Bunched fluxons in coupled Josephson junctions

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The nature of bunched fluxons on coupled long Josephson junctions is investigated theoretically by means of a coupled set of sine-Gordon equations. For cases of inductive and capacitive coupling between the junctions, we analyze the stability of the bunched states for different velocities and perturbation parameters, damping and bias. The internal oscillation frequency is found for the bunched modes and the origin of the stability is discussed for the different types of bound states. Extensive numerical simulations are carried out in order to validate the analytical results against the dynamical behavior of the full system. Excellent agreement between the analysis and the simulations is obtained for all values of coupling parameters and velocities. Finally, we demonstrate numerically that the stability of the bound states can persist even when the two systems are not exactly identical, and we conclude that both inductive and capacitive coupling can give rise to large-locking regimes in synchronization experiments on coupled Josephson junctions.

I. INTRODUCTION

Various mechanisms coupling long Josephson junctions have proven to synchronize and phase-lock fluxon modes under certain conditions.[1–6] Regardless of the specific coupling mechanism, phase locking is an important phenomenon for Josephson technology, since the emitted power from a single oscillator is extremely small.[1] An array of synchronized oscillators can dramatically enhance the emitted power while maintaining the low linewidth of the signal.[1] Theoretically, phase locking has demonstrated many novel phenomena of nonlinear dynamics. In particular systems of long Josephson junctions coupled by spatially distributed coupling mechanisms have shown many interesting phenomena, such as phase locking between fluxon modes,[3,4,7–10] mode-dependent characteristic velocities,[4,11] and hyperradiance.[3,5,12] The coupled system is typically modeled by a set of coupled sine-Gordon equations,[4,13–15] where a magnetic fluxon is represented by the topological kink soliton solution to the unperturbed coupled system. Using the model of Refs. 13, 14, Kivshar and Malomed[16] demonstrated that bound states could form between fluxons on different junctions. It was later demonstrated[4] that this bound state could explain phase locking in sets of Josephson junctions with a mutually inductive-capacitive coupling. Those studies were all based on simple energy balance perturbation techniques. However, more rigorous stability analysis of the bound states revealed more exotic stable states in this system. In Ref. 8 it was demonstrated that the energetically unfavorable bound state of unipolar fluxons could be stable if a certain critical velocity was exceeded.

In this paper, we extend the analysis of Ref. 8 to cover both inductive and capacitive coupling mechanisms. We use the present analysis to investigate the stability properties of all possible bound modes, and we find the internal oscillation frequency of the stable states for *all* values of the translational velocity and for *all* values of coupling strengths. Numerical experiments on the coupled sine-Gordon model are carried out and the agreement with the analytical analysis is excellent for all the obtained results.

The stability analysis is performed for two identical junctions; i.e., the system is symmetric with respect to the two junctions. However, in real experiments it is impossible to obtain this ideal situation. We have therefore performed numerical experiments in order to determine if the stability of the bunched unipolar mode is an artifact of the ideal system. Representing differences between the junctions by a difference in a single parameter (in this case η), we demonstrate that the bunched states are indeed stable for both inductively and capacitively coupled systems. Further, we show the range of difference in parameters (the locking range) for which the bunched state still exists as a stable object.

The model under consideration is,[4,13,14]

$$\begin{aligned}\phi_{xx}-\phi_{tt}-\sin\phi&=\alpha\phi_t-\eta_\phi-\Delta_I\psi_{xx}-\Delta_C\psi_{tt}\ ,\\ \psi_{xx}-\psi_{tt}-\sin\psi&=\alpha\psi_t-\eta_\psi-\Delta_I\phi_{xx}-\Delta_C\phi_{tt}\ ,\end{aligned}\qquad(1)$$

where the variables ϕ and ψ represent the quantum mechanical phase differences over the two junctions. The spatial dimension (x) is normalized to the characteristic Josephson length λ_J, and the temporal dimension (t) is normalized to the inverse plasma frequency ω_p of the junction (see Ref. 4). The dissipative terms $\sim\alpha$ represent tunneling of quasiparticles, and the torques η_ϕ and η_ψ represent the applied bias currents through the junctions. The inductive and capacitive coupling parameters are denoted Δ_I and Δ_C, respectively, where $0\le\Delta_I<1$ and $0\le\Delta_C<1$ for coupled Josephson junctions.[4,14] As described in Ref. 4, the system can also be described with negative coupling parameters, if one of the variables ϕ or ψ (and its bias value) changes sign.

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II. ANALYTICAL ANALYSIS

The fundamental bunched soliton solution to the unperturbed Eq. (1) (i.e., for $\alpha = \eta_\phi = \eta_\psi = 0$) is given by[4]

$$\sigma_\phi \phi = \sigma_\psi \psi = 4\tan^{-1}\left[-\exp\left\{\gamma\left(\frac{\sqrt{1-\sigma\Delta_C}}{\sqrt{1+\sigma\Delta_I}}u\right)\times\frac{x-ut}{\sqrt{1+\sigma\Delta_I}}\right\}\right], \quad (2)$$

where $\sigma_\phi = \pm 1$ and $\sigma_\psi = \pm 1$ are the polarities of the kinks in the systems, and $\sigma \equiv \sigma_\phi\sigma_\psi$. The Lorentz contraction factor is given by $\gamma^{-1}(u) = \sqrt{1-u^2}$, u being the translational velocity of the solution.

Defining the total energy of the system as[4]

$$H = \int\left[\frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_t^2 + 1 - \cos\phi\right]dx + \int\left[\frac{1}{2}\psi_x^2 + \frac{1}{2}\psi_t^2 + 1 - \cos\psi\right]dx + \Delta_I\int\phi_x\psi_x dx - \Delta_C\int\phi_t\psi_t dx, \quad (3)$$

we can insert the solution, Eq. (2), and obtain the energy of the traveling wave as

$$H_{\text{bunched}} = 16\sqrt{1+\sigma\Delta_I}\;\gamma\left(\frac{\sqrt{1-\sigma\Delta_C}}{\sqrt{1+\sigma\Delta_I}}u\right). \quad (4)$$

Note that the expressions Eqs. (2)–(4) are exact for all values of Δ_I and Δ_C. Equation (4) shows how the energy of the traveling wave decreases (increases) for antipolar (unipolar) kinks as Δ_I and/or Δ_C is increased. If the two solitons are well separated in space, the wave profiles and their energies can be found (to first order in Δ_I and Δ_C) from Eqs. (2) and (4) for $\Delta_I = \Delta_C = 0$.[4] For the inductive coupling, this means that two antipolar solitons ($\sigma = -1$) will have higher energy when separated in space than in the bunched state, whereas two unipolar solitons ($\sigma = 1$) will have lower energy when separated in space compared to the bunched mode. However, it was shown in Ref. 8 that for bunched states traveling at high speed it is possible to obtain stable *unipolar* states. The origin of this stability can be found in the double characteristic velocity nature of the coupled equations. These velocities can be immediately identified in the exact solution Eq. (2), where we find the asymptotic velocities for the bunched state to be given by

$$u_{+1}^2 = \frac{1+\Delta_I}{1-\Delta_C} \geq 1 \quad \text{for } \sigma = 1, \quad (5)$$

$$u_{-1}^2 = \frac{1-\Delta_I}{1+\Delta_C} \leq 1 \quad \text{for } \sigma = -1. \quad (6)$$

Writing the equations for the sum, $\mu = (\phi+\psi)/2$, and the difference, $\nu = (\phi-\psi)/2$, we obtain

$$(1+\Delta_I)\mu_{xx} - (1-\Delta_C)\mu_{tt} - \sin\mu\cos\nu = \alpha\mu_t - \frac{1}{2}(\eta_\phi + \eta_\psi), \quad (7)$$

$$(1-\Delta_I)\nu_{xx} - (1+\Delta_C)\nu_{tt} - \sin\nu\cos\mu = \alpha\nu_t - \frac{1}{2}(\eta_\phi - \eta_\psi), \quad (8)$$

where we find the two fields μ and ν to evolve on different time scales given by the characteristic velocities, Eqs. (5) and (6), respectively. Using the solution, Eq. (2), for $\sigma = 1$ [$\sigma = -1$] in Eq. (7) [Eq. (8)], we can apply the adiabatic perturbation technique[17] and obtain the steady state velocity of the bunched state for given perturbation parameters,

$$u^2 = \frac{(1+\sigma\Delta_I)(\pi\eta/4\alpha)^2}{1+(1-\sigma\Delta_C)(\pi\eta/4\alpha)^2}, \quad (9)$$

where $\eta \equiv \eta_\phi = \sigma\eta_\psi$.

Since the exact solution, Eq. (2), is a solution of the unperturbed Eq. (7) [Eq. (8)] for $\sigma = 1$ [$\sigma = -1$], we can then perform linear stability analysis for the unipolar [antipolar] solution by linearizing Eq. (8) [Eq. (7)].

Assuming that we are dealing with the unperturbed system ($\alpha = \eta_\phi = \eta_\psi = 0$), we can write the linearized equation for the deviation $\varrho^{(-)}$ as

$$(1-\sigma\Delta_I)\varrho_{xx}^{(-)} - (1+\sigma\Delta_C)\varrho_{tt}^{(-)} - \varrho^{(-)}\cos\varrho^{(+)} = 0,$$
$$\varrho^{(+)} = \frac{1}{2}(\phi+\sigma\psi), \quad \varrho^{(-)} = \frac{1}{2}(\phi-\sigma\psi). \quad (10)$$

The characteristic scales for the bunched mode give us the new variable

$$\xi = \frac{x-ut}{\sqrt{1+\sigma\Delta_I-(1-\sigma\Delta_C)u^2}}. \quad (11)$$

Inserting Eq. (11) into Eq. (10) and using the identity $\cos\varrho^{(+)} = 1 - 2\text{sech}\xi$, we obtain the eigenvalue problem

$$\lambda\zeta_{\xi\xi} - (\kappa - 2\text{sech}^2\xi)\zeta = 0, \quad (12)$$

where

$$\zeta(\xi) = \varrho^{(-)}(x,t)\exp(\varepsilon\xi - i\omega t), \quad (13)$$

$$\kappa = 1 - \frac{(1-\sigma\Delta_I)(1+\sigma\Delta_C)}{1-\sigma\Delta_I-(1+\sigma\Delta_C)u^2}\omega^2, \quad (14)$$

$$\lambda = \frac{1-\sigma\Delta_I-(1+\sigma\Delta_C)u^2}{1+\sigma\Delta_I-(1-\sigma\Delta_C)u^2}, \quad (15)$$

$$\varepsilon = iu\omega(1+\sigma\Delta_C)\frac{\sqrt{1+\sigma\Delta_I-(1-\sigma\Delta_C)u^2}}{1-\sigma\Delta_I-(1+\sigma\Delta_C)u^2} \quad (16)$$

From Eqs. (15) and (16) we see that the condition $\omega^2 > 0$ must be fulfilled in order to ensure stability of the bunched mode. The eigenvalue problem, Eq. (12), has the *only* localized solution

$$\zeta = (\text{sech}\xi)^{\frac{1}{2}(\sqrt{8/\lambda+1}-1)}, \quad (17)$$

when

$$\omega^2 = \Omega_0^2 \equiv \frac{1-\sigma\Delta_I-u^2(1+\sigma\Delta_C)}{(1-\sigma\Delta_I)(1+\sigma\Delta_C)}\times\left(\frac{1}{2}\sqrt{8\lambda+\lambda^2}-\frac{1}{2}\lambda-1\right). \quad (18)$$

Case $\sigma = 1$: It is easy to see that for all velocities $0 \leq |u| < u_{-1}$, we have $\omega^2 < 0$, and hence an unstable bunched state. However, $u \to u_{-1}$ implies $\Omega_0^2 \to 0_-$, suggesting marginal stability for $u = u_{-1}$. For velocities $u > u_{-1}$, there exists no confined solution to the eigenvalue problem, even though a bunched soliton state is known to exist. This phenomenon is easily understood in terms of the two characteristic velocities present in the system. A small deviation between two unipolar solitons, traveling with $u > u_{-1}$, will be described by Eq. (10) and, hence, travel with $u = u_{-1}$; i.e., the difference between two solitons cannot travel as fast as the solitons themselves, and the stability of the bunched unipolar solitons is then ensured by the different characteristic velocities, since any deviation between the fields will decay as a result of (classical) Cherenkov radiation.

We conclude that the bunched unipolar state is stable if

$$|u| > u_* = u_{-1} \equiv \sqrt{\frac{1-\Delta_I}{1+\Delta_C}}\,, \tag{19}$$

and unstable otherwise. Inserting this threshold condition to the power balance expression, Eq. (9), we obtain a threshold in the bias such that larger bias values will result in stable bunched states; i.e., the bunched state is stable when

$$|\eta| > \eta_* \equiv \frac{2\sqrt{2}}{\pi}\alpha\sqrt{\frac{1-\Delta_I}{\Delta_I+\Delta_C}}\,, \tag{20}$$

and unstable otherwise. From the above expression we can now derive the minimum coupling between the two sine-Gordon systems for which the unipolar bunched mode can be stable. This is obtained from the fact that the kinks only can be stable if $|\eta| < 1$, i.e., if

$$\frac{\Delta_I+\Delta_C}{1-\Delta_I} > 8\left(\frac{\alpha}{\pi}\right)^2 . \tag{21}$$

It is worth noting that since the stability of the fast traveling unipolar bunched state is related to Cherenkov radiation, we can conclude that *no* internal frequency is associated with the stability; i.e., Eq. (18) is only valid for $|u| \leq u_{-1}$.

Case $\sigma = -1$: Since the propagating bunched antipolar state cannot travel with velocities exceeding u_{-1}, given by Eq. (6), it is obvious that $\lambda > 1$ for all relevant values of u. From this it is then straightforward to see that the bunched antipolar state of solitons is stable for *all* relevant velocities.

Unlike the unipolar case, this stability originates from the fact that the binding energy of the bunched state is lower than that of two separate solitons (see Sec. III). An internal frequency Ω_0 is therefore associated with the bunched mode and its value is given by Eq. (18) for $\sigma = -1$. Note that this expression is valid for all relevant values of the velocity and the coupling parameters, $|u| < u_{-1}$, $0 \leq \Delta_I < 1$, and $0 \leq \Delta_C < 1$. For $u = 0$ and $\Delta_I, \Delta_C \ll 1$, the internal frequency is given by $\Omega_0 \approx \sqrt{2\Delta_I/3}$, which is in agreement with the result obtained in Ref. 4.

III. NUMERICAL ANALYSIS AND DISCUSSION

Numerical simulations of the full system, Eq. (1), have been performed in order to verify our analytical results. The simulations were carried out using an explicit second order finite difference scheme in time and space of Eqs. (7) and (8). Since we intend to approximate an infinite system size, periodic boundary conditions were imposed on the system,

$$\phi(0,t) = \phi(L,t) + 2\pi\sigma_\phi\,, \quad \psi(0,t) = \psi(L,t) + 2\pi\sigma_\psi\,, \tag{22}$$

where L is the normalized system length.

In Fig. 1 we show the normalized velocity u of the unipolar bound state ($\sigma = 1$) as a function of the normalized bias current, $\eta = \eta_\phi = \eta_\psi$. The system parameters for this simulation have been chosen to be $\alpha = 0.1$, $L = 40$, $\Delta_I = 0$, and $\Delta_C = 0.2$. For the same simulation with $\Delta_I = 0.2$ and $\Delta_C = 0$, see Ref. 8. The system was initiated with high bias $\eta > \eta_*$, high velocity $u > u_*$, and with a distance $dx = 0.05$ of separation between the solitons. After a transient time of 1600 normalized time units, we measured the steady state velocity of the two solitons. Then, changing the bias η by $\delta\eta = 10^{-3}$, we introduced a small disturbance in the ν field in order to break perfect symmetry between the two lines. In this way we were able to detect when the bunched mode became unstable. As is clearly seen from Fig. 1, bias values greater than a certain critical value sustain soliton propagation with asymptotic velocity u_{+1}, given by Eq. (5). This is a clear indication of the presence of the bunched

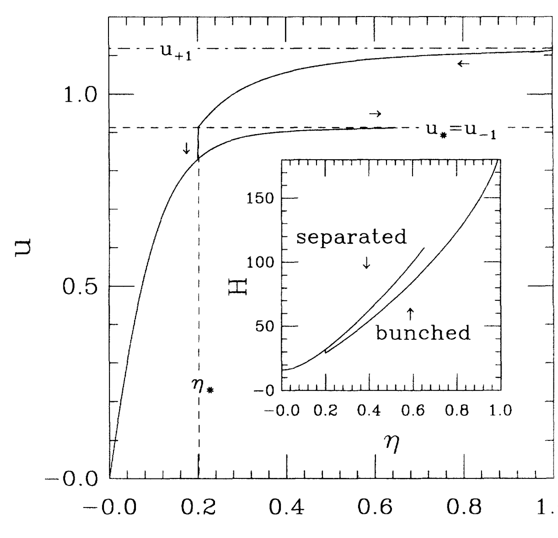

FIG. 1. Normalized soliton velocity u as a function of the normalized bias current density $\eta = \eta_\phi = \eta_\psi$. Parameters are $\alpha = 0.1$, $\Delta_C = 0.2$, $\Delta_I = 0$, $L = 40$, and $\sigma = 1$ (unipolar). The dash-dotted line indicates the asymptotic velocity u_{+1} for the bunched traveling wave solution, Eq. (5), for $\sigma = 1$. The dashed lines indicate the boundary for the stability of the bunched mode, as given by Eqs. (19) and (20). The inset shows the total energy of the system, as given by Eq. (3).

unipolar mode, given by Eq. (2) for $\sigma = 1$. In fact, using the power balance expression Eq. (9) for the velocity as a function of the bias, we would find a curve in Fig. 1 with no detectable difference from the numerical result for $u > u_{-1}$. Decreasing the bias, the bunched state is lost at the critical value η_*, given by Eq. (20). This transition from the bunched to the separated state is clearly seen in Fig. 1 as a jump in the velocity. Once separated, the two solitons stay with an internal distance of $L/2$, even when the bias is increased above the critical value η_*. The asymptotic velocity for this state is given by u_{-1} since the ν field is nonzero for this state [see Eq. (8)]. The inset in Fig. 1 shows the measured energy of the system as calculated from the dynamical simulation. We find that the total energy for the bunched (capacitively coupled) state has *lower* energy than the separated state for the same bias value. Even at the transition point η_*, we find that the total energy jumps up when the system lets the bunched state transform to the separated. This phenomenon is different from the usual behavior, where the total energy seeks a minimum (as is the case for the inductively coupled system[8]).

The fact that the energy of the capacitively coupled bunched unipolar mode is lower than that of the corresponding separated mode can be seen by inserting Eq. (9) into Eq. (4) and thereby obtain the total energy as a function of the bias $\eta = \eta_\phi = \sigma\eta_\psi$,

$$H_{\text{total}} = 16\sqrt{1+\sigma\Delta_I}\left[1 + (1-\sigma\Delta_C)\left(\frac{\pi\eta}{4\alpha}\right)^2\right] . \tag{23}$$

From this we find that the bunched unipolar mode increases (decreases) the total energy for the inductive (capacitive) coupling. For the capacitive coupling, this is somewhat against intuition, but in the following we will show that the *binding* energy between the two solitons behaves properly. The wave momentum for the system is given by

$$P = -\int [\phi_t\phi_x + \psi_t\psi_x - \Delta_C(\phi_t\psi_x + \psi_t\phi_x)]\, dx . \tag{24}$$

Using the exact solution, Eq. (2), we obtain

$$P = 16u\,\gamma\left(\frac{\sqrt{1-\sigma\Delta_C}}{\sqrt{1+\sigma\Delta_I}}u\right)\frac{1-\sigma\Delta_C}{\sqrt{1+\sigma\Delta_I}} . \tag{25}$$

We now define the mass of the bunched mode as

$$M = \sqrt{H^2 - P^2} . \tag{26}$$

For the bunched state we then find the mass from Eqs. (4) and (25),

$$M_b = 16\sqrt{1+\sigma\Delta_I}\,\gamma\left(\frac{\sqrt{1-\sigma\Delta_C}}{\sqrt{1+\sigma\Delta_I}}u\right)\Big/\,\gamma\left(\frac{1-\sigma\Delta_C}{1+\sigma\Delta_I}u\right) , \tag{27}$$

which is valid for all Δ_I, Δ_C, and u. The mass of the separated state is given (to first order in the coupling parameters) for small u by

$$M_s = 16 , \tag{28}$$

which is the usual mass of two solitons in an uncoupled system. The binding energy is given by $M_b - M_s$, where a positive binding energy implies that it is energetically favorable to be in the separated state. From the above expressions it is now easy to see that

$$M_b - M_s \geq 0 \quad \text{for } \sigma = 1 , \tag{29}$$

$$M_b - M_s \leq 0 \quad \text{for } \sigma = -1 . \tag{30}$$

The binding energy between the solitons is negative for antipolar bunched states and positive for unipolar bunched states, even when the total energy Eq. (23) indicates otherwise (for the capacitive coupling). Thus, we find that the nature of the stability of the unipolar mode is not energetically based, since the binding energy of the stable bunched mode is positive.

The stability limit of the bunched unipolar mode has been tested for many different values of system parameters. In Fig. 2 we show the values η_* and u_* for which the system switches from the bunched to the separated mode as a function of the coupling parameters. The results of numerical simulations are shown as markers and the analytical predictions, Eqs. (19) and (20), as solid curves. The open diamonds show the results for $\Delta_I > 0$ and $\Delta_C = 0$, whereas the open squares represent results for $\Delta_I = 0$ and $\Delta_C > 0$. Other system parameters are $L = 40$, $\alpha = 0.05$ (a), and $\alpha = 0.1$ (b). Evidently, the analytical results fit the numerical data almost perfectly for both types of coupling mechanisms in the entire region $0 \leq \Delta_I < 1$ and $0 \leq \Delta_C < 1$. This is not surprising, since the basis of the analytical treatment is an exact solution, valid for all values of coupling parameters. The minor deviations between the dynamical data and the analytical results, seen in the inset of Fig. 2(a) for the inductive coupling, is actually an artifact of a sensitive velocity measurement, due to the finite step size in the bias (see Fig. 1). The deviation between numerical and analytical data seen in Fig. 2(b) for small coupling parameters is due to the approximations in the adiabatic perturbation method, used to obtain Eqs. (9) and (20). This method assumes that the perturbation parameters α and η are relatively small. For small coupling parameters, the critical value η_* increases and the perturbation result for η_* therefore deviates from the true critical value. However, the critical velocity u_* is independent of this perturbation error and therefore no deviations are detected in the insets of Fig. 2 for small values of the coupling parameters. We note that *no* internal oscillations were detected in the numerical simulations shown in Fig. 2. Any small deviation between the positions of the two solitons decayed exponentially in time. This is in agreement with the analytical treatment in Sec. II, where Cherenkov radiation was argued to form the basis for the stability of this mode.

In real experiments, two adjacent Josephson junctions have slightly different system parameters such as characteristic lengths, characteristic frequencies, critical currents, damping parameters, and bias values. This will result in broken symmetry between the two junctions. In the framework of the coupled sine-Gordon model, this means that we do not have an exact solution of the bunched mode [Eq. (2)] on which we can base an analytical stability analysis. We have therefore performed numerical simulations with different bias values of the two systems, in order to break the perfect symmetry of the equations. These simulations also relate to experiments on phase locking,[3] where the range in difference between system parameters (typically the bias current) for which the soliton oscillators are synchronized is studied. In Fig. 3 we show numerical results of the locking range $2\eta_1 = (\eta_\phi - \eta_\psi)$ in bias as a function of the bias point, $\eta_0 = (\eta_\phi + \eta_\psi)/2$, for the unipolar bunched mode. For given system parameters the bunched unipolar solution was initiated with $\eta_1 = 0$. Increasing the bias difference η_1 in steps of $\delta\eta = 10^{-3}$, we detected whether the system was still in the bunched state after a transient time of a few thousand normalized time units. This procedure was repeated until the bunched mode broke into the separated mode. The markers in Fig. 3 represent the largest value of the bias difference η_1 for which the bunched unipolar mode still existed (straight lines are drawn between markers, representing the same coupling parameters). Not surprisingly, we find the general trend that larger coupling parameters give rise to larger locking ranges in bias. By comparing Figs. 3(a) and 3(b) we find that increasing the damping parameter generally decreases the locking range. This is consistent with the locking-range studies of the antipolar state performed in Ref. 4. In fact, except for small values of the bias point (the translational velocity) the locking ranges for the uni- and antipolar bunched modes look quite similar, even though the mechanisms for their stability are different. The optimal bias point for obtaining the largest locking range for a given set of coupling parameters is seen to be somewhere in the center of the scale (depending on the damping α). Intuitively, this can be understood as follows. Increasing the bias point η_0 will increase the coupling strength between the traveling solitons, since faster traveling solitons will Lorentz contract and thereby enhance the coupling term, e.g., $\Delta_C\phi_{tt}$. However, localized objects in a dc-driven sine-Gordon system can only exist for $|\eta_\phi|, |\eta_\psi| < 1$. This condition limits the possible bias difference η_1 for a given bias point η_0 to be $|\eta_1| < 1 - |\eta_0|$. This limit is shown in Fig. 3 as dashed diagonal lines. For larger values of the coupling parameters we find that the locking range can give rise to nonsmooth behavior as a function of the bias point. Detailed investigation revealed that this is caused by strong Cherenkov emission from the bunched mode when $|\eta_1| > 0$. Even for long systems of $L = 160$ this radiation was still strong enough to interfere with soliton propagation in the periodic system. However, increasing the damping parameter suppresses the effect of the radiation [see Fig. 3(a)].

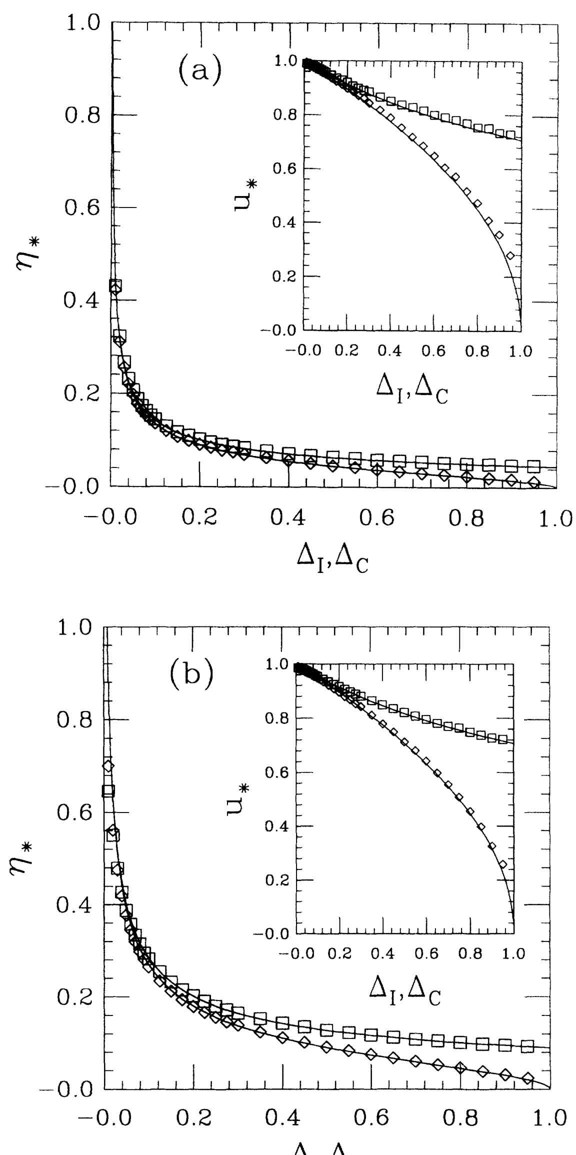

FIG. 2. Lower limit in bias current, η_*, as a function of the coupling strengths Δ_I and Δ_C. Parameters are L =40–80 and $\sigma = 1$ (unipolar). The markers indicate the results of numerical simulations: □, $\Delta_I = 0$; ◇, $\Delta_C = 0$. The solid lines represent the analytical expression Eq. (20). The insets show the comparison between the numerical data (markers) and the analytical expression for the critical velocity Eq. (19): $\alpha = 0.05$ (a), $\alpha = 0.10$ (b).

The insets of Fig. 3 show the internal distance dx between the positions of the two solitons at the critical points shown in Fig. 3. It is evident that the internal distance is *always* relatively small, both for the capacitive and the inductive coupling as well as for different damping parameters. In all cases studied, the critical internal distance was found to decrease exponentially with increasing bias point.

In Fig. 4 we show numerical measurements of the internal oscillation frequency of the energetically stable antipolar mode. The perturbation parameters α, η_ϕ, and η_ψ were set to zero. For $\sigma = -1$, $L = 40$, and given coupling parameters, the system was initiated with Eq. (2) as the wave profiles. In order to detect an internal frequency the two initial solitons were separated by $dx = 0.1$ in normalized space units before the first time step. The oscillation frequency, measured over the first two to ten periods, is displayed in Fig. 4 for different velocities and coupling parameters as markers. Along with the results of the dynamical simulations are solid lines, representing the analytical expression for the frequency, Eq. (18) ($\sigma = -1$). The analytical result is clearly in almost perfect agreement with the numerically obtained data for all values of translational velocities and coupling parameters, even large velocities and coupling parameters. The vertical dashed lines indicate the asymptotic velocity of the antipolar bunched mode for the given coupling parameter.

IV. CONCLUSION

We have performed a comprehensive linear stability analysis of bunched solitons in coupled sine-Gordon systems, where the coupling terms represent mutual induc-

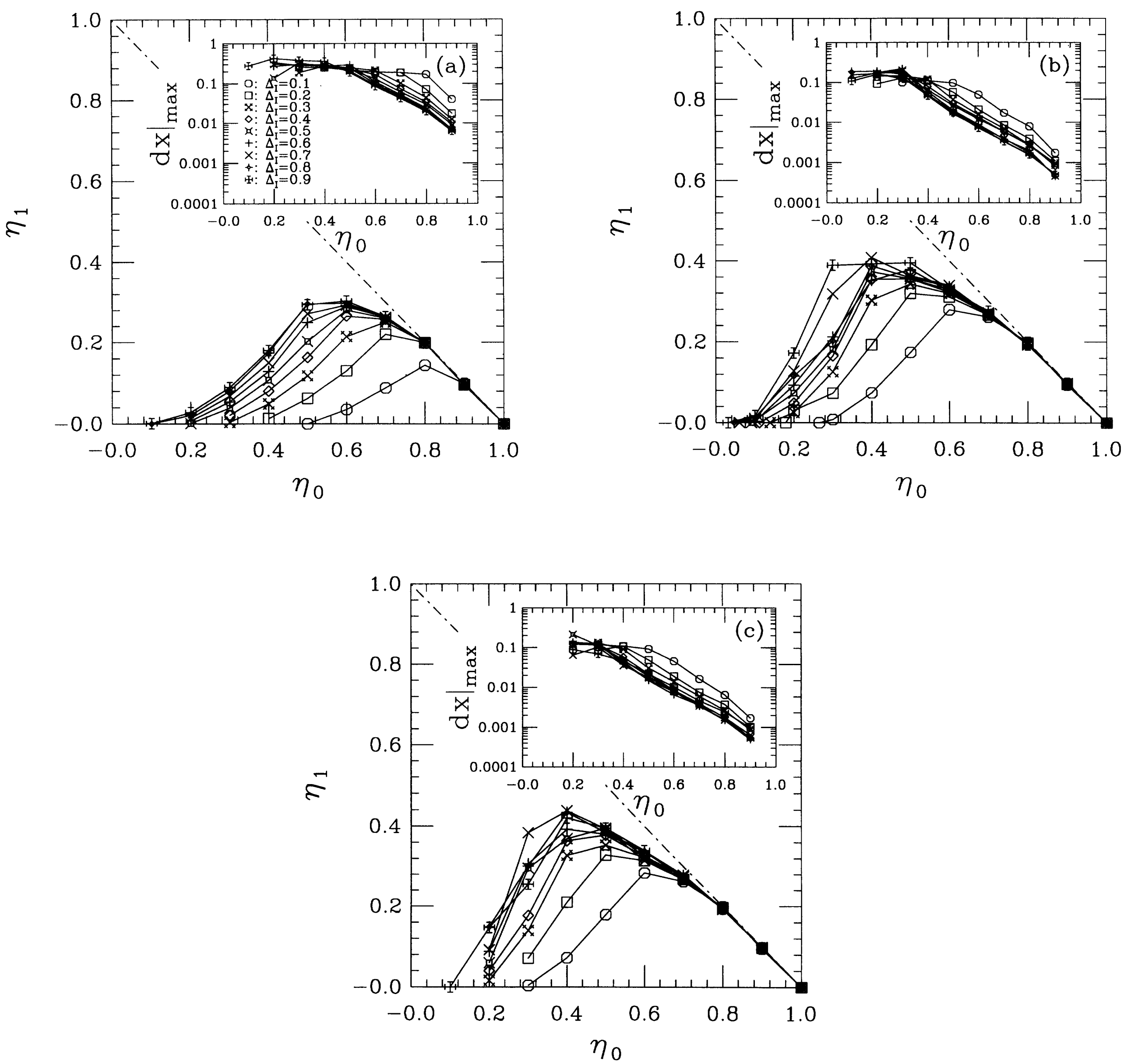

FIG. 3. Maximum value of the difference in bias, $\eta_1 = (\eta_\phi - \eta_\psi)/2$, for which the bunched unipolar ($\sigma = 1$) mode exists as a function of the bias point, $\eta_0 = (\eta_\phi + \eta_\psi)/2$. Markers indicate the numerical results obtained from simulations. The diagonal dashed line indicates the limit $|\eta_1| < 1 - |\eta_0|$. Parameters are $L = 40$–160, (a) $\alpha = 0.1$ and $\Delta_C = 0$, (b) $\alpha = 0.2$ and $\Delta_C = 0$, (c) $\alpha = 0.1$ and $\Delta_I = 0$. Insets show the maximum value of the internal distance in position, dx, of the two unipolar ($\sigma = 1$) solitons as a function of the bias point, $\eta_0 = (\eta_\phi + \eta_\psi)/2$.

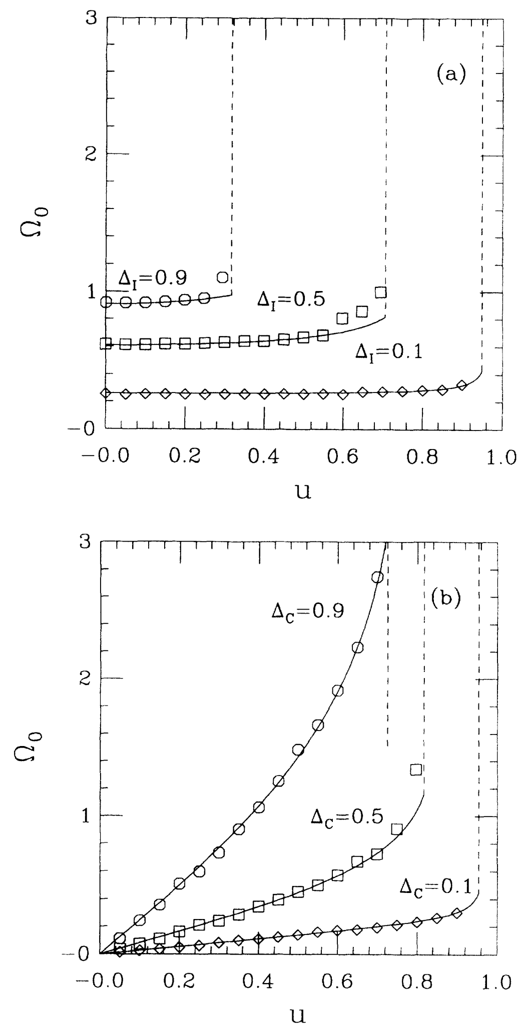

FIG. 4. Internal oscillation frequency Ω_0 as a function of the translational velocity u of the bunched antipolar mode ($\sigma = -1$). Markers indicate results of dynamical simulations, and solid lines represent the analytical expression Eq. (18) for $\sigma = -1$. Vertical dashed lines indicate the asymptotic velocities u_{-1}. Parameters are $\alpha = \eta_\phi = \eta_\psi = 0$, $L = 40$, (a) $\Delta_C = 0$, and (b) $\Delta_I = 0$.

tance and capacitance between long Josephson junctions. We have demonstrated that the antipolar bunched mode is always stable in the case where the two sine-Gordon systems are identical, and the internal oscillation frequency has been calculated. Dynamical results confirm this frequency expression for all relevant values of the coupling parameters as well as the translational velocity of the bunched mode. This almost perfect agreement should not surprise us, since the analytical results are based on a single assumption, namely, that the solitons are unchanged when separated slightly in space. No other assumptions are made regarding the velocity or the coupling parameters. The stability of the antipolar bunched state has been considered in Ref. 4 in the case of small coupling parameters and different bias values of the two systems. We have demonstrated that the energetically repulsive unipolar state of solitons can be stable for both inductive and capacitive coupling if the translational velocity is higher than a critical velocity in the coupled system. This stability has been shown to be a result of (classical) Cherenkov radiation. The velocity criterion has been verified by dynamical simulations and almost perfect agreement has been found between the analysis and the numerical data for all values of coupling parameters. The existence of the unipolar bunched state as a stable object in nonperfect systems has been verified by simulating the system for different values of the bias parameter in the two systems. These simulations have revealed a large range in difference between the two bias values for which the bunched state still exists. The bunched modes discussed in this paper are different in nature from bunched states studied in single sine-Gordon systems.[18] Bunching between fluxons of the same sine-Gordon system is typically caused by details in the friction forces, whereas bunching in the coupled system has been explained here in terms of double characteristic velocities for the unipolar state and a negative binding energy for the antipolar state. For coupled long Josephson junctions this implies that phase locking between two junctions operated in their fluxon modes can be obtained over some interval of the system parameters. In fact, three different types of phase locking can be expected for both inductive and capacitive coupling: the attractive antipolar mode, where the bias currents have opposite signs in order to drive the fluxons in the same direction;[3,4] the separated repulsive unipolar mode, where the bias currents have the same signs and the asymptotic velocity of the mode is u_{-1};[3,7] and the bunched unipolar mode discussed in Sec. II, where the locking range is as shown in Fig. 3. The latter mode can be distinguished from the other modes by the asymptotic velocity u_{+1}. The present analysis has been performed for two interacting sine-Gordon systems (Josephson junctions). It would be interesting to investigate the stability and phase-locking nature of a system with many interacting soliton oscillators.

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