# Dynamical $T = 0$ correlations of the $S = \frac{1}{2}$ one-dimensional Heisenberg antiferromagnet with $1/r^2$ exchange in a magnetic field

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We present a selection rule for matrix elements of local spin operators in the $S = \frac{1}{2}$ Haldane-Shastry model. Based on this rule we extend a recent exact calculation by Haldane and Zirnbauer of the ground-state dynamical spin correlation function $S^{ab}(n,t) = \langle 0|S^a(n,t)S^b(0,0)|0\rangle$ and its Fourier transform $S^{ab}(Q,E)$ of this model to a finite magnetic field. In zero field, only *two-spinon* excitations contribute to the spectral function; in the (positively) partially spin-polarized case, there are two types of elementary excitations: *spinons* ($\Delta S^z = \pm\frac{1}{2}$) and *magnons* ($\Delta S^z = -1$). The magnons are divided into left- or right-moving branches. The only classes of excited states contributing to the spectral functions are (I) two spinons, (II) two spinons + one magnon, (IIIa) two spinons + two magnons (moving in opposite directions), and (IIIb) one magnon. The contributions to the various correlations are $S^{-+}$: (I); $S^{zz}$: (I)+(II); $S^{+-}$: (I)+(II)+(III). In the zero-field limit there are no magnons, while in the fully polarized case, there are no spinons. We discuss the relation of the spectral functions to correlations of the Calogero-Sutherland model at coupling $\lambda = 2$.

## I. INTRODUCTION

In 1988 Haldane and Shastry[1,2] independently introduced a $S = \frac{1}{2}$ one-dimensional (1D) spin model on $N$ sites with an exchange interaction that falls off inversely proportional to the distance between the spins. In the past few years this model has proven to be solvable to a remarkable extent.[3-5] The simple structure of this model even allowed the authors of Ref. 6 to compute the zero-magnetic-field dynamical structure function (DSF) at zero temperature:

$$\langle \text{GS}|S_m^a(t)S_n^b(t')|\text{GS}\rangle = S^{ab}(m-n,t-t') \qquad (1)$$

$(a, b = x, y, z)$. In this paper we extend these results to a dynamical ground-state correlation function in a *nonzero* magnetic field. Although a closed expression is not available, we are able to identify relevant excitations that contribute to these functions. More specifically, when we expand the expression (1) for the dynamical structure functions in a basis of eigenstates of the Hamiltonian $\{|\nu\rangle\}$,

$$\langle \text{GS}|S_m^a(m,t)S_n^b(n,t')|\text{GS}\rangle$$

$$= \sum_\nu \langle \text{GS} \mid S_m^a(t) \mid \nu\rangle\langle \nu \mid S_n^b(t') \mid \text{GS}\rangle \, e^{-\frac{i}{\hbar}(t'-t)(E_\nu - E_0)},$$

$$(2)$$

the set of intermediate states $|\nu\rangle$ that contribute to the sum is finite and contains only states that have very small numbers of elementary excitation added to the ground state, viz., two "spinons" and up to two "magnons." In zero magnetic field the magnon excitations are absent

(see Ref. 6), whereas in very strong magnetic fields (such that all spins are fully polarized in the ground state) only one magnon participates. The set of intermediate states is small as a consequence of a new selection rule for matrix elements of the local spin operators between eigenstates of this model. We will present it below.

A more traditional playground for 1D spin chains is the Heisenberg model with nearest neighbor exchange (NNE).[7] It shares its low energy properties with the Haldane-Shastry model (HSM). However, for the NNE model the number of excited states contributing to its dynamic structure functions is not bounded; consequently these functions are still unknown in both zero and nonzero magnetic field. It seems again that the characterization of the HSM as an "ideal spinon gas"[3] allows one to push forward our understanding of these 1D spin chains much further.

For the remaining part of this presentation, we will continue with a brief reiteration of the relevant properties of the Haldane-Shastry model (HSM) in Sec. II. Section III describes the selection rule. We then use this rule to identify the contributions of intermediate states to the different DSF's in Secs. IV–VI. Finally in Sec. VII we conclude with a comparison to earlier DSF calculations for the NNE model.

## II. YANGIAN SYMMETRIES

The Hamiltonian of the HSM in a magnetic field on $N$ sites is given by

$$H = \frac{\pi v_s}{N^2} \sum_{i \neq j} d^{-2}(i-j)(P_{ij}-1) + h\sum_i S_i^z, \qquad (3)$$

where $d(n) = \sin(\frac{n\pi}{N})$ and $P_{ij}$ is the operator that permutes two spins on sites $i$ and $j$. As described in Refs. 1, 4, and 5, in zero magnetic field the spectrum of the HSM consists of large degenerate multiplets. Responsible for these degeneracies is a symmetry algebra of (3) identified in Ref. 4 as the Yangian $Y(sl(2))$ which is generated by the following two vector operators:

$$\mathbf{J}_0 = \sum_{i=1}^{N} \mathbf{S}_i,$$

$$\mathbf{J}_1 = \sum_{i<j} w_{ij} \mathbf{S}_i \times \mathbf{S}_j, \qquad (4)$$

where $w_{ij} = \frac{z_i + z_j}{z_i - z_j}$ and the $\{z_i\}$, $i = 1, \ldots, N$ are $N$ equally spaced points on the unit circle. Both operators commute with the Hamiltonian. Every degenerate multiplet forms a representation of this algebra and is characterized by a so-called (Yangian) highest weight state (YHWS). That state is the only member of that multiplet which is annihilated by $J_0^+$ and $J_1^+$. The rest of its multiplet is generated when we act upon the highest weight state with the other members of the Yangian algebra. Every Yangian multiplet has an accompanying *Drinfeld polynomial* $P(u)$ of order $\leq N$ in $u$. The roots of this polynomial (which can be chosen to be half-integers in the range $0, \ldots, N$) correspond to the elementary excitations of this model: spinons. A Drinfeld polynomial can be represented pictorially as a sequence $\Gamma$ of $N - 1$ zeros or ones, a generalized occupation number configuration. We tag on two zeros in positions 0 and $N$. As elucidated in Ref. 5 two ones have to be separated by at least one 0. The roots of the Drinfeld polynomial are then located between two consecutive zeros. The locations of the ones, $\{m_i\}$, are the so-called *rapidities*. There are $M = \frac{N - N_{sp}}{2}$ of them, where $N_{sp}$ is the number of spinons. To clarify this with an example for $N = 10$ sites, let us consider the multiplet characterized by the sequence 101000100. It has three rapidities $m_1 = 1, m_2 = 3, m_3 = 7$ and four spinons causing roots in the Drinfeld polynomial at $u = 4\frac{1}{2}$, $5\frac{1}{2}$, $8\frac{1}{2}$, and $9\frac{1}{2}$.

Following Ref. 3, a sequence of $m + 1$ zeros (between ones) is to be interpreted as a single orbital filled with $m$ spinons in a symmetric state. Since spinons have spin $1/2$, this implies that the orbital has total spin $\frac{m}{2}$ (more formally, this sequence gives an "$m$ string" of roots of the Drinfeld polynomial, which, from the representation theory of the Yangian algebra,[8] constitutes a spin-$\frac{m}{2}$ factor in the representation). In the case of the above example multiplet that means that the multiplet has total spin content $1 \otimes 1 = 2 \oplus 1 \oplus 0$ and contains $3 \times 3 = 9$ states. With $M$ ones there are therefore $M + 1$ orbitals—two of which are empty in our example. Notice that the maximum possible $S^z$ and $S^{tot}$ a state in the multiplet can have is $N_{sp}/2$. That state (the YHWS) has all its spinons polarized up.

These spinons have a semionic character as is evidenced by the fact that adding two spinons (and we always have to add two at a time to avoid getting a sequence with consecutive ones) to a sequence reduces the number of available orbitals $M + 1 = \left(\frac{N - N_{sp}}{2} + 1\right)$ by 1, as opposed to 0 for bosons and 2 for fermions.[9] All the states in a multiplet characterized by a sequence $\{m_i\}$ have the same energy and momentum given by

$$P = \sum_{i=1}^{M} m_i \bmod N, \quad \text{in units of } \frac{2\pi}{N},$$

$$E = \sum_{i=1}^{M} \left(\frac{2\pi v_s}{N^2}\right) m_i(m_i - N). \qquad (5)$$

We can also express these quantities in terms of spinon variables. If we label the $M + 1$ orbitals from right to left by spinon momenta $-k_0 \leq k \leq k_0 = \frac{2\pi}{N}\frac{M}{2}$ spaced by $\frac{2\pi}{N}$, we get

$$P = \sum_k k n_{k\sigma} + N k_0 \bmod 2\pi,$$

$$E = \sum_{k\sigma} \epsilon(k) n_{k\sigma} + \frac{1}{N} \sum_{kk',\sigma\sigma'} V(k - k') + E(M, N), \quad (6)$$

where $\epsilon(k) = \frac{v_s}{\pi}(k_0^2 - k^2)$, $V(k) = v_s(k_0 - |k|)$, and $n_{k\sigma}$ is the number of spinons in the orbital with momentum $k$ and spin $\sigma$. $E(M, N)$ only depends on the total number of sites and spinons. We also recognize $v_s$ as the spinon velocity $\frac{d\epsilon(k)}{dk}$ at the zone endpoint, $k_0$, in the ground state.

The action of the Yangian algebra within a multiplet of states is to rotate the spinons individually [rather than all of them through the global SU(2) spin operators]. For this reason $J_0^+$ and $J_1^+$ annihilate a YHWS since it has all its spinons fully polarized. Therefore the YHWS's have also been dubbed fully polarized spinon gas states[3] (FPSG). In a local spin basis $\{|\{n_1, \ldots, n_M\}\rangle\}$ where the $\{n_i\}$ are the locations of the down spins, the wave functions of these FPSG states $\Psi(n_1, \ldots, n_M) = \psi(z_{n_1}, \ldots, z_{n_M})$ are polynomials in the $\{z_{n_i}\}$ of degree $< N$. They can be written as $\chi \Psi_0$, where $\Psi_0$ is the $h = 0$ ground-state Jastrow wave function, and $\chi$ is a polynomial known in the mathematical literature as a Jack polynomial. Algorithms for their construction exist.[10]

To get a more physical idea of spinon states consider the following wave functions in the same basis:[5]

$$\Psi(n_1, \ldots, n_M | \alpha_1, \ldots, \alpha_{N_{sp}})$$

$$= \left(\prod_{i<j}(z_{n_i} - z_{n_j})\right)^2 \left(\prod_{i=1}^{M} z_{n_i}\right) \left(\prod_{i=1}^{M}\prod_{j=1}^{N_{sp}}(z_{n_i} - z_{\alpha_j})\right). \quad (7)$$

The $\{\alpha_i\}$ are the locations of *localized* spinons that can point up or down; the $\{n_i\}$ are the positions of down spins (other than those of possible localized spinons pointing down). Notice that the wave function prevents the $\{n_i\}$ from coinciding with the spinon sites. We call the complement of the set of spinon sites the *condensate*. It is a singlet under the action of total spin. Furthermore for $N_{sp} = 0$ Eq. (7) represents the exact ground-state wave

function for $h = 0$. The usefulness of these states is limited by the fact that they are not mutually orthogonal and, worse, overcomplete. However, based on numerical evidence for up to 12 spinons, it is clear that the space spanned by states (7) with a fixed number $N_{sp}$ of *localized* spinons contains only eigenstates of the Hamiltonian belonging to Yangian multiplets with $N_{sp}$ or fewer spinons.

The subspace of states that have a fixed number of $N_{sp}$ localized spinons, all polarized, has the pleasant property that it only contains eigenstates of $H$ with *precisely* $N_{sp}$ spinons. This is clear from the fact that these local spinon wavefunctions—although not eigenstates of the Hamiltonian—are annihilated by both $J_0^+$ and $J_1^+$ (see Appendix A). This is consistent with the fact that fully polarized spinon eigenstates are supposed to be of a polynomial form with degree $< N$ in the $\{z_{n_i}\}$,[3] just like wave functions (7). In that same article it is also shown that these fully polarized localized spinon states are *complete* as well and span *all* YHWS's.

In a nonzero magnetic field the term $h \sum_i S_i^z$ in the Hamiltonian (3) will give a Zeeman splitting within the Yangian multiplets, although its members remain eigenstates of $H$. As a consequence, for increasing magnetic field, the ground state will contain more and more (fully polarized) spinons. In the thermodynamic limit their number is given by

$$\frac{N_{sp}}{N} = 2\sigma = 1 - \sqrt{1 - \frac{h}{h_c}}, \quad h_c = \frac{\pi}{2}v_s, \qquad (8)$$

where $\sigma$ denotes the ground-state magnetization. For $h \geq h_c$ the ground state is completely ferromagnetic.

The occupation sequence characterizing the Yangian multiplet that contains the ground state will have the spinons "condense" into the left- and rightmost orbital (in equal numbers), in accordance with the spinon dispersion (6) relation which assumes a minimum at $\pm k_0$. So for a typical magnetic field below the critical value, the ground state would be the YHWS of a Yangian multiplet that is described by a sequence like 000010101010000. For higher fields the 1010101 pattern shrinks as more spinons go into left and right orbitals. These wave functions happen to be known analytically:[1]

$$\Psi(n_1,\ldots,n_M) = \prod_{i<j} \left(z_{n_i} - z_{n_j}\right)^2 \prod_{i=1}^{M}(z_{n_i})^{\frac{N}{2}-M+1}. \qquad (9)$$

## III. STRUCTURE FUNCTIONS AND THE SELECTION RULE

The dynamic structure function $S^{ab}(m - n, t - t') = \langle GS|S^a(m,t)S^b(n,t')|GS\rangle$ with $S^a(m,t) = \exp(-itH)S_m^a \exp(itH)$ measures the response of the system to excitations created by flipping and/or imposing a certain spin on a particular site $n$ in state $|GS\rangle$ at time $t'$ and measuring its effect at time $t$ on site $m$. Obviously, since $H$ conserves $S^z$, only $S^{-+}$, $S^{zz}$, and $S^{+-}$ are nonzero. At zero temperature the Fourier transform

$S(Q,E)$ of its expansion (2) in a basis of eigenstates $\{|\nu\rangle\}$ with energy $E_\nu$ and momentum $p_\nu$ looks like

$$S(Q,E) = \sum_\nu M_\nu^a \delta(E - (E_\nu - E_0))\, \delta(Q - (p_\nu - p_0)),$$

$$M_\nu^a = 2\pi \left|\langle \nu| S^a(Q) |GS\rangle\right|^2, \qquad (10)$$

with $S^a(Q) = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} \exp(-inQ)S_n^a$, and $|GS\rangle$ the ground state of the model. That is, the support of $S^{ab}(Q,E)$ in the $(Q,E)$ plane is zero except when $(Q,E)$ corresponds to the excitation energy and momentum of a state contained in $S^a(Q)|GS\rangle$.[11]

From numerical evidence up to $N = 16$ sites it has become clear that there is only an unexpectedly small number of nonzero matrix elements $M_\nu^a$. To resolve parity and other accidental degeneracies between the Yangian multiplets we split these degeneracies by actually diagonalizing $H + \lambda H_3$, where $H_3$ is the second integral of the motion for this model as presented in Refs. 4 and 12. The eigenvalues of this operator allowed the Yangian occupations sequences to be unambiguously identified. States in the multiplets are partially resolved by fixing $S^z$ and $S^{tot}$. (A unique resolution of states would be obtained by adding another term $\mu \mathbf{J_0} \cdot \mathbf{J_1}$ to the Hamiltonian. This would correspond to a basis of states within the Yangian multiplet obtained through the algebraic Bethe ansatz.[13])

Now let us denote the eigenstates of this model as $|\Gamma,\mu\rangle$ where $\Gamma$ labels a Yangian multiplet through an occupation sequence and $\mu$ labels the state within the multiplet. Furthermore define $M^{\Gamma\Gamma'}(S_i^a)_{\mu\mu'} \equiv \langle \Gamma\mu| S_i^a |\Gamma'\mu'\rangle$. Then the observation made above implies that the matrix $\underline{\mathbf{M}}^{\Gamma\Gamma'}(S_i^a)$ vanishes if the occupation sequences $\Gamma$ and $\Gamma'$ differ "too much" in a sense made precise below. This situation is analogous to an *ideal* gas, where if $\hat{O}$ is a one-body operator, $\langle \alpha| \hat{O} |\beta\rangle = 0$ if the occupation number configurations of $|\alpha\rangle$ and $|\beta\rangle$ differ on more than one orbital. E.g., $\hat{O} = \rho(x) = \sum_{kk'} e^{i(k-k')x} c_k^\dagger c_{k'}$ can add or take out a single particle from an orbital in an ideal gas, but in an *interacting* gas it could add unlimited numbers of particle-hole pairs.

However, if $\Gamma$ and $\Gamma'$ do *not* differ too much, according to the rule there will always be a pair of states $\mu$ and $\mu'$ in either multiplet for which the matrix element is nonzero.

**Rule 1.** If $\pi(\Gamma,m,n)$ *is the total number of ones in positions $m$ through $n$ in Yangian occupation sequence $\Gamma$, then* $\underline{\mathbf{M}}^{\Gamma\Gamma'}(S_i^a) \neq \underline{\mathbf{0}}$ iff. $|\pi(\Gamma,m,n) - \pi(\Gamma',m,n)| \leq 1$ *for any* $1 \leq m < n \leq N - 1$.

The rule is illustrated in Fig. 1. A general consequence

$$\langle \Gamma = 100010010000, \mu| S_i^a |\Gamma' = 100010001000, \mu'\rangle \neq 0$$
$$\langle \Gamma = 100\underline{01001}0000, \mu| S_i^a |\Gamma'' = 001\underline{00000}1000, \mu''\rangle = 0$$

FIG. 1. According to the selection rule, there will be states $\mu$ in multiplet $\Gamma$ that are connected to others $\mu'$ in $\Gamma'$ through a local spin operator. In $\Gamma$ and $\Gamma''$ there are none, e.g., since $|\pi(\Gamma,4,8) - \pi(\Gamma',4,8)| = 2 - 0 = 2 > 1$

of this rule is that when we choose $m = 1$ and $n = N - 1$, it follows that the total number of ones in a sequence cannot change by more than one; i.e., the total number of spinons can only change by $+2$, 0, or $-2$. It is remarkable that according to the rule this also holds on any corresponding subsequences of the occupation number sequences.

The zero-magnetic-field DSF has been computed in Ref. 6. The particular structure function computed happened to be $S^{-+}(Q, E)$ (the others are identical because of rotational invariance). This function is governed by excitations present in $S_i^+ |\mathrm{GS}\rangle$. Since the zero-field ground state contains no spinons, the rule tells us that we can only expect excitations with zero or two spinons. As $S_i^+ |\mathrm{GS}\rangle$ has $S^z = +1$ the former is ruled out and in the states in the multiplets with two spinons, both must be polarized. This was to be expected since we can expand $S_i^+ |\mathrm{GS}\rangle$ in a set of localized spinon wave functions (7) containing *two* polarized spinons:

$$\langle n_1, \ldots, n_{M-1} | S_i^+ |\mathrm{GS}\rangle$$

$$= \sum_{m=1}^{N-1} \frac{2}{N} \frac{1 - (-)^m}{1 - z^{-m}} \Psi(n_1, \ldots, n_{M-1} | i, i + m). \quad (11)$$

So one of the spinons seems to be sitting on the site on which $S_i^+$ acted and the other is an even number of sites removed from it.

Since we know the spinon dispersion relation, we can demarcate the support of $S^{-+}(Q, E)$ in the $(Q, E)$ plane in Fig. 2. The main steps of the computation of the

matrix elements $M_\nu^+$, i.e., the weight of the DSF at a point $(Q, E)$ on the plot, are the following: Since $S_i^+ |\mathrm{GS}\rangle$ only contains states with two fully polarized spinons, it must be built out of YHWS's. These wave functions are functionally identical to eigenfunctions of the Calogero-Sutherland model at coupling $\lambda = 2$ of particles moving on a ring. Since both wave functions are of a polynomial form with degree $< N$, the computation of a sum over sites is identical to taking an integral over the ring in the continuum model. The action of $S_i^+$ in the spin chain is translated into a particle destruction operator $\Psi(x, t)$ ($S_i^+$ removes a down spin). So the $S^{-+}(Q, E)$ DSF reduces to the Green's function $\langle \mathrm{GS}| \Psi^\dagger(x, t)\Psi(0, 0) |\mathrm{GS}\rangle$ in the Calogero-Sutherland model. It can be computed in the thermodynamic limit, in which case it can be mapped unto a Gaussian Hermitian matrix model correlator. The result is

$$S^{-+}(Q, E) = \frac{1}{8v_s} \left( \frac{(v_1 - v_2)^2}{(v_s^2 - v_1^2)(v_s^2 - v_2^2)} \right)^{\frac{1}{2}}, \quad (12)$$

with $Q = -\frac{\pi}{2v_s}(v_1 + v_2)$ and $E = \epsilon(v_1) + \epsilon(v_2)$. The DSF matrix element is parametrized by $v_1, v_2$, which are quickly identified with the velocities of the two spinons in the excited state. The $\langle \Psi^\dagger \Psi \rangle$ Green's function has been obtained recently at finite $N$ as well.[14]

For $h \neq 0$ the three different structure functions $S^{-+}(Q, E)$, $S^{zz}(Q, E)$, and $S^{+-}(Q, E)$ will not be equal, since $|\mathrm{GS}\rangle$ is no longer a singlet. In fact it has $S^z = S^{\mathrm{tot}} = \frac{N_{\mathrm{sp}}}{2} \equiv S_0$, where $N_{\mathrm{sp}} = N_{\mathrm{sp}}(h)$ is given by (8). This difference between the three correlation functions is also expressed in two additional global SU(2) selection rules,[11] which rule out certain matrix elements based on the total spin $S^{\mathrm{tot}}$ and $S^z$ of the final state $\mu$ *inside* the Yangian multiplet $\Gamma$.

In the first place there is the Wigner-Eckart theorem for vector operators such as $S_i^\alpha$, which tells us that in order for $\langle \Gamma, \mu| S_i^\alpha |\mathrm{GS}\rangle$ to be nonzero the total spin $S$ of $|\Gamma, \mu\rangle$ must satisfy $S_0 - 1 \leq S \leq S_0 + 1$. Second, for any state in a multiplet with $N_{\mathrm{sp}}$ spinons, we have $\frac{N_{\mathrm{sp}}}{2} \geq S \geq S^z$, where equality only holds for the YHWS, which has all its spinons up. Now put this together with the fact that $S_i^+$ raises $S^z$ by $+1$, $S_i^-$ lowers it by 1, and $S_i^z$ leaves it the same. Classifying states according to their $S^z$ and $S^{\mathrm{tot}}$ as types (i)–(vi), following Müller *et al.*[11] we find the following contributions:

$S_i^+ |\mathrm{GS}\rangle$ contains states with $S = S_0 + 1$ and $\Delta N_{\mathrm{sp}} = +2$ [type (iii)].

$S_i^z |\mathrm{GS}\rangle$ contains states with $S = S_0 + 1$ and $\Delta N_{\mathrm{sp}} = +2$ [type (i)] or $S = S_0$ and $\Delta N_{\mathrm{sp}} = +2, 0$ [type (ii)$^{a,b}$].

$S_i^- |\mathrm{GS}\rangle$ contains states with $S = S_0 + 1$ and $\Delta N_{\mathrm{sp}} = +2$ [type (iv)] or $S = S_0$ and $\Delta N_{\mathrm{sp}} = +2, 0$ [type (v)$^{a,b}$], or $S = S_0 - 1$ and $\Delta N_{\mathrm{sp}} = +2, 0, -2$ [type (vi)$^{a,b,c}$].

Since we have an additional quantum number to label states, $N_{\mathrm{sp}}$, we added Latin superscripts $a, b, c$ to the Roman numerals. All ten contributions are summarized in Table I.

We will now investigate all three structure functions individually following these selection rules.
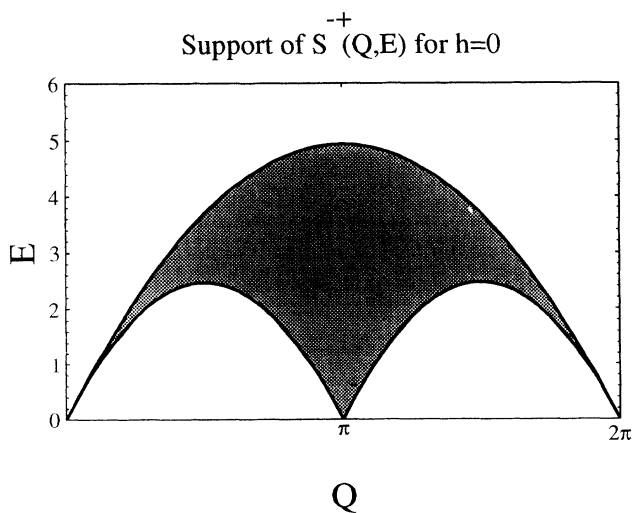


Support of S$^{-+}$(Q,E) for h=0

FIG. 2. The shaded region shows where $S^{-+}(Q, E)$ is nonzero for $h = 0$. The top boundary corresponds to excitations with two spinons that have identical momentum; on the bottom boundary one of the spinons has fixed momentum $\pm\pi$. $E$ is given in units of $v_s/\pi$.

TABLE I. List of matrix elements contributing to the DSF. The correlation functions in the last column refer to those in the Calogero-Sutherland model as they appear in Eqs. (12), (15), (16), (19), (21), and (22). The entries marked with a (†) do not survive in the thermodynamic limit. For the example occupation sequences we act on a ground state 000|101010101|000, with the | delimiting the center region. In the column under "Excitation" $S$ denotes a spinon and $\mathcal{M}$ denotes a magnon.

| DSF | Type | $S^{tot}$ | $\Delta N_{sp}$ | Typical contributing Yangian multiplet | Excitation | Matrix element |
|---|---|---|---|---|---|---|
| $S^{-+}(Q,E)$ $(S^z = S_0 + 1)$ | (iii) | $S = S_0 + 1$ | $\Delta N_{sp} = +2$ | 000|100100101|000 | $2S$ | $\langle \Psi^\dagger \Psi \rangle$ |
| $S^{zz}(Q,E)$ $(S^z = S_0)$ | (i)† | $S = S_0 + 1$ | $\Delta N_{sp} = +2$ | 000|100101001|000 | $2S$ | $\frac{1}{2(S_0+1)}\langle \Psi^\dagger \Psi \rangle$ |
| | (ii)ᵃ | $S = S_0$ | $\Delta N_{sp} = +2$ | 000|100101001|000 | $2S$ | |
| | (ii)ᵇ | $S = S_0$ | $\Delta N_{sp} = 0$ | 100|101001001|000 | $2S + \mathcal{M}$ | $\langle \rho\rho \rangle$ |
| $S^{+-}(Q,E)$ $(S^z = S_0 - 1)$ | (iv)† | $S = S_0 + 1$ | $\Delta N_{sp} = +2$ | 000|100100101|000 | $2S$ | $\frac{1}{(S_0+1)(2S_0+1)}\langle \Psi^\dagger \Psi \rangle$ |
| | (v)ᵃ† | $S = S_0$ | $\Delta N_{sp} = +2$ | 000|010010101|000 | $2S$ | |
| | (vi)ᵃ | $S = S_0 - 1$ | $\Delta N_{sp} = +2$ | 000|010010101|000 | $2S$ | |
| | (v)ᵇ† | $S = S_0$ | $\Delta N_{sp} = 0$ | 000|100100101|010 | $2S + \mathcal{M}$ | $\frac{2}{S_0}\langle \rho\rho \rangle$ |
| | (vi)ᵇ | $S = S_0 - 1$ | $\Delta N_{sp} = 0$ | 100|100101001|000 | $2S + \mathcal{M}$ | |
| | (vi)ᶜ | $S = S_0 - 1$ | $\Delta N_{sp} = -2$ | 100|100100101|010 | $2S + 2\mathcal{M}, \mathcal{M}$ | $\langle \Psi\Psi^\dagger \rangle$ |

## IV. $S^{-+}(Q,E)$

Type (iii): $\Delta N_{sp} = +2$, $\Delta S = +1$. For the occupation sequence of the ground state in a given magnetic field (e.g., 000010101010000) let us label the zeros in the leftmost orbital as the left spinon condensate and the ones in the rightmost orbital as the right spinon condensate. From Table I we learn that action of $S_i^+$ on the ground state only produces states with two more spinons, i.e., one less 1. This 1 has to come out of the center . . . 10101 . . . region. We cannot take more than a single 1 out of the center region—and stow it into the left or right spinon condensate—since this would imply a violation of rule 1 applied to the center region. Taking out a 1 in the center region is equivalent to inserting two spinons there. A typical nonzero matrix element would be $\langle 0001001001000, \mu | \, S_i^+ \, | 0001010101000 \rangle$, with the two spinons residing in orbitals 2 and 3.

All this means is that we get a simple two-spinon spectrum, as in the zero-magnetic-field case. The only difference is that now the momenta of the spinons can only vary from $-k_0$ to $k_0$, where $k_0 = \frac{\pi}{4N}(N - N_{sp})$ decreases with increasing magnetic field as $N_{sp} = N_{sp}(h)$ according to Eq. (8). The support of $S^{-+}(Q,E)$ is essentially a squeezed version of Fig. 2.

As for the weight associated with two-spinon excitations, the calculation for the zero-magnetic-field case (12) carries over without problems. The reason for this is twofold. In the first place, the nonzero-magnetic-field ground-state wave function (9) is of the same Jastrow form as the zero-field one, with just an extra phase factor $\prod_i z_{n_i}^{\frac{N}{2} - M}$ appended. When we take the matrix element, the phase factors from ket and bra parts cancel each other. Second, the excited states are again of the YHWS type $(S = \frac{N_{sp}}{2})$ and have to be polynomials. The mapping onto a Calogero-Sutherland model matrix ele-

ment remains therefore legitimate.

The contribution of just two-spinon YHWS excitations was to be expected since we can expand any fully polarized localized spinon wave function with $N_{sp}$ spinons acted upon with $S_i^+$ in terms of a set containing $N_{sp} + 2$ spinons:

$$(S_i^+ \Psi_{\alpha_i})(n_1, \ldots, n_{M-1})$$

$$= \sum_{p\in V} \prod_{r\in V} \frac{(z_r - z_i)}{(z_r - z_p)} \Psi(n_1, \ldots, n_{M-1} | \alpha_1, \ldots, \alpha_{N_{sp}}, i, p).$$

$$(13)$$

Here $V$ is a set of $M$ random sites on the circle excluding the spinon sites $\{\alpha_i\}$ (see Appendix B). Equation (11) is a special case of this expansion with $V$ equal to the sites that are an even number of steps removed from the site on which the local spin operator acts. We could also have realized that the number of spinons cannot go up by more than 2 when we consider that $S^+$ and $J_1^+$ annihilate $S_i^+ |GS\rangle$, indicating that the latter must consist of pure YHWS's *with* $S^z = S_0 + 1$.

## V. $S^{zz}(Q,E)$

For the DSF $S^{zz}(Q,E)$ we find similar simple excitations that contribute, although at present not all resulting matrix elements can be computed. From the combined selection rules we find three types of excitations: (i), (ii)ᵃ, and (ii)ᵇ, in Table I. They all have in common that $\Delta N_{sp} = 0$ or $+2$. This is not surprising: Let us consider the state $J_1^+ (S_i^z |GS\rangle)$. This state is annihilated by $J_1^+$ and $S^+$, and so it must be a YHWS (see Appendix C). Therefore, since the first action of $J_1^+$ does not change the number of spinons, $S_i^z |GS\rangle$ must be a mixture of

states that contain not more than $2S_0 + 2$ spinons, where $2S_0$ is the number of spinons in the ground state.

We will now discuss the individual types and where possible compute the values of the matrix elements.

Type (i): $\Delta N_{\rm sp} = +2$, $\Delta S = +1$. Having identical selection rules, these states sit in the exact same Yangian and spin multiplets as the type (iii) states, and therefore they also contain two additional spinons. However their $S^z = S^{\rm tot} - 1$, and so they are no longer a YHWS like the type (iii) states. Nevertheless they are related by a simple application of $S^-$:

$$\left| \Gamma, \Delta N_{\rm sp} = +2, \begin{array}{l} S^{\rm tot} = S_0 + 1 \\ S^z = S_0 \end{array} \right\rangle = \frac{1}{\sqrt{2(S_0 + 1)}} S^- \left| \Gamma \right\rangle, \tag{14}$$

where $\left| \Gamma \right\rangle$ denotes the YHWS of the multiplet with occupation sequence $\Gamma$ of type (iii). This allows us to reduce a type (i) matrix element to one that is a multiple of a type (iii) given in Eq. (12):

$$\left| \left\langle \Gamma, \Delta N_{\rm sp} = +2, \begin{array}{l} S^{\rm tot} = S_0 + 1 \\ S^z = S_0 \end{array} \right| S_i^z | {\rm GS} \right\rangle \right|^2$$
$$= \frac{1}{2(S_0 + 1)} | \langle \Gamma | [S^+, S_i^z] | {\rm GS} \rangle |^2$$
$$= \frac{1}{2(S_0 + 1)} | \langle \Gamma | S_i^+ | {\rm GS} \rangle |^2, \tag{15}$$

where we used that $S^+$ annihilates the ground state.

Type (ii)$^a$: $\Delta N_{\rm sp} = +2$, $\Delta S = 0$. This type of state is a member of the same kind of *Yangian* multiplet as types (iii) and (i)—i.e., with two extra spinons—but it sits in a *spin* multiplet that does not contain the YHWS. We expect the associated matrix element to be proportional to a type (iii) matrix element as well, but we lack the necessary operator that would step us from this state to the YHWS. This operator would have to be some member of the Yangian algebra.

Type (ii)$^b$: $\Delta N_{\rm sp} = 0$, $\Delta S = 0$. Since this type of state has $\Delta N_{\rm sp} = 0$, the number of ones in its occupation sequence must be identical to that in the ground state. As with the previous three types we

can delete just a single 1 from the center, leaving behind two spinons pointing up. This 1 then must go into either the left or the right spinon condensate. Therefore a typical nonzero matrix element would be $\langle 00100|1001001|00000| \, S_i^z \, |00000|1010101|00000 \rangle$ where the $|$ just helps to draw attention to the center region. Rule 1 rules out any additional ones leaving the center region. The additional 1 on the left or right can be interpreted as a *magnon* with $S^z = -1$. The limiting case where the magnon "fuses" with two spinons at a boundary between a condensate and the center region gives us the ground state.

These states must be a YHWS since they have $\frac{N_{\rm sp}}{2} = S = S^z$, like $|{\rm GS}\rangle$. This fact allows us to calculate the corresponding matrix elements. Since both ground state and excited state are a YHWS, a mapping onto the Calogero-Sutherland (CS) model is valid. In this case we need a ground-state density-density correlator $\langle {\rm GS} | \rho(x,t) \rho(x',t') | {\rm GS} \rangle$ since $S_i^z$ measures the presence or absence of a down spin (i.e., a particle in the CS model). This calculation has been done by Simons *et al.*,[15] in the thermodynamic limit, by studying the repulsion of energy levels in a random matrix model under a varying perturbation. The energy levels are identified with the positions of the particles and the strength of the perturbation corresponds to imaginary time. Their original expression depends on three parameters (called $\lambda, \lambda_1$, and $\lambda_2$), the latter two of which are compact, and the first one is unbounded. This is precisely what we expect from our selection rule: two spinons restricted to the center region with momenta in the range $-k_0, \ldots, +k_0$ and a magnon that can go off all the way to the right or left (i.e., $\pm\infty$ in the thermodynamic limit). In terms of the velocity $v$ of the magnon—with dispersion relation (5)—and spinon velocities $v_1, v_2$ their result is as follows:

$$|\langle v, v_1, v_2 | \rho(Q) | {\rm GS} \rangle|^2$$

$$= \frac{(v - v_s)(v + v_s)}{(v - v_1)^2 (v - v_2)^2} [(v - v_1) + (v - v_2)]^2. \tag{16}$$

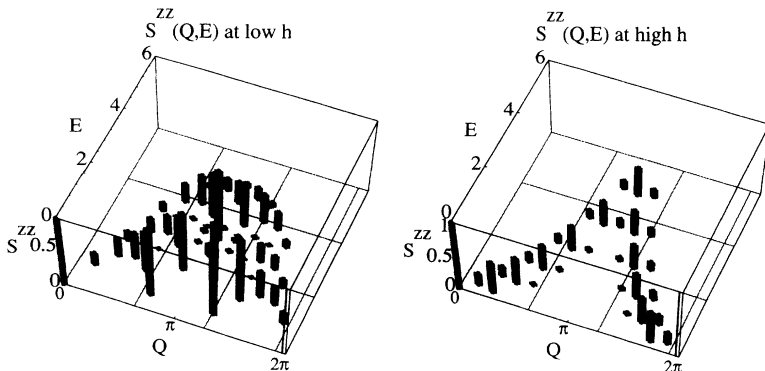For completeness we give the relation between the $v$'s and the $\lambda$'s :



FIG. 3. $S^{zz}(Q, E)$ for small $h$ ($\sigma = \frac{S_z}{N} = 0.05$) and large $h$ ($\sigma = 0.4$) on $N = 14$ sites. $E$ is in units of $v_s/\pi$.

Support of $S^{zz}(Q,E)$ for low h
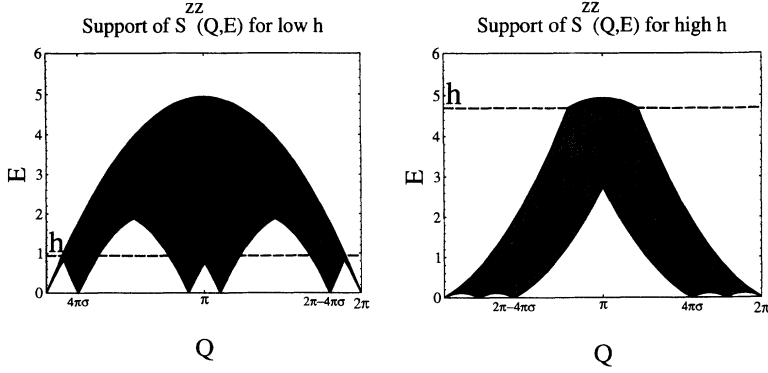
Support of $S^{zz}(Q,E)$ for high h

FIG. 4. The regions where $S^{zz}(Q,E)$ is nonzero for low and high $h$. The area shaded dark contains the contributions from the excitations with $\Delta N_{\text{sp}} = +2$ [types (i) and (ii)$^a$]. The excitations with $\Delta N_{\text{sp}} = 0$—type (ii)$^b$—can also occupy the lightly shaded area.

$$\lambda = v/v_s,$$

$$\lambda_1\lambda_2 = \frac{(v_1+v_2)}{2v_s},$$

$$(1-\lambda_1^2)(1-\lambda_2^2) = \left(\frac{(v_1-v_2)}{2v_s}\right)^2. \quad (17)$$

Numerical data for $S^{zz}(Q,E)$ can be found in Fig. 3 at values of $h$ close to 0 and $h_c$. Figure 4 shows the corresponding support of $S^{zz}(Q,E)$ in the $(Q,E)$ plane, as predicted from the selection rule, in the thermodynamic limit. We notice that for finite-size systems some of the features near the lower boundary are absent.

## VI. $S^{+-}(Q,E)$

Finally we discuss the $S^{+-}(Q,E)$ DSF which is governed by the excitations of types (iv)–(vi)$^c$ that are present in $S_i^-|\text{GS}\rangle$.

Type (iv): $\Delta N_{\text{sp}} = +2$, $\Delta S = +1$. This type of state is very similar to types (iii) and (i); as a matter of fact they all reside in identical Yangian and spin multiplets. Therefore type (iv) states differ from the ground state only by two extra spinons in the center region. They are related to their YHWS $|\Gamma\rangle$ [of type (iii)] through

$$\left|\Gamma,\Delta N_{\text{sp}}=+2,\ \begin{matrix}S^{\text{tot}}=S_0+1\\ S^z=S_0-1\end{matrix}\right\rangle$$

$$= \frac{1}{\sqrt{(4S_0+2)(2S_0+2)}}(S^-)^2|\Gamma\rangle. \quad (18)$$

Analogous to the calculation for type (i) states, a matrix element of type (iv) can now easily be reduced to one involving type (iii):

$$\left|\left\langle\Gamma,\Delta N_{\text{sp}}=+2,\ \begin{matrix}S^{\text{tot}}=S_0+1\\ S^z=S_0-1\end{matrix}\ S_i^z|\text{GS}\right\rangle\right|^2$$

$$= \frac{1}{(4S_0+2)(2S_0+2)}|\langle\Gamma|(S^+)^2 S_i^-|\text{GS}\rangle|^2$$

$$= \frac{1}{(2S_0+1)(S_0+1)}|\langle\Gamma|S_i^+|\text{GS}\rangle|^2. \quad (19)$$

The last matrix element in this equation has already been computed for the $S^{-+}(Q,E)$ DSF. However, the energy of the corresponding excited state here is shifted by $2h$ in comparison, because of the Zeeman term in the Hamiltonian.

Types (v)$^a$ and (vi)$^a$: $\Delta N_{\text{sp}} = +2$, $\Delta S = 0, -1$. States of types (v)$^a$ and (vi)$^a$ contain a two-spinon excitation as well—like types (iii), (i), and (iv). However, since they do not reside in the spin multiplet of their YHWS $(\frac{N_{\text{sp}}}{2} > S)$, the associated matrix elements are unknown.

Types (v)$^b$: $\Delta N_{\text{sp}} = 0$, $\Delta S = 0$. Type (v)$^b$ states are very similar to those of type (ii)$^b$; they only differ in $S^z$ by $-1$. Therefore both contain two excited spinons and a single left- or right-moving magnon. They are related to each other by

$$\left|\Gamma,\Delta N_{\text{sp}}=0,\ \begin{matrix}S^{\text{tot}}=S_0\\ S^z=S_0-1\end{matrix}\right\rangle = \frac{1}{\sqrt{2S_0}}S^-|\Gamma\rangle, \quad (20)$$

and $|\Gamma\rangle$ is the type (ii)$^b$ YHWS of the multiplet with occupation sequence $\Gamma$. We can now trivially relate the matrix elements of types (ii)$^b$ and (v)$^b$:

$$\left|\left\langle\Gamma,\Delta N_{\text{sp}}=0,\ \begin{matrix}S^{\text{tot}}=S_0\\ S^z=S_0-1\end{matrix}\ |S_i^-|\text{GS}\right\rangle\right|^2$$

$$= \frac{1}{2S_0}|\langle\Gamma|[S^+,S_i^-]|\text{GS}\rangle|^2$$

$$= \frac{2}{S_0}|\langle\Gamma|S_i^z|\text{GS}\rangle|^2. \quad (21)$$

The last matrix element is listed in Eq. (16).

Type (vi)$^b$: $\Delta N_{\text{sp}} = 0$, $\Delta S = -1$. These states reside in the same Yangian multiplets as the previous type; however, they are not in the spin multiplets of the YHWS: $\frac{N_{\text{sp}}}{2} > S = S_0 - 1$. Since we lack the operator that steps us up to the YHWS, we are not able to compute the matrix element.

Type (vi)$^c$: $\Delta N_{\text{sp}} = -2$, $\Delta S = -1$. This last class of states has $\Delta N_{\text{sp}} = -2$ and therefore the number of ones in its occupation sequence goes up by one compared to the ground state. The selection rule I only allows the extra 1 to go into the left or right spinon condensate. As before we are also allowed to take a 1 out of the center region and bring it into the left or right spinon condensate. Notice however that the rule forbids both
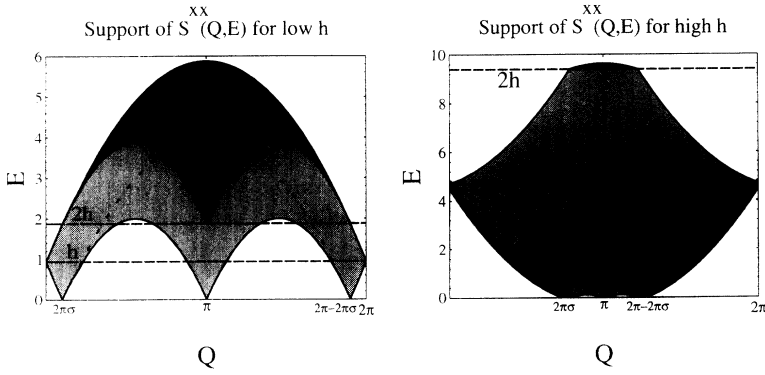
Support of $S^{xx}(Q,E)$ for low h

Support of $S^{xx}(Q,E)$ for high h

FIG. 5. Nonzero region of $4S^{xx}(Q,E)$ $=S^{+-}(Q,E)+S^{-+}(Q,E)$ in the $(Q,E)$ plane for low and high $h$. The contributions of $\Delta N_{\rm sp} = +2$ [types (iv), (v)$^a$, and (vi)$^a$] live in the dark shaded region. These excitations will survive in the limit $h \to 0$. The $\Delta N_{\rm sp} = 0, -2$ can also occupy the area shaded light. For $h \to h_c$ only the one magnon contributions survive, as can be seen in Fig. 6. These high field magnon excitations are indicated by the thick lines.

of the ones to go into the *same* condensate: One has to be left moving and the other must be right moving. The result is an excited state with two magnons and two spinons. A typical nonzero matrix element would be $\langle 0100|100101001|0001|\, S_i^-\, |0000|101010101|0000\rangle$. The $|$ just helps to guide the eye. Also present are YHWS of multiplets from the limiting cases where one of the magnons fuses with the two spinons in the center; this leaves a multiplet with nothing but one magnon; example: $\langle 0000|101010101|0010|\, S_i^-\, |0000|101010101|0000\rangle$. This single magnon excitation is familiar from the strong field regime.

Since type (vi)$^c$ states are YHWS $(\frac{N_{\rm sp}}{2} = S)$, as is the ground state, we can repeat the calculation of the matrix elements by a mapping onto the Calogero-Sutherland model. Because $S_i^-$ creates a down spin, it corresponds to a particle creation operator in the CS model. The relevant correlation function is therefore $\langle {\rm GS}|\, \Psi(x,t)\Psi^\dagger(x',t')\, |{\rm GS}\rangle$. As in the $\langle \Psi^\dagger \Psi \rangle$ case for type (iii) states, a further mapping onto a Gaussian Hermitian matrix model allows one to calculate the Fourier transform of this correlation function.[16] The result is parametrized by four variables, two of which are compact, $v_1, v_2$, and two are noncompact, $v, v'$:

$$|\langle v, v', v_1, v_2 | S^-(Q) | {\rm GS} \rangle|^2 \propto \frac{(v^2 - v_s^2)(v'^2 - v_s^2)|v_1 - v_2|}{(v_s^2 - v_1^2)(v_s^2 - v_2^2)(v^2 - v_1^2)(v^2 - v_2^2)(v'^2 - v_1^2)(v'^2 - v_2^2)}. \tag{22}$$

Energy and momentum in terms of the $v$'s are given by $Q = \frac{\pi}{v_s}[v + v' - \frac{1}{2}(v_1 + v_2)]$ and $E = \frac{\pi}{2v_s}[v^2 + v'^2 - \frac{1}{2}(v_1^2 + v_2^2) - v_s^2]$. It is obvious that the compact parameters are to be identified with the spinon velocities and the noncompact ones with the magnon velocities.

Since we now know all possible excitations contributing to $S^{+-}(Q,E)$, we can draw its support in Fig. 5 for low and high values of $h$. Figure 6 shows numerical data on $S^{+-}(Q,E)$ for those values of $h$. Table I summarizes the selection rules and available information on matrix elements.

## VII. COMPARISON TO THE BETHE ANSATZ MODEL

In 1981 Müller *et al.*[11] did a similar calculation of DSF's for the nearest neighbor Heisenberg chain. They identified certain types of states called *spin wave continuum* (SWC) states as carrying the dominant contribution to the DSF's. These SWC states can be described in a Bethe ansatz (BA) rapidity language by occupation sequences, just like the Yangian multiplets in the HSM. As it turns out, these SWC states correspond to *exactly the*
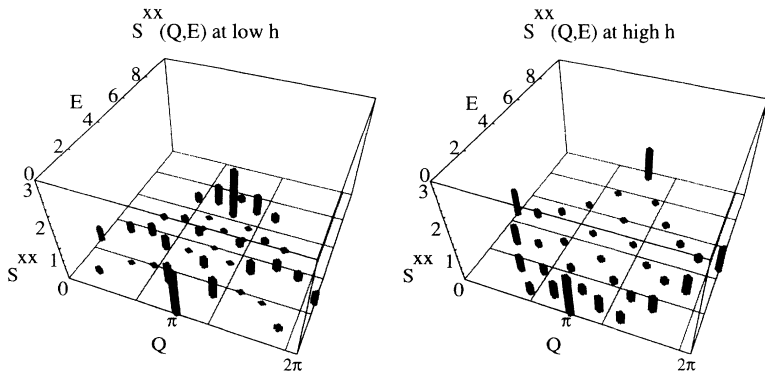


$S^{xx}(Q,E)$ at low h

$S^{xx}(Q,E)$ at high h

FIG. 6. $S^{xx}(Q,E)$ for low and high $h$, from numerical simulations on a ten-site chain. $E$ is in units of $v_s/\pi$.

*same* rapidity sequences that are favored by our selection rules in the HSM. Although the dispersion relations for the BA rapidities are different from those in the HSM, the authors find the support of the DSF's in the nearest neighbor model to have essentially the same shape as we do in the the the inverse exchange case.

However, in the nearest neighbor Heisenberg chain there are some "anomalous" states, characterized by a change of more than $\pm 2$ spinons, which contribute to a lesser degree to the structure functions and do not lie within the bounds found by the authors. In the HSM these contributions are completely absent and once again we find this model to have a surprisingly clean structure. So in this sense the Haldane-Shastry model is an ideal spinon gas, whereas in the NNE Heisenberg chain the spinons interact.

Müller *et al.* also gave general rules, based on comparing Clebsch-Gordon coefficients, determining which matrix elements will survive in the thermodynamic limit. Their conclusion is that the only surviving ones have $S^{\mathrm{tot}} = S^z$. This means only excitations (iii) for $S^{-+}(Q,E)$, (ii)$^a$ and (ii)$^b$ for $S^{zz}(Q,E)$, and (vi)$^a$, (vi)$^b$, and (vi)$^c$ for $S^{+-}(Q,E)$ remain relevant. (Exceptions are single excitations with $Q=0$, since these correspond to $S^+$, $S^z$, and $S^-$ which give macroscopic contributions, as we can see in the figures.) These contributions are trivial to compute.

## VIII. CONCLUSION

We found a remarkably simple selection rule for nonzero matrix elements of local spin operators between eigenstates of the Haldane-Shastry model, which is reminiscent of the ideal gas single-particle selection rules. One of the consequences of this rule is that the *total* number of spinons can only change by 0, $\pm 2$. Within the occupation sequences this holds *locally* as well.

In the particular case that one of the states in the matrix element is also the ground state in a magnetic field (i.e., fully polarized spinons, condensed into the left- and rightmost orbitals in equal amounts), the general selection rule only allows excitations with no more than two spinons (the rule applied to the center region) and one left- and one right-moving magnon (the rule applied to the condensates on the left and right).[17] This implies that the structure functions based on the matrix elements involving these states have a finite support in a region dictated by convolving the dispersion relations of these particles (Figs. 4 and 5). These regions have the same shape as those for the nearest neighbor Heisenberg chain. The latter model carries some weight outside these regions as well. Therefore the HSM has a much cleaner spinon structure than the BA model.

Matrix elements that connect a number of the states in these regions to the ground state through the local action of a spin operator have been presented. However, information is lacking on those types that involve states, not in the spin multiplet of the YHWS. This is particularly bothersome for types (ii)$^a$, (vi)$^a$, and (vi)$^b$ since these will survive in the thermodynamic limit. Their cal-

culation would allow a full reconstruction of the $h \neq 0$ DFS for the HSM. A more algebraic treatment involving Yangian operators should provide more insight.

## ACKNOWLEDGMENTS

## APPENDIX A

We show that localized spinon wave functions with $N_{\mathrm{sp}}$ spinons all pointing up are necessarily linear combinations of YHWS's with exactly $N_{\mathrm{sp}}$ spinons. This follows easily from the fact that both $S^+$ and $J_1^+$ annihilate these states. If we write $\Psi(n_1,\ldots,n_M) = \psi(z_{n_1},\ldots,z_{n_M})$ where $z_n = \exp(\frac{2\pi i n}{N})$ and

$$\psi(w_1,\ldots,w_M) = \left(\prod_{i<j}(w_i-w_j)^2\right)\left(\prod_i w_i\right)$$
$$\times\left(\prod_{i=1}^{M}\prod_{j=1}^{N_{sp}}(w_i-z_{\alpha_j})\right), \quad \text{(A1)}$$

then

$$(S^+\Psi)(n_1,\ldots,n_{M-1})$$
$$=\sum_{j=1}^{N}\Psi(n_1,\ldots,n_{M-1},j)=\sum_{j=1}^{N}\psi(n_1,\ldots,n_{M-1},e^{\frac{2\pi i j}{N}})$$
$$=\psi(z_{n_1},\ldots,z_{n_M},0)=0 \quad \text{(A2)}$$

(where the $\sum_j$ was recognized as the zero mode of a Fourier expansion). And with $w_{jk} = \frac{z_j+z_k}{z_j-z_k}$,

$$(J_1^+\Psi)(n_1,\ldots,n_{M-1})$$

$$\sim -\sum_{j=1}^{M}\sum_{i\neq n_j}^{N}w_{n_j,i}\Psi(n_1,\ldots,n_{M-1},i), \quad \text{(A3)}$$

where we used the fact that $2S_i^z = 1 - 2\sum_{j=1}^{M}\delta_{i,n_j}$ and that $\Psi$ is a symmetric function which vanishes when two of its arguments coincide. We now use that when convolving a polynomial $P(z)$ of degree less than $N$ (such as $\psi$) with $w_{jk}$ we have

$$\sum_{i=1}^{N}w_{ij}P(z_i) = NP(z_j) - 2z_jP'(z_j) - NP(0) \quad \text{(A4)}$$

($P'$ is the derivative of $P$). Since $\psi$ has a double zero when two of its arguments coincide and it vanishes at $z=0$, we have $J_1^+\Psi = 0$.

## APPENDIX B

In this appendix we want to prove Eq. (13). Let us first introduce the following identity which holds for any set of distinct complex numbers $\{\omega_i\}$:

$$\prod_{j=1}^{M-1}(Z_j - z) = \sum_{k=1}^{M}\prod_{l(\neq k)}^{M}\frac{\omega_l - z}{\omega_l - \omega_k}\prod_{i=1}^{M}(Z_i - \omega_k). \quad (B1)$$

The right-hand side (RHS) is just the Lagrange interpolation formula applied to the function in $z$ on the LHS.

Say we want to write $S_i^+\Psi_{\{\alpha_i\}}$ as a linear combination of localized spinon wave functions with two more spinons than $\Psi_{\{\alpha_i\}}$, and all spinons pointing up. We fix one of the additional two spinons at $i$, the site on which $S_i^+$ acts, i.e.,

$$(S_i^+\Psi_{\{\alpha_i\}})(n_1,\ldots,n_{M-1})$$

$$\equiv \Psi(n_1,\ldots,n_{M-1},i|\alpha_1,\ldots,\alpha_{N_{\text{sp}}})$$
$$= \sum_{p(\neq i,\{\alpha_k\})} a_p\Psi(n_1,\ldots,n_{M-1}|\alpha_1,\ldots,\alpha_{N_{\text{sp}}},i,p).$$

$$(B2)$$

Using Eq. (7) we can divide out common factors of $(z_{n_k} - z_{n_l})$, etc., and we are left with

$$z_i\prod_j(z_{n_j} - z_i) = \sum_{p\neq(i,\{\alpha_k\})}a_p\prod_j(z_{n_j} - z_p). \quad (B3)$$

The result follows when we apply the identity (B1) to this equation with $z = z_i$ and $a_p = z_i\prod_k\frac{z_k-z_i}{z_k-z_p}$ where the $z_k$ are randomly chosen distinct sites which do not coincide with the localized spinons.

## APPENDIX C

We show that $S_i^z|\text{GS}\rangle$ can only have zero or two more spinons than $|\text{GS}\rangle$ where $|\text{GS}\rangle$ is a YHWS ground state in a nonzero magnetic field with

$$\langle n_1,\ldots,n_M|\text{GS}\rangle = \Psi_0(n_1,\ldots,n_M)$$
$$= \prod_{i<j}(z_{n_i} - z_{n_j})^2\prod_i z_{n_i}^{\frac{N}{2}-M+1}. \quad (C1)$$

The proof hinges on the fact that in either of those cases ($\Delta N_{\text{sp}} = 0$ or $\Delta N_{\text{sp}} = +2$) acting on $S_i^z|\text{GS}\rangle$ *twice* with $J_1^+$ will annihilate that state. Potential $\Delta N_{\text{sp}} = 4,\ldots$ contributions should survive as they are at least two levels from the top of their Yangian multiplet. Now

$$(2S_i^z\Psi_0)(n_1,\ldots,n_M) = \left(1 - 2\sum_{j=1}^{M}\delta_{i,n_j}\right)$$
$$\times \Psi_0(n_1,\ldots,n_M) \quad (C2)$$

and

$$(2J_1^+S_i^z\Psi_0)(n_1,\ldots,n_{M-1})$$

$$\sim -\sum_{k=1}^{M-1}\sum_{l\neq n_k}^{N}w_{n_k,l}$$

$$\times\left[\Psi_0(n_1,\ldots,n_{M-1},l) - 2\right.$$
$$\times\left.\sum_{j=1}^{M-1}\delta_{i,n_j}\Psi_0(n_1,\ldots,n_{M-1},l)\right]$$
$$-2\sum_{k=1}^{M-1}w_{n_k,i}\Psi_0(n_1,\ldots,n_{M-1},i). \quad (C3)$$

The first two terms vanish when we apply the convolution theorem with $w_{kl}$, as in Appendix A; only the last term survives. Notice that the $\{n_i\}$, $i = 1\ldots,M-1$, cannot be equal to $i$ anymore. This state is trivially annihilated by $S^+$ as it vanishes at $z = 0$. Furthermore,

$$\left[(J_1^+)^2 S_i^z\Psi_0\right](n_1,\ldots,n_{M-2})$$

$$= \sum_{p=1}^{M-2}\sum_{q\neq i,n_p}^{N}w_{n_p,q}\sum_{k=1}^{M-2}w_{n_k,i}\Psi_0(n_1,\ldots,n_{M-2},q,i)$$
$$+ \sum_{p=1}^{M-2}\sum_{q\neq i,n_p}^{N}w_{n_p,q}w_{q,i}\Psi_0(n_1,\ldots,n_{M-2},q,i). \quad (C4)$$

With the help of the convolution theorem we can set the first term to zero (we can stick in the extra term with $q = i$, which is missing, at no cost since $\Psi_0$ vanishes when $q = i$). For the second term we use the identity that lies at the heart of the integrability of the Haldane-Shastry type models, $w_{ij}w_{jk} + w_{jk}w_{ki} + w_{ki}w_{ij} = -1$:

$$\left[(J_1^+)^2 S_i^z\Psi\right](n_1,\ldots,n_{M-2})$$

$$= -\sum_{p=1}^{M-2}\sum_{q\neq i,n_p}^{N}(1 + w_{i,n_p}w_{n_p,q} + w_{q,i}w_{i,n_p})$$
$$\times\Psi(n_1,\ldots,n_{M-2},q,i). \quad (C5)$$

The first term is zero since $\Psi$ is a polynomial that vanishes at the origin, whereas terms 2 and 3 can also be put to zero with the help of the convolution theorem.

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[17] The particular application of our selection rule to matrix elements of the form $\langle \Gamma | \, S_i^a \, | GS \rangle$ where $|GS\rangle$ is the ground state in a magnetic field, $a = +$, and $|\Gamma\rangle$ is a YHWS, has now been confirmed mathematically in two recent papers (Ref. 14) using the theory of Jack polynomials. Additional evidence for the $S_i^z$ case can be found in those references as well.
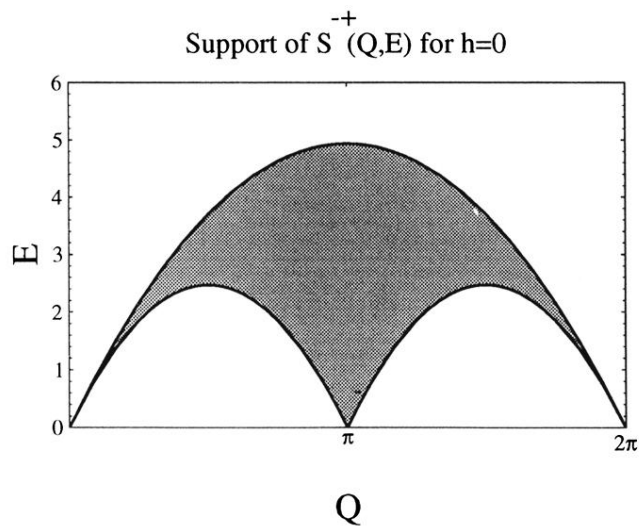
Support of $S^{-+}(Q,E)$ for h=0

FIG. 2. The shaded region shows where $S^{-+}(Q, E)$ is nonzero for $h = 0$. The top boundary corresponds to excitations with two spinons that have identical momentum; on the bottom boundary one of the spinons has fixed momentum $\pm\pi$. $E$ is given in units of $v_s/\pi$.

Support of $S^{zz}(Q,E)$ for low h

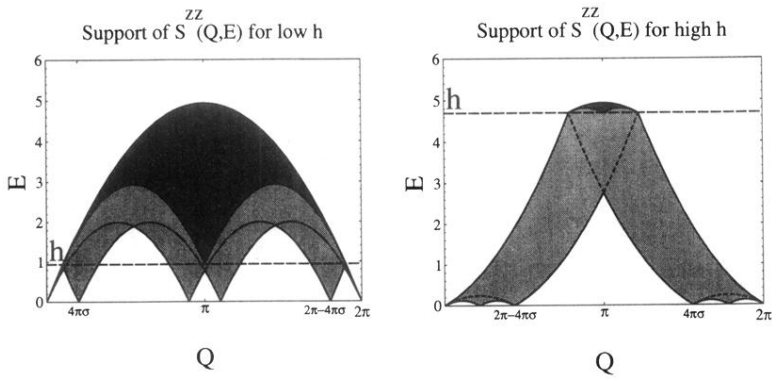Support of $S^{zz}(Q,E)$ for high h



FIG. 4. The regions where $S^{zz}(Q,E)$ is nonzero for low and high $h$. The area shaded dark contains the contributions from the excitations with $\Delta N_{\mathrm{sp}} = +2$ [types (i) and (ii)$^a$]. The excitations with $\Delta N_{\mathrm{sp}} = 0$—type (ii)$^b$—can also occupy the lightly shaded area.

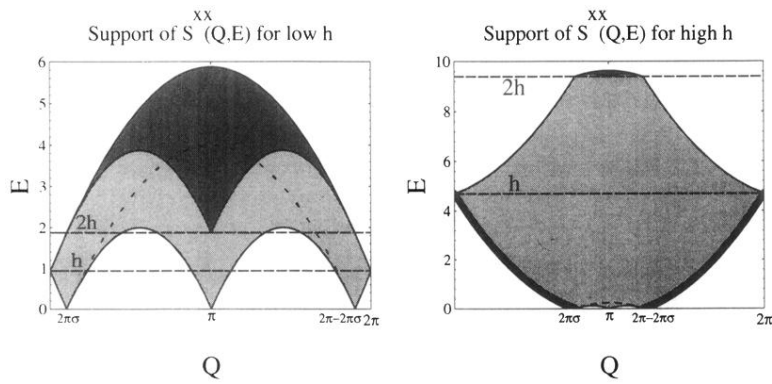Support of $S^{xx}(Q,E)$ for low h

Support of $S^{xx}(Q,E)$ for high h

FIG. 5. Nonzero region of $4S^{xx}(Q,E)$ $=S^{+-}(Q,E)+S^{-+}(Q,E)$ in the $(Q,E)$ plane for low and high $h$. The contributions of $\Delta N_{\mathrm{sp}} = +2$ [types (iv), (v)$^a$, and (vi)$^a$] live in the dark shaded region. These excitations will survive in the limit $h \to 0$. The $\Delta N_{\mathrm{sp}} = 0, -2$ can also occupy the area shaded light. For $h \to h_c$ only the one magnon contributions survive, as can be seen in Fig. 6. These high field magnon excitations are indicated by the thick lines.