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Ballistic charge transport in superconducting weak links

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Ballistic charge transport through mesoscopic superconductor–normal-metal–superconductor (SNS) microbridges is investigated within the framework of the Keldysh technique. At low voltages we find a peak in the conductance $G \sim (\ell_{\rm in}/2a)(1/R_0)$, where $\ell_{\rm in}$ is the inelastic mean free path, $2a$ is the normal layer thickness, and R_0 is the Sharvin resistance. In this limit the dissipative current is carried by quasiparticles which are trapped in the pair potential well of height Δ, suffering multiple Andreev reflections and relaxing inside the N layer. For larger voltages $eV \gtrsim (2a/\ell_{\rm in})\Delta$ quasiparticle are accelerated out of the N layer and relax in superconducting banks. The current-voltage characteristic of the system in this regime is calculated both analytically and numerically. In the intermediate voltage range $V \sim \Delta/e$ we analyze the subharmonic gap structure on the I-V curve and show that this structure is most pronounced in the temperature range $T \sim \Delta/k_B$ and vanishes for both $T \to 0$ and $T \to T_C$. We explain this behavior taking into account the contribution of thermally excited quasiparticles with momenta opposite to the direction of total current flow.

I. INTRODUCTION

Two superconductors (S) weakly linked by a normal (N) or semiconducting (Sm) region in a wealth of different geometrical sample configurations[1] exhibit a number of features in their experimental current–voltage characteristics (CVC's) which cannot be described by the simple resistively shunted junction (RSJ) model. As listed by Likharev[1] these are the "excess current" at high voltages $eV \gg \Delta$, a high-low voltage conductance ("bump" or "foot"), a "negative slope region on the I-V curves" at low temperatures which may result in hysteretic $V(I)$ dependence, and "subharmonic gap structures" (SGS's) at voltages $V = 2\Delta/en$, n an integer; Δ is the pair potential in the superconducting banks. After their measurement in weak links with conventional superconductors[2–8] most of these features have also been observed in weakly linked high-temperature superconductors.[9–12]

An interpretation of these features based on the phenomenon of Andreev reflection at NS interfaces[13] evolved from the theory of Artemenko, Volkov and Zaitsev[14,15] of the excess current. Subsequent microconstriction studies of Klapwijk *et al.*,[16] Blonder *et al.*,[17] and Octavio *et al.*,[18] the latter combining the "generalized semiconductor model" of the former with a Boltzmann equation approach, showed that the SGS is directly related to multiple Andreev reflections at NS interfaces. The Boltzmann equation approach allows the inclusion of elastic scattering at interface barriers. This elastic scattering enhances the number of quasiparticles with momenta opposite to the current direction by inverting the momenta of the quasiparticles accelerated out of the condensate by the electric field. Thus, more quasiparticles which cause the SGS because of multiple Andreev scattering are trapped in the pair potential well and the SGS is enhanced by elastic scattering at the interfaces. In this paper, where inelastic scattering is taken into account only, we will show that increasing the temperature up to $T \sim \Delta/k_B$ has a similar effect. Elastic scattering in a semiconductor, coupling a superconductor to a reservoir, can also enhance the number of Andreev reflections and the differential conductance of the sample.[19] A review of mesoscopic superconductor-semiconductor heterostructures has been given recently by Klapwijk.[20]

The quasiparticle trajectories in the clean normal region of a SNS junction under the influence of an electric field and multiple Andreev reflections have been described in Ref. 21 by accelerated wave packet solutions of the time-dependent version of the Bogoliubov–de Gennes equations. Subsequently this approach has been extended to describe the response of the superconducting condensate to the charge and momentum transfer due to Andreev scattering processes.[17] Based on the accelerated wave packet solutions CVC's were calculated[22,23] with inelastic scattering being taken into account by a phenomenological relaxation time. This model allows for a simple physical interpretation of the above mentioned features in the CVC in terms of multiple Andreev reflections. A deficiency of the phenomenological approach[23] is the necessity to make assumptions about the quasiparticle distribution. The simplest thing to do, and done in Ref. 23, is to assume in a Drude-like model that after each relaxation cycle all quasiparticles return to their initial (equilibrium) energy distribution. On the other hand—perhaps except for the cases in which the linear response theory can be applied—in the presence of an external electric field inside the N layer the distribution function of a SNS junction essentially deviates from its equilibrium value. Such nonequilibrium effects cannot be described within the phenomenological description[23] and a more general technique is needed.

The general method which allows for a self-consistent quantum statistic description of nonequilibrium phenomena is the Keldysh technique.[24] With this technique, adapted to superconductivity,[25] one can treat the problem without making *ad hoc* assumptions about quasiparticle distribution and initial states. Within the framework of the Keldysh formalism the problem of nonstationary and nonequilibrium ballistic charge transport through clean SNS microbridges was first studied in Refs. 26, 27 in which the general expressions for the Green functions of clean SNS structures and CVC's were calculated. For small ac voltages the expression for the current was found within the linear response approximation, and microwave stimulation of the dc Josephson current was studied. In the limit of large voltages it was shown that (in contrast to dirty systems[28,29]) the ballistic excess current is completely independent of the normal layer thickness and coincides with that found by Zaitsev for short superconducting microconstrictions.[15] For NS microjunctions noise and current fluctuations due to Andreev reflections at voltages of order of the gap have been discussed by Khlus.[30]

The complicated structure of the expressions for the Green functions found in Refs. 26, 27 makes it quite difficult to evaluate the current due to ballistic charge transport for the intermediate voltage range as well as for small dc voltages. In this paper we present a formalism which allows us to avoid these technical difficulties by a factorization of the Green functions as products of wave fields depending on one time variable only. With the aid of this technique we provide a complete description of the system behavior for any value of the external voltage. We demonstrate that within the framework of our model this behavior is essentially determined by the competition of two processes: acceleration of quasiparticles due to multiple Andreev reflections in the electric field and their relaxation due to inelastic electron-phonon scattering inside the N layer. For a given voltage V across the weak link a quasiparticle accelerated from the Fermi surface suffers $n \sim \Delta/eV$ Andreev reflections until it reaches the top of the pair potential well. It means that a quasiparticle spends the time of order $\tau \sim 2an/v_F$ in the N layer and then escapes from the pair potential well into a superconducting bank. Provided this time exceeds the inelastic relaxation time $\tau \gg \tau_{\rm in}$ or, equivalently, in the limit $eV \ll (2a/v_F\tau_{\rm in})\Delta$ quasiparticles cannot reach the energy Δ and remain trapped in the N layer. This results in a substantial increase of the low voltage conductance $G \sim (v_F\tau_{\rm in}/2a)(1/R_0)$ over the normal state conductance $1/R_0$. For larger voltages $eV > (2a/v_F\tau_{\rm in})\Delta$ inelastic relaxation inside the N layer plays no role and quasiparticles are freely accelerated out of the N layer. The current-voltage characteristic of the system in this regime is expressed by a sum of the standard Ohmic term and an additional term caused by multiple Andreev reflections. The last term weakly depends on V in a wide voltage interval and coincides with the excess current[15,26] in the limit $eV \gg \Delta$. As we already discussed in the intermediate voltage range $V \sim \Delta/e$ the I-V curve has the so-called subharmonic gap structure which manifests itself in a set of upwards peaks in the differential junction resistance at voltages $V_n = 2\Delta/en$, where n is a positive integer number. Our calculation shows that this structure is most pronounced in the temperature range $T \sim \Delta/k_B$ and vanishes for both $T \to 0$ and $T \to T_C$. We demonstrate that both the presence of SGS's and its nontrivial temperature dependence can account for the contribution of thermally excited quasiparticles with momenta opposite to the direction of the total current flow.

The structure of the paper is as follows. Section II is devoted to the description of the formalism. In Sec. III we specify the model of the voltage-biased clean SNS junction, calculate the new wave fields, and construct the expressions for the retarded, advanced, and Keldysh Green functions. The dc current-voltage characteristic and differential resistance are calculated in Sec. IV. Emphasis is put on the analytic description of current enhancement by multiple Andreev reflections and the temperature dependence of the SGS. Discussion of our results and comparison with experiment are presented in Sec. V.

II. METHOD

Nonequilibrium effects in superconducting structures represent a nontrivial combination of quantum mechanical and kinetic phenomena. A general self-consistent description of both these parts of the problem can be provided within the framework of the Keldysh technique.[24] Applied to superconductors this technique yields a system of Gor'kov equations for the Keldysh matrix Green functions. A significant simplification of the formalism was achieved by Larkin and Ovchinnikov[25] who combined the Keldysh technique with the Eilenberger quasiclassical approach[31] and formulated the system of equations for quasiclassical Green functions:

$$\hbar \mathbf{v}_F \nabla \check{G} + \hat{\sigma}_3 \hbar \frac{\partial}{\partial t} \check{G} + \hbar \frac{\partial}{\partial t'} \check{G} \hat{\sigma}_3 + \hat{H}(\mathbf{r},t)\check{G} - \check{G}\hat{H}(\mathbf{r},t') + i\left(\check{\Sigma}\check{G} - \check{G}\check{\Sigma}\right) = \check{0}. \quad (1)$$

Here $\check{G} = \check{G}(\mathbf{v}_F, \mathbf{r}, t, t')$ is the 4×4 matrix in Keldysh space,

$$\check{G} = \begin{pmatrix} \hat{G}^R & \hat{G}^K \\ \hat{0} & \hat{G}^A \end{pmatrix}. \quad (2)$$

It consists of energy-integrated retarded, advanced, and Keldysh Green functions $\hat{G}^R$, $\hat{G}^A$, and $\hat{G}^K$. They depend on one space and two time coordinates as well as the Fermi velocity $\mathbf{v}_F$. Each of these matrices in turn is a 2×2 matrix in the Nambu space, defined by

$$\hat{G}^{R,A,K} = \begin{pmatrix} G^{R,A,K} & F^{R,A,K} \\ -(F^{R,A,K})^+ & -G^{R,A,K} \end{pmatrix}. \quad (3)$$

The matrix $\check{G}$ obeys the normalization condition

$$\check{G}\check{G} = \check{1}\delta(t-t'); \tag{4}$$

$\check{1}$ is the unit matrix. The self-energy matrix $\check{\Sigma}$ describes scattering of electrons, e.g., by impurities and phonons. It has the same structure in Keldysh-Nambu space as the matrix $\check{G}$; $\hat{\sigma}_i$, $i = 1, 2, 3$, are the Pauli matrices.

Products of $\check{G}$ functions and $\check{\Sigma}$ are integrals over the internal time variable. The matrix

$$\hat{H}(\mathbf{r},t) = -i[U(\mathbf{r},t)\hat{1} + (e/c)\mathbf{v}_F\mathbf{A}(\mathbf{r},t)\hat{\sigma}_3 + \hat{\Delta}(\mathbf{r},t)] \tag{5}$$

contains the scalar potential $U(\mathbf{r},t)$, the vector potential $\mathbf{A}(\mathbf{r},t)$, and the complex pair potential $\Delta(\mathbf{r},t)$ in the matrix

$$\hat{\Delta} = \begin{pmatrix} 0 & -\Delta \\ \Delta^* & 0 \end{pmatrix}. \tag{6}$$

All information about physical quantities is contained in the Keldysh Green function $\hat{G}^K$. The quantities we are interested in are the current density

$$\mathbf{j}(\mathbf{r},t) = -\frac{emp_F}{4\pi\hbar^3}\int (d\Omega_{v_F}/4\pi)\,\mathrm{Sp}\hat{\sigma}_3\mathbf{v}_F\hat{G}^K(\mathbf{v}_F,\mathbf{r},t,t) \tag{7}$$

and the scalar potential

$$U(\mathbf{r},t) = -\frac{\pi}{4}\int (d\Omega_{v_F}/4\pi)\,\mathrm{Sp}\hat{G}^K(\mathbf{v}_F,\mathbf{r},t,t). \tag{8}$$

The integral over $d\Omega_{v_F}$ represents the average over all directions of the Fermi velocity $\mathbf{v}_F$ and Sp is the trace of the matrix. The system of equations (1)–(8) together with Maxwell equations and the self-consistency equations for the pair potential Δ and the self-energy $\check{\Sigma}$ determines completely the behavior of inhomogenous superconductors in arbitrary, time-dependent electromagnetic fields.

Although the system of equations (1) turns out to be much simpler to deal with than the system of exact Gor'kov equations, the structure of the Green functions $\check{G}$ still remains quite complicated, because these functions depend on two time variables. Further simplification of the formalism was achieved in the work of Panyukov and one of the authors.[32] Following Ref. 32 let us introduce the two-component time-dependent wave fields

$$\hat{u}^{+\,R(A)} = \begin{pmatrix} u^{+\,R(A)} \\ v^{+\,R(A)} \end{pmatrix}, \quad \hat{u}^{\,R(A)} = \left(u^{\,R(A)}\,;\,v^{\,R(A)}\right),$$

$$\hat{u}^{+} = \begin{pmatrix} u^{+} \\ v^{+} \end{pmatrix}, \quad \hat{u} = (u\,;\,v)\,. \tag{9}$$

The $\hat{u}^{\,R(A)}$ and $\hat{u}^{+R(A)}$ obey the equations

$$\hbar\mathbf{v}_F\nabla\hat{u}^{\,R(A)} + \hbar\frac{\partial}{\partial t}\hat{u}^{\,R(A)}\,\hat{\sigma}_3 - \hat{u}^{\,R(A)}\hat{H}(\mathbf{r},t) - i\hat{u}^{\,R(A)}\hat{\Sigma}^{\,R(A)} = \hat{0}, \tag{10}$$

$$\hbar\mathbf{v}_F\nabla\hat{u}^{+R(A)} + \hat{\sigma}_3\,\hbar\frac{\partial}{\partial t}\hat{u}^{+R(A)} + \hat{H}(\mathbf{r},t)\hat{u}^{+R(A)} + i\hat{\Sigma}^{R(A)}\hat{u}^{+R(A)} = \hat{0}. \tag{11}$$

Here integration over the internal time coordinate is again implied in products of u functions with the self-energy terms

$$\hat{u}^{\,R(A)}\hat{\Sigma}^{R(A)} = \int_{-\infty}^{+\infty} dt_1\,\hat{u}^{\,R(A)}(\mathbf{v}_F,\mathbf{r},t_1)\hat{\Sigma}^{R(A)}(\mathbf{v}_F,\mathbf{r},t_1,t) \tag{12}$$

and

$$\hat{\Sigma}^{R(A)}\hat{u}^{+R(A)} = \int_{-\infty}^{+\infty} dt_1\,\hat{\Sigma}^{R(A)}(\mathbf{v}_F,\mathbf{r},t,t_1)\,\hat{u}^{+R(A)}(\mathbf{v}_F,\mathbf{r},t_1). \tag{13}$$

Let us furthermore define the four-component functions

$$\check{u}^{+} = \begin{pmatrix} \hat{u}^{+} \\ \hat{u}^{+A} \end{pmatrix}, \qquad \check{u} = \left(\hat{u}^{\,R}\,;\,\hat{u}\right), \tag{14}$$

which satisfy the set of equations

$$\hbar\mathbf{v}_F\nabla\check{u} + \hbar\frac{\partial}{\partial t}\check{u}\,\hat{\sigma}_3 - \check{u}\hat{H}(\mathbf{r},t) - i\check{u}\check{\Sigma} = \check{0}, \tag{15}$$

$$\hbar\mathbf{v}_F\nabla\check{u}^{+} + \hat{\sigma}_3\,\hbar\frac{\partial}{\partial t}\check{u}^{+} + \hat{H}(\mathbf{r},t)\check{u}^{+} + i\check{\Sigma}\check{u}^{+} = \check{0}. \tag{16}$$

It was shown in Ref. 32 that the solution of the system of equations (1) for the matrix Green function $\check{G}$ can be expressed in terms of the u and v functions of Eqs. (9)–(16) as follows:

$$\check{G}(\mathbf{v}_F,\mathbf{r},t,t') = \sum_{\sigma=\pm 1}\int_{-\infty}^{+\infty}(d\varepsilon/2\pi)\,\check{\mathcal{U}}^{+}_{\sigma}(\varepsilon,\mathbf{v}_F,\mathbf{r},t) \times\check{\mathcal{U}}_{\sigma}(\varepsilon,\mathbf{v}_F,\mathbf{r},t'), \tag{17}$$

where $\check{\mathcal{U}}^{+}$ and $\check{\mathcal{U}}$ are 4×4 matrices,

$$\check{\mathcal{U}}^{+} = \begin{pmatrix} \hat{\mathcal{U}}^{+R} & \hat{\mathcal{U}}^{+} \\ \hat{0} & \hat{\mathcal{U}}^{+A} \end{pmatrix}, \tag{18}$$

$$\check{\mathcal{U}} = \begin{pmatrix} \hat{\mathcal{U}}^{\,R} & \hat{\mathcal{U}} \\ \hat{0} & \hat{\mathcal{U}}^{\,A} \end{pmatrix}. \tag{19}$$

The 2×2 matrices $\hat{\mathcal{U}}^{+R(A)}$, $\hat{\mathcal{U}}^{+}$, $\hat{\mathcal{U}}^{\,R(A)}$, and $\hat{\mathcal{U}}$ consist of the solutions of Eqs. (10)–(16):

$$\hat{\mathcal{U}}^{+R(A)} = (\hat{u}^{+R(A)};\,0), \quad \hat{\mathcal{U}}^{\,R(A)} = \begin{pmatrix} \hat{u}^{\,R(A)} \\ 0 \end{pmatrix}, \tag{20}$$

$$\hat{\mathcal{U}}^{+} = (\hat{u}^{+};\,0), \qquad \hat{\mathcal{U}} = \begin{pmatrix} \hat{u} \\ 0 \end{pmatrix}. \tag{21}$$

The index σ in Eq. (17) labels two independent solutions of Eqs. (10)–(16). In the stationary case ε determines the time dependence of the solutions of Eqs. (10)–(16) according to

$$\hat{u}^{\,R(A)},\ \check{u} \sim \exp(-i\,\varepsilon t/\hbar),$$

$$\hat{u}^{+R(A)},\ \check{u}^{+} \sim \exp(+i\,\varepsilon t/\hbar). \tag{22}$$

In the nonstationary case, ε characterizes the solutions—presented in Sec. III A—according to their asymptotic boundary conditions. These conditions, valid at all times t and t', are given by the Fourier transform of $\check{G}(\mathbf{v}_F, \mathbf{r}, t, t')$ with respect to $t - t'$, with $\mathbf{r}$ being deep in the superconducting banks where $\check{G}(\mathbf{v}_F, \mathbf{r}, t, t') = \check{G}(t-t')$ is the well-known Green function of the homogeneous, field-free superconductor. Provided the solutions, Eq. (20), satisfy the normalization condition

$$\sum_{\sigma=\pm 1} \sigma \, \hat{\mathcal{U}}_\sigma^{+R(A)}(\varepsilon) \times \hat{\mathcal{U}}_\sigma^{R(A)}(\varepsilon) = \hat{1}, \tag{23}$$

the normalization condition (4) for the Green functions is satisfied automatically. It is sufficient to check the validity of the normalization condition (23) only in the part of the system in which the quasiparticle distribution function does not deviate from equilibrium, e.g., deep in the superconductors. As this condition represents the first integral of Eqs. (10)–(16) it will be then automatically satisfied for the solutions of these equations. Due to that it is also not necessary to define extra normalization conditions for $\hat{u}^+$ and $\hat{u}$. In equilibrium these functions are linked to the $\hat{u}^{+A}$ and $\hat{u}^R$ functions by means of the equations

$$\hat{u}_\sigma^+(\varepsilon) = -\tanh\left(\frac{\varepsilon}{2k_BT}\right) \hat{u}_\sigma^{+A}(\varepsilon),$$

$$\hat{u}_\sigma(\varepsilon) = +\tanh\left(\frac{\varepsilon}{2k_BT}\right) \hat{u}_\sigma^{R}(\varepsilon). \tag{24}$$

Equations (9)–(24) allow one to reduce the problem of the calculation of the Keldysh Green function (which in the nonstationary case depends on two times rather than on the time difference only) to the much simpler problem of the solution of Eqs. (10)–(16) for wave fields which depend only on one time variable. We will follow this formalism in the analysis presented below.

In a stationary case the structure of the equations resembles that of the Bogoliubov–de Gennes equations reduced within the framework of the quasiclassical approximation.[13] Note however that in the nonstationary case the u and v functions (9) also contain information about the distribution function of the problem and by no means can be interpreted as coefficient functions of the Bogoliubov transformation for the BCS Hamiltonian. Corresponding equations can be formulated also for a nonquasiclassical situation.[32]

III. QUASIPARTICLE DYNAMICS IN CLEAN WEAK LINKS

We consider SNS junctions, i.e., systems which consist of a normal layer (N) between two superconducting bulks (S). We shall mostly concentrate our attention on the description of SNS microbridges and constrictions. In the low voltage limit (see below) our analysis can be also applied to planar SNS structures and sandwiches. We assume very low concentration of impurities so that elastic scattering can be neglected and inelastic scattering occurs only via electron-phonon interaction. Furthermore we disregard any differences in Fermi energies and effective masses in the normal (N) and the superconducting (S) regions and suppose perfect transparency of the NS interfaces. Thus the only scattering process at the interfaces considered is Andreev reflection. We choose the direction of the z axis perpendicular to the NS interfaces and assume translational invariance in the x and y directions. For the sake of simplicity we stick to the standard square well pair potential, the absolute value of which is given by

$$\Delta(z) = \Delta\Theta(|z| - a). \tag{25}$$

As already mentioned we focus on a voltage-biased situation where there is a constant voltage drop V between the two superconducting bulks. We assume that the electric field does not penetrate into the superconductors. This assumption is valid in systems where we have thick superconducting bulks connected via a normal conducting region of small cross section area. Beside this we solve our problem without any further assumptions on the field distribution in the normal layer. Below we shall show that the expression for the current is essentially independent from the particular choice of such distribution. What matters is the energy which quasiparticles acquire during the process of multiple Andreev reflections in the normal layer. This energy depends only on the total voltage V applied to the system being insensitive to its spatial distribution in the junction. Explicit forms of the field distribution in this layer will be presented in Sec. V. Neglecting the magnetic field from the current through our junction and choosing the gauge $\nabla \mathbf{A} = 0$ we can put the vector potential equal to zero. Then the scalar potential can be written in the form

$$U(z) = -\frac{eV}{2}\Theta(-z-a) + U_N(z)\Theta(a-|z|) + \frac{eV}{2}\Theta(z-a), \tag{26}$$

where $U_N(z)$ is, at the moment, an arbitrary function. According to the Josephson equation we have a phase difference $\Delta\Phi = (2eV/\hbar)t$ between the right and the left superconductors. Within these approximations our problem becomes effectively one dimensional and we have

$$\mathbf{v}_F \nabla \to s v_{zF} \frac{\partial}{\partial z}, \quad s = \pm 1, \quad v_{zF} = |\mathbf{e}_z \mathbf{v}_F| > 0. \tag{27}$$

In the relaxation time approximation[14,25] the self-energy has the form

$$\hat{\Sigma}^{R,A,K}(\mathbf{v}_F, \mathbf{r}, t, t') = i \int_{-\infty}^{+\infty} (dE/2\pi) \; \hat{S}(z,t) e^{+iEt/\hbar} \, \hat{\zeta}^{R,A,K}(z,E) e^{-iEt'/\hbar} \hat{S}^+(z,t'), \tag{28}$$

where

$$\hat{\zeta}^R(z,E) = \zeta_1(z,E)\,\hat{\sigma}_3 + \zeta_2(z,E)\,i\hat{\sigma}_2 = -\hat{\zeta}^A(z,E), \quad (29)$$

$$\hat{\zeta}^K(z,E) = 2\tanh\left(\frac{E}{2k_BT}\right)\hat{\zeta}^R(z,E), \quad (30)$$

with $\zeta_{1,2}(z,E) = \hbar/2\tau_{1,2}(z,E)$; $\tau_{1,2}$ are relaxation times due to inelastic electron-phonon scattering and

$$\hat{S}^{(+)}(z,t) = \exp\left(\overset{-}{\scriptstyle(+)} i\frac{eV}{2}\frac{t}{\hbar}\hat{\sigma}_3\right)\Theta(-z-a) + \exp\left(\overset{+}{\scriptstyle(-)} iU_N(z)\frac{t}{\hbar}\hat{\sigma}_3\right)\Theta(a-|z|) + \exp\left(\overset{+}{\scriptstyle(-)} i\frac{eV}{2}\frac{t}{\hbar}\hat{\sigma}_3\right)\Theta(z-a), \quad (31)$$

where terms of the form $\exp(X\hat{\sigma}_3)$ are defined as

$$\exp(X\hat{\sigma}_3) = \begin{pmatrix} \exp(X) & 0 \\ 0 & \exp(-X) \end{pmatrix}. \quad (32)$$

A. u and v functions

In the superconductors appropriate linear combinations of solutions of Eq. (10) from which the Green function of Eq. (17) can be built up are given by

$$\hat{u}_\sigma^{R(A)}(\varepsilon, s, v_{zF}, z, t) = \int_{-\infty}^{+\infty} (dE/2\pi)\,\hat{u}_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}, z)e^{-iEt/\hbar}\hat{S}^{+}(z,t), \quad (33)$$

where

$$\hat{u}_\sigma^{R(A)}(\varepsilon, s, v_{zF}, E, z) = A_\sigma^{R(A)}(\varepsilon, E, s, v_{zF})(1;\gamma^{R(A)}(E))\exp\left(+is\frac{W^{R(A)}}{\hbar v_{zF}}z\right) + B_\sigma^{R(A)}(\varepsilon, E, s, v_{zF})(1;[\gamma^{R(A)}(E)]^{-1})\exp\left(-is\frac{W^{R(A)}}{\hbar v_{zF}}z\right), \quad (34)$$

with

$$\gamma^{R(A)}(E) = \frac{[E \overset{+}{\scriptstyle(-)} i\zeta_1(E)] - W^{R(A)}}{\Delta \overset{+}{\scriptstyle(-)} i\zeta_2(E)} \quad (35)$$

and

$$W^{R(A)} = \sqrt{[E \overset{+}{\scriptstyle(-)} i\zeta_1(E)]^2 - [\Delta \overset{+}{\scriptstyle(-)} i\zeta_2(E)]^2}. \quad (36)$$

$A_\sigma^{R(A)}$ and $B_\sigma^{R(A)}$ are integration constants which are determined by the asymptotic boundary conditions deep in the superconducting banks and by the boundary conditions at the NS interfaces. Because of the assumed high transparency of the interfaces, the solutions have to be continuous across them. In deriving Eqs. (33) and (34) ζ_1 and ζ_2 were taken to be spatially constant.

In the N region the solutions of Eq. (10) are given by

$$\hat{u}_\sigma^{R(A)}(\varepsilon, s, v_{zF}, z, t) = \int (dE/2\pi)\,\hat{u}_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}, z)e^{-iEt/\hbar}, \quad (37)$$

where

$$\hat{u}_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}, z) = \left\{A_\sigma^{R(A)}(\varepsilon, E, s, v_{zF})\,(1;0)\exp\left(+is\frac{E \overset{+}{\scriptstyle(-)} i\zeta_1}{\hbar v_{zF}}z\right) + B_\sigma^{R(A)}(\varepsilon, E, s, v_{zF})\,(0;1)\exp\left(-is\frac{E \overset{+}{\scriptstyle(-)} i\zeta_1}{\hbar v_{zF}}z\right)\right\}\exp\left(-is\frac{\tilde{\Phi}(z)}{\hbar v_{zF}}\right), \quad (38)$$

and $\tilde{\Phi}(z)$ is the integral of the scalar potential $U_N(z)$ in the normal layer,

$$\tilde{\Phi}(z) = \int_{z_0}^{z} dz'\,U_N(z'); \quad (39)$$

z_0 is associated with an arbitrary integration constant. To simplify our analysis we neglect ζ_2 and consider ζ_1 to be E independent inside the normal layer. This choice can be easily justified, e.g., for relatively thick normal layers in which case we can neglect the proximity effect while calculating the relaxation time $\tau_1 = \tau_{\text{in}}$ due to inelastic electron phonon scattering. In this approximation τ_{in} reduces to that of a normal metal, $\tau_{\text{in}} \sim \hbar^3\omega_D^2/k_B^3T^3$, where ω_D is the Debye frequency. Note that in the SNS structures considered here, with conventional superconductors and normal layer thicknesses $2a$ of the order of the BCS coherence length ξ_0, the inelastic mean free path $\ell_{\text{in}} = v_F\tau_{\text{in}}$ is large compared to $2a$ [typical values are $(\ell_{\rm in}/2a) \sim 10^3$–$10^4$]. The same is valid in the superconductors.

The solutions of Eq. (11) in the superconductors are

$$\hat{u}_\sigma^{+R(A)}(\varepsilon, s, v_{zF}, z, t) = \int (dE/2\pi)\ \hat{S}(z,t) e^{+iEt/\hbar} \hat{u}_\sigma^{+R(A)}(\varepsilon, E, s, v_{zF}, z), \tag{40}$$

where

$$\hat{u}_\sigma^{+R(A)}(\varepsilon, E, s, v_{zF}, z) = A_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}) \begin{pmatrix} 1 \\ -\gamma^{R(A)}(E) \end{pmatrix} \exp\left(-is\frac{W^{R(A)}}{\hbar v_{zF}} z\right) + B_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}) \begin{pmatrix} 1 \\ -[\gamma^{R(A)}(E)]^{-1} \end{pmatrix} \exp\left(+is\frac{W^{R(A)}}{\hbar v_{zF}} z\right). \tag{41}$$

In the normal region we have

$$\hat{u}_\sigma^{+R(A)}(\varepsilon, s, v_{zF}, z, t) = \int (dE/2\pi)\ e^{+iEt/\hbar} \hat{u}_\sigma^{+R(A)}(\varepsilon, E, s, v_{zF}, z), \tag{42}$$

where

$$\hat{u}_\sigma^{+R(A)}(\varepsilon, E, s, v_{zF}, z) = \left\{ A_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \exp\left(-is\frac{E \overset{+}{(-)} i\zeta_1}{\hbar v_{zF}} z\right) + B_\sigma^{R(A)}(\varepsilon, E, s, v_{zF}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \exp\left(+is\frac{E \overset{+}{(-)} i\zeta_1}{\hbar v_{zF}} z\right) \right\} \exp\left(is\frac{\tilde{\Phi}(z)}{\hbar v_{zF}}\right). \tag{43}$$

Let us now calculate the functions $\hat{u}$ and $\hat{u}^+$. From Eqs. (15) and (16) we have

$$\hbar \mathbf{v}_F \nabla \hat{u} + \hbar \frac{\partial}{\partial t} \hat{u}\ \hat{\sigma}_3 - \hat{u}\, \hat{H}(\mathbf{r}, t) - i\hat{u}\, \hat{\Sigma}^A = i\hat{u}^{\,R} \hat{\Sigma}^K, \tag{44}$$

$$\hbar \mathbf{v}_F \nabla \hat{u}^+ + \hat{\sigma}_3\, \hbar \frac{\partial}{\partial t} \hat{u}^+ + \hat{H}(\mathbf{r}, t) \hat{u}^+ + i\hat{\Sigma}^R \hat{u}^+ = -i\hat{\Sigma}^K \hat{u}^{+A}. \tag{45}$$

Appropriate solutions of Eq. (44) are the superposition

$$\hat{u}_\sigma(\varepsilon, s, v_{zF}, z, t) = \hat{u}_\sigma^{\rm Sp}(\varepsilon, s, v_{zF}, z, t) + \hat{u}_\sigma^{H}(\varepsilon, s, v_{zF}, z, t), \tag{46}$$

of a special solution of this equation,

$$\hat{u}_\sigma^{\rm Sp}(\varepsilon, s, v_{zF}, z, t) = \int_{-\infty}^{+\infty} dt_1\ \hat{u}_\sigma^{R}(\varepsilon, s, v_{zF}, z, t_1) \hat{f}(s, v_{zF}, z, t_1, t), \tag{47}$$

and a solution $\hat{u}_\sigma^{H}$ of the homogeneous equation (10). The distribution function matrix $\hat{f}(s, v_{zF}, z, t_1, t)$ has the simple form

$$\hat{f}(s, v_{zF}, z, t_1, t) = \int_{-\infty}^{+\infty} (dE/2\pi)\ \hat{S}(z, t_1) e^{+iEt_1/\hbar} \tanh\left(\frac{E}{2k_BT}\right) e^{-iEt/\hbar} \hat{S}^+(z, t) \tag{48}$$

in the superconductors, whereas in the normal region it is given by

$$\hat{f}(s, v_{zF}, z, t_1, t) \equiv \hat{f}(s, v_{zF}, z, t_1 - t) = \int_{-\infty}^{+\infty} (dE/2\pi)\ e^{+iE(t_1 - t)/\hbar} \hat{f}(s, v_{zF}, E, z). \tag{49}$$

Here

$$\hat{f}(s, v_{zF}, E, z) = \tanh\left(\frac{E}{2k_BT}\right) \hat{1} + \delta\hat{f}(s, v_{zF}, E, z) \exp\left(2s\frac{2a}{\hbar v_{zF}} \zeta_1 \frac{z}{2a} \hat{\sigma}_3\right), \tag{50}$$

with

$$\delta\hat{f}(s, v_{zF}, E, z) = 2s\frac{2a}{\hbar v_{zF}} \zeta_1 \hat{\sigma}_3 \int_{z_0}^{z} (dz'/2a)\ \exp\left(-2s\frac{2a}{\hbar v_{zF}} \zeta_1 \frac{z'}{2a} \hat{\sigma}_3\right) \times \begin{pmatrix} \left[\tanh\left(\frac{E}{2k_BT}\right) - \tanh\left(\frac{E - U_N(z')}{2k_BT}\right)\right] & 0 \\ 0 & \left[\tanh\left(\frac{E}{2k_BT}\right) - \tanh\left(\frac{E + U_N(z')}{2k_BT}\right)\right] \end{pmatrix}; \tag{51}$$

z_0 has the same meaning as before, and $\hat{u}^{H}{}_{\sigma}$ is determined by the asymptotic boundary condition for $\hat{u}_\sigma$, Eq. (24), and by the continuity condition of $\hat{u}_\sigma$ at the NS interfaces.

Calculation of the function $\hat{u}^+$ is completely analogous to that of $\hat{u}$. We again represent $\hat{u}^+$ as a sum,

$$\hat{u}^+_\sigma(\varepsilon,s,v_{zF},z,t) = \hat{u}^{+^{\rm Sp}}_\sigma(\varepsilon,s,v_{zF},z,t) + \hat{u}^{+^{H}}_\sigma(\varepsilon,s,v_{zF},z,t), \tag{52}$$

where

$$\hat{u}^{+^{\rm Sp}}_\sigma(\varepsilon,s,v_{zF},z,t) = -\int_{-\infty}^{+\infty} dt_1\, \hat{f}(s,v_{zF},z,t,t_1)\hat{u}^{+^{A}}_\sigma(\varepsilon,s,v_{zF},z,t_1) \tag{53}$$

is a special solution of Eq. (45), and $\hat{u}^{+^{H}}_\sigma$ is a solution of the homogeneous equation (11).

B. Green functions

Making use of the solutions for the u and v functions given in the preceding subsection and Eqs. (17)–(23) it is straightforward to obtain the expressions for the Green functions of our problem. In the normal layer the retarded and advanced Green functions are

$$\hat{G}^{R(A)}(s,v_{zF},z,t,t') = \int (dE_1/2\pi)\int (dE_2/2\pi)\, e^{+iE_1t/\hbar} \times \hat{G}^{R(A)}(s,v_{zF},z,E_1,E_2)e^{-iE_2t'/\hbar}, \tag{54}$$

with

$$\hat{G}^{R(A)}(s,v_{zF},z,E_1,E_2) = \exp\left(-is\frac{2a}{\hbar v_{zF}}E_1\frac{z}{2a}\hat{\sigma}_3\right) \times\left[b_1^{R(A)}(s,v_{zF},E_1,E_2)\hat{\sigma}_3 + b_2^{R(A)}(s,v_{zF},E_1,E_2)i\hat{\sigma}_2\right] \times \exp\left(is\frac{2a}{\hbar v_{zF}}E_2\frac{z}{2a}\hat{\sigma}_3\right).$$

The coefficients $b_1^{R(A)}$ and $b_2^{R(A)}$ for the retarded and advanced Green functions are defined by

$$b_1^{R(A)}(s,v_{zF},E_1,E_2) = \underset{(-)}{+}\delta(E_1-E_2)\underset{(-)}{+}2\sum_{n=1}^{\infty}\left\{\prod_{l=1}^{2n}\exp\left(\underset{(-)}{+}i\frac{2a}{\hbar v_{zF}}(E_{1(2)}-sleV+seV/2\underset{(-)}{+}i\zeta_1)\right) \times\prod_{l=1}^{2n}\gamma^{R(A)}(E_{1(2)}-sleV+seV/2)\right\}\delta(E_{1(2)}-s\,2n\,eV-E_{2(1)}) \tag{55}$$

and

$$b_2^{R(A)}(s,v_{zF},E_1,E_2) = \underset{(-)}{+}2\sum_{n=1}^{\infty}\left\{\prod_{l=1}^{2n-1}\exp\left(\underset{(-)}{+}i\frac{2a}{\hbar v_{zF}}(E_{1(2)}-sleV+seV/2\underset{(-)}{+}i\zeta_1)\right) \times\prod_{l=1}^{2n-1}\gamma^{R(A)}(E_{1(2)}-sleV+seV/2)\right\}\delta(E_{1(2)}-s(2n-1)eV-E_{2(1)}). \tag{56}$$

The first and the second indices in $E_{1(2)}$ or $E_{2(1)}$ and analogous the upper and the lower signs of "$\underset{(-)}{+}$" refer to the coefficients of the retarded and the advanced functions, respectively. To evaluate the Keldysh Green function we make use of the solutions of Eqs. (10)–(16) which in combination with Eq. (17) yield

$$\hat{G}^K(s,v_{zF},z,t,t') = \hat{G}^K_{\rm Sp}(s,v_{zF},z,t,t') + \hat{G}^K_H(s,v_{zF},z,t,t'), \tag{57}$$

where $\hat{G}^K_{\rm Sp}$ contains the special solutions (47) and (53) of the inhomogeneous equations (44) and (45). In the normal layer it has the form

$$\hat{G}^K_{\rm Sp}(s,v_{zF},z,t,t') = \int (dE_1/2\pi)\int (dE_2/2\pi)\, e^{+iE_1t/\hbar}\left\{\hat{G}^R(s,v_{zF},z,E_1,E_2)\tanh\left(\frac{E_2}{2k_BT}\right) - \tanh\left(\frac{E_1}{2k_BT}\right)\hat{G}^A(s,v_{zF},z,E_1,E_2)\right\}e^{-iE_2t'/\hbar}. \tag{58}$$

Here we have neglected small terms of order $2a/v_F\tau_{\rm in} \ll 1$. $\hat{G}^K_H$ contains the functions $\hat{u}_\sigma^{-H}$ and $\hat{u}_\sigma^{+H}$. In the normal layer it is given by

$$\hat{G}^K_H(s, v_{zF}, z, t, t') = \int (dE_1/2\pi) \int (dE_2/2\pi)\, e^{+iE_1 t/\hbar} \exp\left(-is\frac{2a}{\hbar v_{zF}}(E_1 + i\zeta_1)\frac{z}{2a}\hat{\sigma}_3\right)$$
$$\times \left[b_1^K(s, v_{zF}, E_1, E_2, \hat{\sigma}_3 + b_2^K(s, v_{zF}, E_1, E_2) i\hat{\sigma}_2\right] \exp\left(is\frac{2a}{\hbar v_{zF}}(E_2 - i\zeta_1)\frac{z}{2a}\hat{\sigma}_3\right) e^{-iE_2 t'/\hbar}. \tag{59}$$

The coefficients b_1^K and b_2^K are defined in the Appendix.

Note that our solution for the Keldysh Green function (57)–(59) can be also expressed in the standard form (see, e.g., Ref. 25),

$$\hat{G}^K(\mathbf{v}_F, \mathbf{r}, t, t') = \int_{-\infty}^{+\infty} dt_1 \left\{\hat{G}^R(\mathbf{v}_F, \mathbf{r}, t, t_1)\hat{\mathcal{N}}(\mathbf{v}_F, \mathbf{r}, t_1, t') - \hat{\mathcal{N}}(\mathbf{v}_F, \mathbf{r}, t, t_1)\hat{G}^A(\mathbf{v}_F, \mathbf{r}, t_1, t')\right\}, \tag{60}$$

where $\hat{\mathcal{N}}$ represents the matrix of the distribution functions. This matrix obeys the kinetic equation which follows from Eq. (1). In the situation considered here, this kinetic equation does not contain $\hat{G}^R$ and $\hat{G}^A$ functions, and the matrix $\hat{\mathcal{N}}$ can be easily expressed in terms of the solutions of Eqs. (10)–(16) for the u and v functions.

IV. CALCULATION OF THE CURRENT

In order to calculate the current $I_z(t, V)$ in the junction as a function of the constant voltage drop V between the two superconductors we substitute the expressions for the Keldysh Green function (57)–(59) into Eq. (7) multiplied by the cross section $\mathcal{A}$ of the junction. Then we obtain

$$I_z(t, V) = \bar{I}_z(V) + \sum_{n=1}^{\infty}[I_{1,n}(V)\cos(2n\,eVt/\hbar) + I_{2,n}(V)\sin(2n\,eVt/\hbar)], \tag{61}$$

where the time averaged current $\bar{I}_z$ is

$$\bar{I}_z(V) = \frac{V}{R_0} + I_{\rm AR}(V), \tag{62}$$

with $R_0 = 4\pi^2\hbar^3/(e^2 p_F^2 \mathcal{A})$ being the Sharvin resistance and

$$I_{\rm AR}(V) = \sum_{n=1}^{\infty} I_{A,n}(V), \tag{63}$$

$$I_{A,n}(V) = -\frac{1}{2eR_0}\int_0^1 d(v_{zF}/v_F)(v_{zF}/v_F)\int_{-\infty}^{+\infty} dE\, e^{+\alpha/2}$$
$$\times \sum_{s=\pm 1} s e^{-n\alpha} A_{n,s}(E + seV/2)\Delta f(s, v_{zF}, E + seV/2 - sneV) \tag{64}$$

is the current due to multiple Andreev reflections. Here $\alpha = (2a/\ell_{in,N})(v_F/v_{zF})$, where $\ell_{{\rm in},N} = v_F\tau_{{\rm in},N}$ is the inelastic mean free path in the normal layer,

$$A_{n,s}(E) = \prod_{l=1}^{n} A(E - sleV), \tag{65}$$

$$A(E) = \gamma^R(E)\gamma^A(E) \tag{66}$$

is the probability for n-fold Andreev reflection of quasiparticles moving in ($s = 1$) and opposite ($s = -1$) to the current direction,[21,23] and

$$\Delta f(s, v_{zF}, E) = 2\left\{\left[\tanh\left(\frac{E - seV}{2k_BT}\right) - \tanh\left(\frac{E - seV/2}{2k_BT}\right)\right]e^{-\alpha} - \left[\tanh\left(\frac{E}{2k_BT}\right) - f_1(s, v_{zF}, E - seV/2, z = -sa)\right]\right\}; \tag{67}$$

f_1 is the 11-component of the $\hat{f}$ matrix defined in Eq. (50). The factor $e^{-n\alpha}$ in Eq. (64) describes energy relaxation by inelastic scattering within the normal layer. Note that the distribution of the electric field in the normal layer enters only in a small term of order $\alpha \ll 1$ in the function f_1; see Eq. (51). In the following we will concentrate ourselves on the dissipative current of Eqs. (62)–(67).

A. Current enhancement due to multiple Andreev reflections

In the limit of low voltages $eV \ll (2a/\ell_{\text{in}})\Delta$ Eq. (62) for the current can be evaluated analytically. In this limit relaxation prevents the quasiparticles from being accelerated out of the pair potential well. Therefore they are trapped in the normal layer giving the dominating contribution to the current. For simplicity we neglect Andreev reflection for energies $|E| > \Delta$. Then we have from Eqs. (66) and (35)

$$A(E) \approx \left[1 - \left(\frac{\bar{\lambda}}{a}\frac{\ell_{\text{in},N}}{\ell_{\text{in},S}}\right)\frac{2a}{\ell_{\text{in},N}}\right]\Theta(\Delta - |E|), \qquad (68)$$

where $\bar{\lambda} \sim \xi_0$ and $\ell_{\text{in},S}$ are the effective quasiparticle penetration depth and the inelastic mean free path in the superconducting banks, respectively. The deviation of $A(E)$ from 1 is due to inelastic relaxation of quasiparticles with energies $|E| < \Delta$ penetrating into the superconductors over a distance of order $\bar{\lambda}$. For the low voltages of interest the expression for Δf simplifies considerably and the integral over dE can be calculated analytically. Then the sum over n reduces to a geometrical series which can easily be evaluated. Finally we get

$$\bar{I}_z(V) = GV, \qquad (69)$$

where

$$G = \frac{1}{R_0}\frac{\ell_{\text{in},N}}{2a}\frac{4}{3}\frac{a}{a + \bar{\lambda}(\ell_{\text{in},N}/\ell_{\text{in},S})}\tanh(\Delta/2k_BT) \qquad (70)$$

is the junction conductance in the small voltage limit. The explicit form of the electric field in the normal layer appears only in small terms of the order $1/R_0$ which have been neglected here. For relatively thick normal layers $2a \gg \xi_0(\ell_{\text{in},N}/\ell_{\text{in},S})$ charge transport is determined by relaxation within this layer and the low voltage conductance will be proportional to $\ell_{\text{in},N}/2a$. In the opposite limit $2a \ll \xi_0$ nonequilibrium quasiparticles with energies below the gap penetrate into the superconductors and relax there. In this case the conductance will be proportional to $\ell_{\text{in},S}/2a$. As $\ell_{\text{in},N/S} \gg 2a$, the low voltage conductance G (70) is high and yields a steep rise of the current at small V which is shown in the inlet of Fig. 1. Such effect (the so-called "foot" on the I-V curve) has been observed in many experiments with superconducting bridges and SNS junctions; see, e.g., Refs. 2–12. Note that for superconducting bridges close to T_C this effect has been interpreted in terms of the supercurrent stimulation due to Josephson oscillations of the order parameter inside the bridge (see, e.g., Ref. 33). The ac field stimulation of the supercurrent also takes place in SNS structures.[26,27] In both these systems such nonequilibrium stimulation effect is pronounced at $T \sim T_C$ and becomes unimportant at lower T. Thus we believe that — at least for temperatures somewhat below T_C (see, e.g., Refs. 5, 6, 8–10, 12) — there is another physical reason for a steep rise of the current at small V. According to our analysis it lies in the combination of multiple Andreev reflections and inelastical relaxation of quasiparticles trapped in the weak link. Before any of these quasiparticles are inelastically scattered inside the junction it undergoes $n \sim \ell_{\text{in}}/2a$ Andreev reflections giving an n times larger contribution to the current than a quasiparticle which crosses the junction only once.

The result (69) and (70) is valid as long as the number of Andreev reflections n is limited by inelastic scattering in the junction. For voltages $eV > (2a/\ell_{\text{in}})2\Delta$ a quasiparticle gains enough energy to leave the pair potential well before it gets scattered. Then inelastic relaxation in the N region plays no role and the effective number of Andreev reflections becomes of order $n \approx 2\Delta/eV$. Thus we can neglect α in every term in Eq. (64) and (67), and the current becomes completely independent of the field distribution in the normal layer. Neglecting also Andreev reflection for energies $|E| > \Delta$ we can proceed analytically in the evaluation of Eqs. (63) and (64) and get the expression

$$I_{\text{AR}}(V) = \frac{2k_BT}{eR_0}\left\{n_{c0}\ln\left[\cosh\left(\frac{\Delta + eV}{2k_BT}\right)\right] - (n_{c0} - 1)\ln\left[\cosh\left(\frac{\Delta}{2k_BT}\right)\right] - \ln\left[\cosh\left(\frac{\Delta - n_{c0}eV}{2k_BT}\right)\right]\right\}, \qquad (71)$$

where $n_{c0} = [1 + 2\Delta/eV]$ is the maximum number of possible Andreev reflections. In the limit of $k_BT \ll \Delta, eV$ Eq. (71) yields

$$I_{\text{AR}}(V) = \frac{2\Delta}{eR_0}. \qquad (72)$$

Note that strictly speaking it is legitimate to neglect the contribution of Andreev-reflected quasiparticles with energies $|E| > \Delta$ only provided the voltage V is inside the window $(2a/\ell_{\text{in}})\ \Delta \ll eV \ll \Delta$. However, it is easy to check that also for larger voltages quasiparticles with such energies give no significant contribution to the current I_{AR} and the results (71), (72) remain valid apart from a numerical factor of order 1. Indeed coming back to the exact expressions (62)–(64) in the limit $eV \gg \Delta$ we reproduce the well-known result for the excess current

$$I_{\text{exc}} = \frac{8\Delta}{3eR_0}\tanh(eV/2kT), \qquad (73)$$

previously derived in Refs. 15, 26, 27. At $eV \ll k_B T$ this result differs from (72) only by a numerical prefactor of order 1. Our analytic arguments are supported by exact numerical calculation of Eqs. (62)–(64) presented in Fig. 1. This calculation demonstrates that at temperatures $T \ll \Delta/k_B$ the current $I_{\rm AR}$ shows only a slight dependence on V reaching the value (73) in the limit $eV \gg \Delta$. It is also interesting to point out that for $eV \gg (2a/\ell_{\rm in})\Delta$ the current $I_{\rm AR}$ (63) does not depend on the thickness $2a$ of the normal layer. This means that the result (62)–(64) remains valid also in the limit $a \to 0$, i.e, for a short superconducting constriction without any normal metal inclusions. We will come back to that in Sec. V.

B. Subharmonic gap structure

At $T = 0$ K the only contribution to the current through our system comes from quasiparticles accelerated out of the condensate; see Fig. 2. At finite temperatures quasiparticles with energies above the superconducting gap provide an additional essentially negative contribution to this current. Due to that the subharmonic gap structure (SGS) appears in the current-voltage characteristic. This structure manifests itself in a set of upward peaks in the differential resistance at voltages

$$V_n = \frac{2\Delta}{en}, \quad n = 1, 2, 3, ..., \tag{74}$$

and has been detected in many experiments; see, e.g., Refs. 2–7, 9, 10, 12. These peaks have been also obtained by the Boltzmann equation approach of Octavio *et al.* within the "generalized semiconductor model."[18] Here we describe them by means of a rigorous quantum statistical analysis and provide a transparent physical interpretation of this phenomenon. Making use of Eqs. (62)–(64) we calculate the differential resistance $(dV/dI)(V)$ numerically. The corresponding results for various temperatures are presented in Fig. 3. These results clearly show upward peaks of (dV/dI) at the voltages V_n of Eq. (74). To evaluate the heights $\Delta(dV/dI)(V_n)$ of these peaks and their temperature dependence let us for simplicity neglect again Andreev reflection at energies $|E| > \Delta$. Then the (dV/dI) curves become discontinuous at V_n, and their jumps are roughly a measure of the height of the peaks in the continuous (dV/dI) curves of Fig. 3. With the definition $\Delta(dV/dI)(V_n) = (dV/dI)(V_n - 0) - (dV/dI)(V_n + 0)$ we get from Eqs. (62) and (71)

$$\Delta(dV/dI)(V_n) = n\, R_0 \frac{\tanh\left(\frac{(1+2/n)\Delta}{2k_B T}\right) - \tanh\left(\frac{\Delta}{2k_B T}\right)}{1 + n\left[\tanh\left(\frac{(1+2/n)\Delta}{2k_B T}\right) - \tanh\left(\frac{\Delta}{2k_B T}\right)\right]}. \tag{75}$$

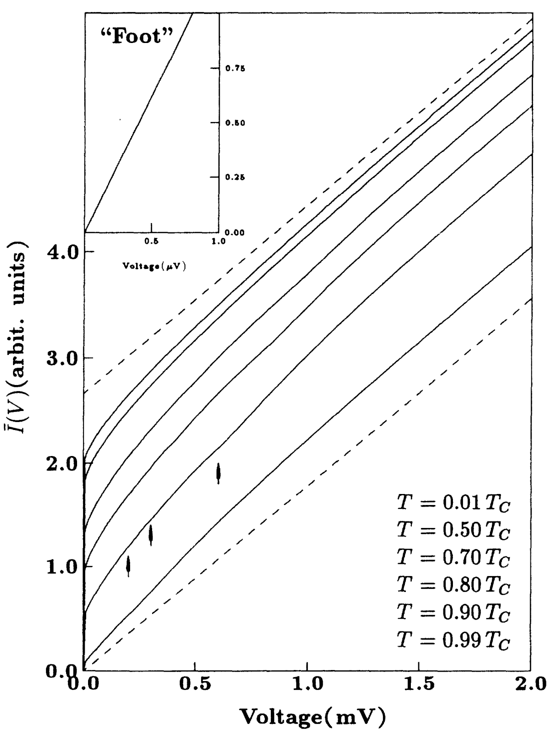

FIG. 1. The solid curves represent the current-voltage characteristics of a SNS junction at different temperatures (the lowest temperature corresponds to the uppermost curve), $\ell_{\rm in}/2a \approx 10^4$. The lower and the upper dashed lines represent the Sharvin current and the extrapolation of the excess current at $T = 0$ K to $V = 0$, respectively. The arrows indicate the subharmonic gap structures at voltages $V_n = 2\Delta(T)/ne$, $n = 1, 2, 3$. The inset shows the steep rise of the current at voltages $V \ll (2a/\ell_{\rm in})\Delta/e$ corresponding to the high-low voltage conductance of Eq. (70), $T = 0$ K.

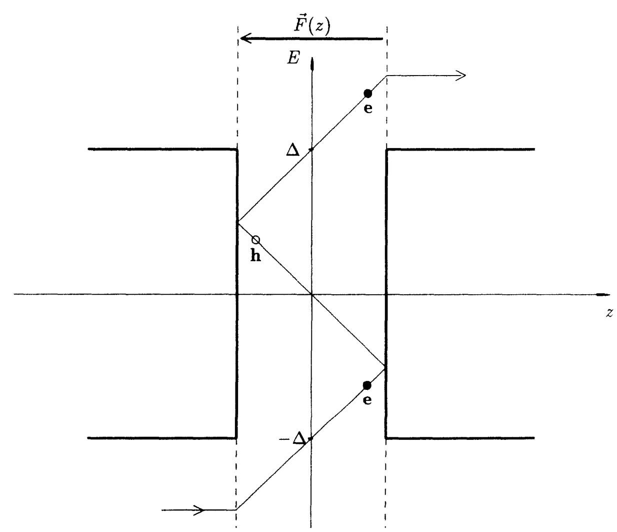

FIG. 2. Trajectory of a quasiparticle which is accelerated out of the condensate by the electric field suffering multiple Andreev reflections. The trajectories of the quasiparticle wave packets (for a constant electric field in the N region) have been calculated in Ref. 23. In this formalism quasiparticles with negative energies represent the condensate. At $T = 0$ K these quasiparticles yield the only contribution to the current $I_{\rm AR}$.

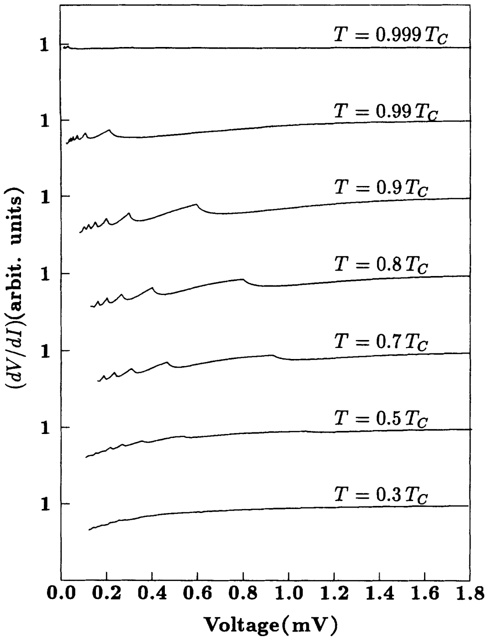

FIG. 3. Differential resistance at different temperatures, $\ell_{\rm in}/2a \approx 10^4$ with subharmonic gap structure at voltages $V_n = 2\Delta(T)/ne$, $n = 1, 2, 3, \ldots$.

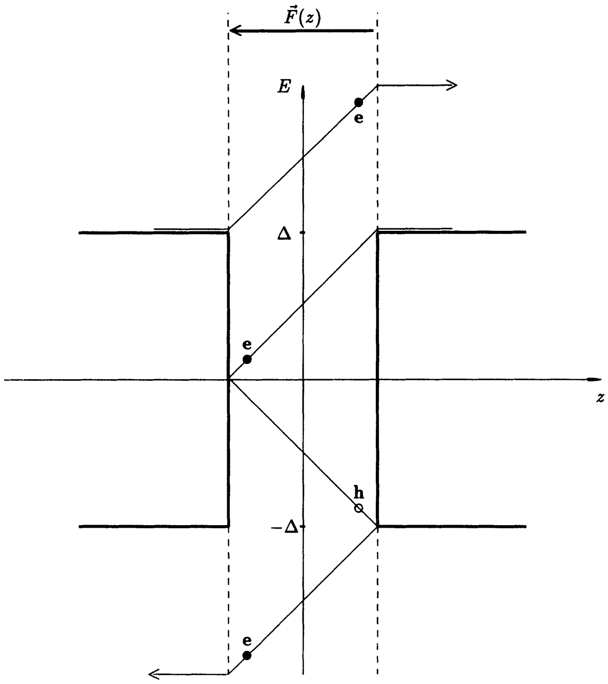

FIG. 4. Thermally excited quasiparticles above the gap Δ which move from the superconducting banks to the NS interfaces. Depending on their momentum direction they can gain energy in the electric field (quasiparticle from the left) and just cross the normal layer, or they can lose energy (quasiparticle from the right) and become trapped in the pair potential well suffering multiple Andreev reflections; these quasiparticles yield a negative contribution to the current. The effective number of Andreev reflections of any one of these quasiparticles with initial energy $\varepsilon_p \simeq \Delta$ changes at voltages $V_n = 2\Delta/ne$, $n = 1, 2, 3, \ldots$, and the current changes correspondingly. This causes the subharmonic gap structure in Fig. 3.

The amplitude of the main peak $n = 1$ reaches its maximum at $T \simeq 1.2\Delta/k_B$ whereas it vanishes for $T \to 0$ K and $T \to T_C$. The amplitudes of other peaks have a similar temperature dependence.

As already indicated the SGS in the current-voltage characteristic is caused by quasiparticles with energies above the superconducting gap. Depending on the direction of the velocity of such quasiparticles they can gain or lose energy in the electric field in the normal layer; see Fig. 4. This has been analyzed in detail by the accelerated wave packet methods of Refs. 21, 23. As a result part of the quasiparticles becomes trapped in the pair potential well and suffer multiple Andreev reflection. Because the momentum direction of such quasiparticles is opposite to that for the quasiparticles accelerated out of the condensate (see Figs. 2 and 4), they give a negative contribution to the current. At relatively low temperatures the initial energy ε_p of such quasiparticles is close to the superconducting energy gap Δ. The effective number of Andreev reflections for these quasiparticles is given by the integer part of $2\Delta/eV$. This number changes at $V = V_n$, given by Eq. (74), resulting in a rapid change of the current; and differential resistance peaks at these values of the applied voltage. At $k_BT \ll \Delta$ the number of thermally excited quasiparticles above the gap is exponentially small and the amplitudes of the resistance peaks are negligible. With increasing temperature the number of such quasiparticles grows and the SGS peaks become more pronounced. On the other hand at temperatures $k_BT \gg \Delta$ quasiparticles with different energies above the gap (and not only at $\varepsilon_p \simeq \Delta$) contribute to the counter current; see Fig. 5. For different quasiparticle energies

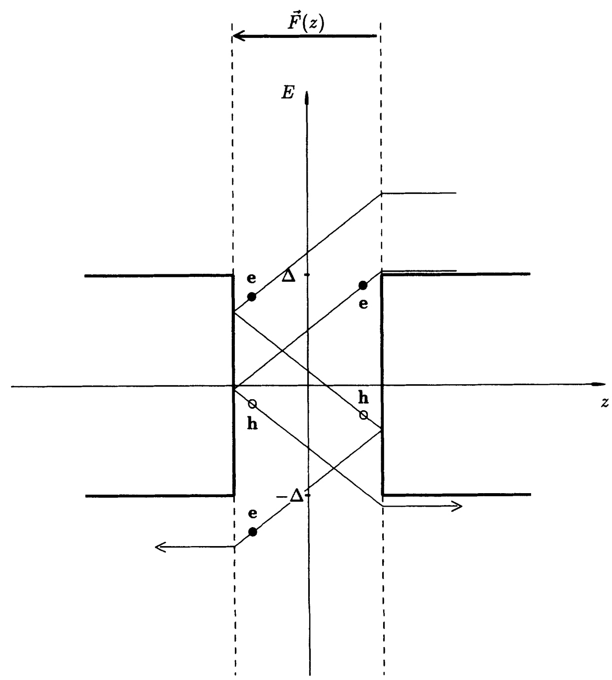

FIG. 5. For temperatures $T \gg \Delta/k_B$ there are many thermally excited quasiparticles at all energies $\varepsilon_p > \Delta$. Their effective number of possible Andreev reflections changes at different voltages so that the subharmonic gap structure vanishes for $T \to T_C$.

the effective number of Andreev reflections changes at different values of V and the integral contribution to the current does not contain SGS peaks. Therefore the amplitudes of SGS peaks should reach their maximum values at $k_B T \sim \Delta$. This is what our calculations yield indeed.

The vanishing of the SGS for temperatures $T \to 0$ K is tied to the fact that we have perfect Andreev reflection at the NS boundaries. Due to interface barriers or a mismatch of Fermi velocities, e.g., in superconductor-semiconductor-superconductor (SSmS) junctions,[38,39] normal reflection at the phase boundaries appears. Such inhomogeneities can be taken into account within the framework of the quasiclassical approach by effective boundary conditions for Green functions[40] or the u and v functions.[32] Qualitatively normal reflection at the interfaces will enhance the number of quasiparticles with momentum opposite to the current direction by inverting the positive z momentum of the quasiparticles accelerated out of the condensate by the electric field. This has a similar effect as increasing the temperature and leads to an enhancement of the SGS even for low temperatures.[18]

V. DISCUSSION

In this paper we have evaluated the Green functions for SNS systems without any assumption about the spatial variation of the scalar potential $U_N(z)$ in the normal layer. Furthermore, it turns out that the current through our junction is essentially independent of the field distribution inside the N layer. Making use of the expressions for the scalar potential (8) and the Keldysh Green function $\hat{G}^K$ (59) in the limit of small voltages $eV \ll (2a/\ell_{\rm in})\Delta$ we get

$$U_N(z) = eV \frac{z}{2a} \tanh\left(\frac{\Delta}{2k_B T}\right) \frac{a}{a + \bar{\lambda}(\ell_{{\rm in},N}/\ell_{{\rm in},S})}. \quad (76)$$

From this equation we can read off that in the limit $k_B T \ll \Delta$ and $(\bar{\lambda}/a)(\ell_{{\rm in},N}/\ell_{{\rm in},S}) \ll 1$ the complete drop of the voltage V occurs linearly in the normal layer: The quasiparticles are trapped in the pair potential well and relax due to inelastic scattering there. In this case we can neglect the penetration of the electric field into the superconductors independently of the considered geometry of small contact areas and our calculation also can be used for the description of planar structures and sandwiches. At higher temperatures $k_B T > \Delta$ when a considerable number of quasiparticles is above the gap, our analysis is only applicable to microbridges and constrictions in which case (part of) the voltage drop $V - U_N(z = +a) + U_N(z = -a)$ is due to the Sharvin resistance and takes place at the NS interfaces. For higher voltages $eV \gg (2a/\ell_{\rm in})\Delta$ inelastic relaxation in the normal layer plays no role and the total potential drop occurs at the two NS interfaces.[26,27] In this case, neglecting terms of the order $2a/\ell_{\rm in}$, we obtain from Eq. (8) that $U_N(z)$ vanishes inside the normal layer.

As we already discussed, the expression for the current, Eqs. (62)–(64), does not depend on the thickness of the normal layer $2a$ and, therefore, can be also applied to short superconducting constrictions $a \to 0$ without normal metal inclusions. In other words, even at $T = 0$ and $eV \ll \Delta$ a *dissipative* current can flow through a part of a *superconducting* system in which a nonzero external electric field is present. The possibility of such an effect has been first pointed out in Ref. 26. The physical meaning of this — at the first sight counterintuitive — result can be easily understood with the aid of Fig. 6 in which we present typical diagrams describing the dissipative contribution to the current in SNS junctions [Fig. 6(a)] and short superconducting constrictions [Fig. 6(b)]. In SNS systems an electric field leads to an acceleration of quasiparticles due to multiple Andreev reflection mechanism. As a result even for $eV \ll \Delta$ such quasiparticles gain the energy 2Δ and leave the pair potential

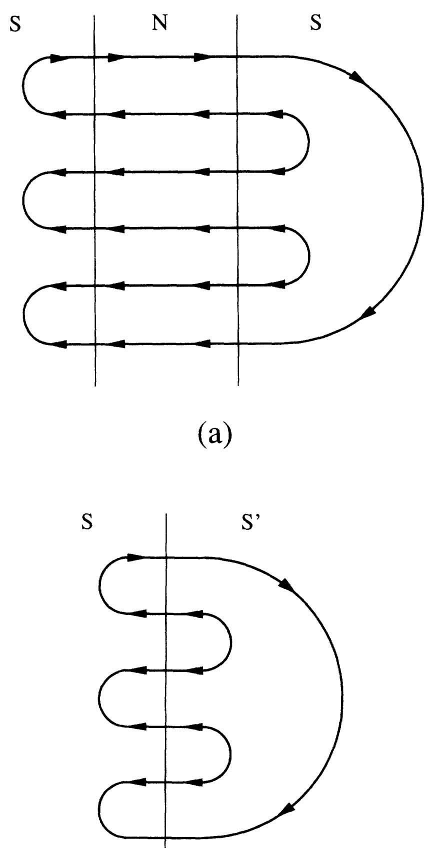

FIG. 6. Typical diagrams describing (a) multiple Andreev reflections in SNS junctions and (b) high order electron transfer processes in tunnel junctions [the so-called bubble diagrams Ref. (34)] and short superconducting constrictions (b). Both diagrams describe electron acceleration by the electric field inside the junction and contribute to a low voltage subgap conductance in the G channel. Analogous diagrams can be drawn for the F channel. Note that the diagrams (a) and (b) are essentially identical in the limit of zero normal layer thickness $2a \to 0$;$\succ\!\!\rightarrow\!\!- \equiv G$ and $\leftarrow\!\!\succ\!\!- \equiv F$.

well giving a dissipative contribution to the current. In the case of superconducting tunnel junctions and short superconducting constrictions quasiparticles can be virtually created under the gap. This process is described by the "many bubble" diagrams depicted in Fig. 6(b) (see also Ref. 34). In the region of a weak link (be it a tunnel junction or a short constriction) such quasiparticles are accelerated by the electric field and gain the energy neV where n is the diagram order (or, in other words, the number of times a quasiparticle crosses the weak link). For $n > 2\Delta/eV$ this process results in pair breaking and causes a dissipative contribution to the current. For tunnel junctions between two superconductors such a contribution is proportional to D^n where D is a (small) transparency of a tunnel barrier and $n \sim 2\Delta/eV$. Therefore at $T \ll \Delta$ and $eV \ll \Delta$ this contribution is negligible for superconducting tunnel junctions with $D \ll 1$ and remains finite for superconducting constrictions with $D \approx 1$. Let us also note that this pair breaking effect in superconducting weak links is to much extent analogous to the effect of vacuum polarization by the electric field in quantum electrodynamics. As for superconducting tunnel junctions high order vacuum polarization diagrams are small because of the small electromagnetic coupling constant. In contrast to that for a superconducting constriction with $D \approx 1$ all high order diagrams are of the same order and their summation leads to the I-V curve calculated above.

In this paper a quantum statistical analysis of ballistic charge transport has been carried through to the point where the current has been calculated for all values of the dc voltage externally applied to the SNS junction. A substantial simplification of the problem was achieved due to the factorization of the Green functions in terms of the u and v functions obeying equations which structure formally resembles that of the Bogoliubov–de Gennes equations. We calculated CVC's and differential resistance curves both analytically and numerically. In the limit of small voltages $eV \ll (2a/\ell_{\rm in})\Delta$ the current is carried by quasiparticles which are effectively localized in the pair potential well due to multiple Andreev reflections and inelastic electron-phonon scattering. These quasiparticles yield the high-low voltage conductance. At higher voltages, $eV \gg (2a/\ell_{\rm in})\Delta$, the quasiparticles gain enough energy to leave the pair potential well and relax in the superconductors. The change in the number of multiple Andreev reflections with voltage of those thermally excited quasiparticles which yield current contributions opposite to the direction of the total current flow explains the subharmonic gap structure and its temperature dependence. The differential resistance computed by us agrees qualitatively and quantitatively with the one of Octavio *et al.*[18] except for the low voltage regime. Here the current contribution of quasiparticles originating from the energy range $|E| < \Delta$ is important. Taking into account this contribution results in a slower decrease of current with voltage and thus a higher differential resistance than in the case where this contribution is omitted.[18]

As we already mentioned in the Introduction, some experiments, especially measurements at low Ohmic SNS junctions,[35] at relatively low temperatures $T < 0.5T_C$ show at voltages $V < \Delta/e$ a hysteretic behavior for increasing and decreasing currents[5,35] or negative differential conductance.[9,10] Such features do not (and cannot) appear in our calculations carried out within the voltage biased model in which case the supercurrent contribution to the I-V curve is absent. Self-heating effects in the contact area[1,5,35–37] might serve as a simplest explanation for these features in SNS junctions. Another possible explanation for a hysteretic behavior of SNS junctions can be obtained within the framework of the current-biased model taking into account the effect of capacitance renormalization (see, e.g., Ref. 34). Although the geometric capacitance of SNS junctions C is usually very small (see, e.g., Ref. 1) their effective capacitance $C_{\rm eff}$ can be large enough to provide a hysteretic I-V curve similarly to that for a simple current-biased RSJ model. Indeed analogously to the case of tunnel junctions[34] making use of the nonlocal in time current-phase relation $I[\varphi(t)]$ one can easily obtain an estimate

$$C_{\rm eff} = C + \delta C, \qquad \delta C \sim \hbar/\Delta R_0, \tag{77}$$

R_0 is the Sharvin resistance. Substituting this estimate in the expression for the McCumber parameter $\beta = 2eI_C R_0^2 C_{\rm eff}/\hbar$ and taking the supercurrent I_C to be of order $I_C \sim \Delta/eR_0$ (this is legitimate at relatively low temperatures and not very thick N layers) we get $\beta \sim 1$. This is a standard criterion for hysteretic behavior of the current-biased RSJ junctions. We can also add that the stray capacitance of external leads can cause an extra renormalization of $C_{\rm eff}$ making the tendency to the hysteretic behavior even more pronounced. A more detailed discussion of this problem goes beyond the scope of the present paper.

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APPENDIX: COEFFICIENTS OF THE KELDYSH GREEN FUNCTION IN THE NORMAL LAYER

The coefficients b_1^K and b_2^K in Eq. (59) are given by

$$\begin{aligned} b_1^K(s,v_{zF},E_1,E_2) &= \left[b_1^R(s,v_{zF},E_1,E_2)\delta_{E,E_2} - b_1^A(s,v_{zF},E_1,E_2)\delta_{E,E_1}\right] \\ &\quad \times e^{-\frac{2a}{\hbar v_{zF}}\zeta_1} h_1(s,v_{zF},z=+sa,E-seV/2,E) + N(s,v_{zF},E_1,E_2) \end{aligned} \tag{A1}$$

and

$$
\begin{aligned}
b_2^K(s,v_{zF},E_1,E_2) = & \Big[b_2^R(s,v_{zF},E_1,E_2)e^{-\frac{2a}{\hbar v_{zF}}\zeta_1}h_2(s,v_{zF},z=-sa,E_2-seV/2,E_2) \\
& - b_2^A(s,v_{zF},E_1,E_2)e^{\frac{2a}{\hbar v_{zF}}\zeta_1}h_1(s,v_{zF},z=-sa,E_1+seV/2,E_1)\Big] \\
& +\gamma^A(E_2-seV/2)e^{-i(E_2-seV/2-i\zeta_1)\frac{2a}{\hbar v_{zF}}} \\
& \times\left[b_1^R(s,v_{zF},E_1,E_2-seV)\delta_{E,E_2-seV/2} - b_1^A(s,v_{zF},E_1,E_2-seV)\delta_{E,E_1+seV/2}\right] \\
& \times\Big[e^{-\frac{2a}{\hbar v_{zF}}\zeta_1}h_1(s,v_{zF},z=+sa,E-seV,E-seV/2) \\
& - e^{+\frac{2a}{\hbar v_{zF}}\zeta_1}h_1(s,v_{zF},z=-sa,E,E-seV/2)\Big] \\
& +\gamma^A(E_2-seV/2)e^{-i(E_2-seV/2-i\zeta_1)\frac{2a}{\hbar v_{zF}}}N(s,v_{zF},E_1+seV/2,E_2-seV/2),
\end{aligned}
\tag{A2}
$$

where the coefficients $b_1^{R(A)}$ and $b_2^{R(A)}$ are given in Eqs. (55) and (56). We have used the abbreviations

$$
h_{1,2}(s,v_{zF},z,E_1,E_2) = \tanh\left(\frac{E_1}{2k_BT}\right) - f_{1,2}(s,v_{zF},z,E_2)
\tag{A3}
$$

and

$$
N(s,v_{zF},E_1,E_2) = \sum_{n=1}^{\infty} e^{in(E_1-E_2+2i\zeta_1)\frac{2a}{\hbar v_{zF}}} \times \tilde{A}_{n,s}(E_1,E_2)\Delta\tilde{f}(s,v_{zF},E_1-sneV,E_2-sneV),
\tag{A4}
$$

with

$$
\begin{aligned}
\Delta\tilde{f}(s,v_{zF},E_1,E_2) = & \left[b_1^R(s,v_{zF},E_1-seV/2,E_2-seV/2)\delta_{E,E_2}\right. \\
& \left. - b_1^A(s,v_{zF},E_1-seV/2,E_2-seV/2)\delta_{E,E_1}\right] \\
& \times\Big[e^{-\frac{2a}{\hbar v_{zF}}\zeta_1}h_1(s,v_{zF},z=+sa,E-seV,E-seV/2) \\
& - e^{+\frac{2a}{\hbar v_{zF}}\zeta_1}h_1(s,v_{zF},z=-sa,E,E-seV/2)\Big]
\end{aligned}
\tag{A5}
$$

and

$$
\tilde{A}_{n,s}(E_1,E_2) = \prod_{l=1}^{n}\gamma^R(E_1-sleV)\gamma^A(E_2-sleV).
\tag{A6}
$$

The functions f_1 and f_2 are the 11- and the 22-components of the right-hand side of Eq. (50).

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