Phys. Rev. B **44**, 4387 (1991).
[15]F. Pazmandi and Z. Domanski, J. Phys. Condens. Matter **5**, L117 (1993).
[16]T. K. Kopec, G. Buttner, and K. D. Usadel, Phys. Rev. B **41**, 9221 (1990).
[17]G. Buttner, T. K. Kopec, and K. D. Usadel, Phys. Lett. A **149**, 248 (1990).
[18]G. Buttner and K. D. Usadel, Europhys. Lett. **14**, 165 (1991).
[19]G. Buttner and K. D. Usadel, Z. Phys. B **83**, 131 (1991).
[20]K. Walasek and K. Lukierska-Walasek, Phys. Lett. A **159**, 97 (1991).
[21]L. Decesare, K. Lukierska-Walasek, I. Rabuffo, and K. Walasek, Phys. Rev. B **45**, 1041 (1992).
[22]For a review, see P. Sen and B. K. Chakrabarti, Int. J. Mod. Phys. B **6**, 2439 (1992).
[23]V. Dobrosavljeric and R. M. Stratt, Phys. Rev. B **36**, 8484 (1987).
[24]R. Pirc, B. Tadic, and R. Blinc, Phys. Rev. B **36**, 8607 (1987).
[25]J. Miller and D. A. Huse, Phys. Rev. Lett. **70**, 3147 (1993).
[26]J. Ye, S. Sachdev, and N. Read, Phys. Rev. Lett. **70**, 4011 (1993).
[27]S. Sachdev and J. Ye, Phys. Rev. Lett. **70**, 3339 (1993).
[28]W. Wu, B. Ellman, T. F. Rosenbaum, G. Aeppli, and D. H. Reich, Phys. Rev. Lett. **67**, 2076 (1991).
[29]W. Wu, D. Bitko, T. F. Rosenbaum, and G. Aeppli, Phys. Rev. Lett. **71**, 1919 (1993).
[30]Y. Moritomo and Y. Tokura, Phys. Rev. Lett. **71**, 2833 (1993).
[31]O. J. Kwon and J. J. Kim, Phys. Rev. B **48**, 6639 (1993).
[32]Y. Y. Goldschmidt, Phys. Rev. B **41**, 4858 (1990).
[33]V. Dobrosavljevic and D. Thirumalai, J. Phys. A **23**, L767 (1990).
[34]B. Derrida, Phys. Rev. Lett. **45**, 79 (1980); Phys. Rev. B **24**, 2613 (1981).
[35]D. J. Gross and M. Mezard, Nucl. Phys. B **240**, 431 (1984).
[36]N. Sourlas, *Stastical Mechanics of Neural Network,* edited by L. Garrido (Springer-Verlag, Berlin, 1990), p. 317.
[37]E. Gardner, J. Phys. A **20**, 3453 (1987).
[38]E. Gardner, B. Derrida, and P. Mottishaw, J. Phys. (Paris) **48**, 741 (1987).
[39]J. J. Hopfield, Proc. Natl. Acad. Sci. **79**, 2554 (1982); **81**, 3088 (1984).
[40]See, for example, D. Amit, *Modeling Brain Function* (Cambridge University Press, New York, 1989); B. Muller and J. Reinhardt, *Neural Networks: An Introduction* (Springer, Berlin, 1991); in *Physics of Neural Networks,* edited by J. L. Van Hemmen, E. Domany, and K. Schulten (Springer, Berlin, 1990).
[41]J. L. Van Hemmen, Phys. Rev. Lett. **49**, 409 (1982).
[42]J. P. Provost and G. Vallee, Phys. Rev. Lett. **50**, 598 (1983).
[43]D. C. Mattis, Phys. Lett. A **56**, 421 (1976).
[44]M. Suzuki, *Quantum Monte Carlo Methods* (Springer, Berlin, 1986).
[45]K. Binder and A. P. Young, Rev. Mod. Phys. **58**, 801 (1986).
[46]R. M. Stratt, Phys. Rev. B **33**, 1921 (1986).
[47]See, e.g., C. J. Thompson, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, New York, 1972), Vol. 1, p. 177.