Eigenstates for an Anderson model of an ordered-lattice–disordered-lattice junction

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(Received 21 March 1994; revised manuscript received 23 May 1994)

We analyze the eigenstates in one dimension of a semi-infinite ordered lattice in contact with a semi-infinite disordered lattice described by an Anderson model. Specifically, we study the site wave functions in the disordered region at distances N from the interface which are small compared to the localization length ξ_0. The wave functions at these length scales are relevant for conduction in the metallic regime of a disordered lattice of finite size. From a perturbation expansion for weak disorder we obtain qualitatively different results for the random (N-dependent) rates of exponential growth of wave functions in different domains corresponding to the energy band of an infinite nondisordered chain. Their mean values are anomalous near the band center and near the band edges, while corresponding to a fixed central limit (ξ_0^{-1}) between these limits, up to oscillatory terms. The study of the relative rms deviations of the above rates shows that they are self-averaging in a range of finite $N \gg 1$, at any energy. At the intermediate energies the weak disorder expansion is valid for any length scale, while near the band center and near the band edges it ceases to converge beyond a characteristic length ξ_1. The length ξ_1 defines the border between scales $N \ll \xi_1$, where the wave functions are weakly perturbed Bloch amplitudes, and scales $N \gg \xi_1$, where the weak disorder acts as a strong perturbation, leading to localized states with stationary positive Lyapunov exponents (ξ_0^{-1}). We find that ξ_1 scales with the disorder in the same way as does ξ_0, while being less than an order of magnitude smaller than ξ_0. Finally, we relate our results for wave-function growth rates at finite length scales to the resistance of a quasimetallic sample, using a simple Ansatz for the transmission coefficient. The resistance is found to be Ohmic, but anomalous, near the band center and strongly non-Ohmic near the band edges.

I. INTRODUCTION

The residual resistance of a statically disordered one-dimensional conductor has been successfully described by means of the Landauer formula:[1]

$$\rho_L = g_L^{-1} = \frac{|r_L|^2}{|t_L|^2}, \quad |r_L|^2 + |t_L|^2 = 1 , \tag{1}$$

where ρ_L (g_L) denotes the dimensionless resistance (conductance) of a sample of length L, $|r_L|^2$ is the reflection coefficient for an electron wave of energy E (Fermi energy) which is incident at one edge of the conductor, and $|t_L|^2$ is the corresponding transmission coefficient. Recalling that in an infinite disordered system all states are localized, there exist two regimes of conduction depending on the relative magnitudes of the sample length L and the localization length ξ_0: for (macroscopic) lengths $L \gg \xi_0$ the eigenstates are localized within envelopes of width $2\xi_0$ in the interior of the sample, which causes it to be insulating; for $L \ll \xi_0$ the states appear to be delocalized on the scale of the sample's size, so that one expects (quasi) metallic conduction. This residual metallic conduction has been assumed to be given by Ohm's law ($\rho_L \propto L$) in the scaling theory of localization.[2]

The localization of eigenstates in a long sample ($L \gg \xi_0$) is well understood for Anderson's one-orbital tight-binding lattice model with site-diagonal disorder. The most detailed studies of localization have been performed for a semi-infinite system[3] described by the Schrödinger equation

$$\Psi_{n+1}^0 + \Psi_{n-1}^0 + \varepsilon_n \Psi_n^0 = E\Psi_n^0, \quad n = 1,2,\ldots, \tag{2}$$

with the free end boundary condition

$$\Psi_0^0 = 0 . \tag{3}$$

Here E is the energy, and Ψ_n^0 and ε_n are the wave function and the atomic site energy at the nth site, respectively. E and ε_n are expressed in units of a constant hopping rate between nearest-neighbor sites. The atomic energies are assumed to be mutually independent random variables with a common probability distribution. Starting the iteration of (2) from value (3) and a fixed amplitude Ψ_1^0 at the edge site (which corresponds to choosing the values of the wave function and its derivative at one point in the case of a continuum) the localization length ξ is defined from the rate of exponential growth of the wave function (real part of a complex rate γ^0) at sites N situated at distances ($N-1 \sim N$ in units of the lattice constant) much larger than ξ itself:

$$\mathrm{Re}\gamma^0 = \xi^{-1} = \lim_{N\to\infty} N^{-1} \ln|\Psi_N^0| . \tag{4}$$

While the localization rate (and the corresponding localization length ξ^{-1}) for a given realization of the disorder is a random quantity, it follows from theorems of Fürstenberg[4] and Oseledec[5] that, for the large distances of interest in Eq. (4), a law of large numbers applies, which leads to a well-defined stationary (i.e., N-independent) central value for ξ^{-1}. In other words, the asymptotic exponent $\mathrm{Re}\gamma^0$ is a self-averaging quantity and the study of the localization of eigenstates reduces in

0163-1829/94/50(8)/5295(10)/$06.00

essence to the determination of its average over the disorder. Thus the inverse localization length $\xi_0^{-1}=\langle \mathrm{Re}\gamma^0\rangle$ has been studied in many papers and review articles which have appeared over the last two decades.[3,6-15] In particular, anomalous dependencies of the localization length on the disorder have been found more recently for energies in the neighborhood of the center and of the edges of the energy band for an infinite nondisordered lattice, for weak disorder.[8-13]

Besides the exponential growth rate of the amplitude, the phase of an asymptotic wave function of energy E,

$$\mathrm{Im}\gamma^0=\lim_{N\to\infty} N^{-1}\mathrm{Im}\ln\Psi_N^0 , \tag{5}$$

is also of interest since it determines the cumulative density of states,

$$N(E)=1-\pi^{-1}\mathrm{Im}\gamma^0 , \tag{6}$$

as may be seen, e.g., by generalizing the Herbert-Jones-Thouless[6,16] formula for a tight-binding chain with free end boundary conditions to slightly complex energies.[14] The dependence of the averaged density of states on energy and disorder has also been discussed in several papers, and anomalous behavior at the special energies above has been demonstrated[8-15,17] for weak disorder.

The analysis of the eigenstates of the disordered system at the finite length scales involved in a metallic sample $L\ll\xi_0$ is much more delicate. Although these states are directly relevant to the understanding of conduction in the quasimetallic regime, which has been studied recently by Pichard,[18] we are not aware of any detailed analytical study of them. Turning to the semi-infinite tight-binding system above, the wave functions of interest for metallic conduction are those at distances $N\ll\xi_0$ from the edge, whose amplitudes may be characterized by an N-dependent growth rate defined by the exponent

$$\mathrm{Re}\gamma_N^0=N^{-1}\ln|\Psi_N^0| . \tag{7}$$

Now, since at the finite length scales considered a law of large numbers (we assume $N\gg 1$) may not be valid, one may need the full probability distribution for describing the random exponent $\mathrm{Re}\gamma_N^0$, rather than just its mean value. Moreover, the mean value (as well as the higher moments) will generally be nonstationary, i.e., it varies with N and thus describes a more complicated exponential growth process.

The purpose of this paper is to present an analytical study of statistical properties of wave functions in weakly disordered one-dimensional systems described by the Anderson model at short length scales (N). Part of our motivation is the application of these results to electrical transport in the metallic regime, i.e., for samples of length $L=N\ll\xi_0$. However, rather than studying the actual probability distribution of the growth rate for finite N (which indeed seems very difficult), we restrict ourselves to the analysis of its mean value and its variance, which measures the fluctuations. The importance of fluctuations in the metallic regime is emphasized by Pichard's numerical results[18] showing that the Ohm's law scaling for the mean resistance may be masked by giant fluctuations. Pichard has derived Ohm's law for the mean metallic resistance analytically for weak disorder, for energies far from the band edges and from the band center of a pure system. In contrast to Pichard's work, our analysis is carried out for the three characteristic domains of the energy band of a nondisordered system: the vicinity of the band center, the vicinity of the band edges, and the intermediate domain between these two limits. Here we are motivated by the existence of anomalies of the localization length ξ_0 near the band center and band edges,[8-13] for weak disorder. One expects of course that corresponding anomalies (most likely of a quite different form) will occur in the wave functions at finite length scales in these energy domains.

When studying the resistance (or the conductance) of a disordered sample, one has to incorporate explicitly the nondisordered leads which connect the sample to the measuring apparatus. Thus the resistance of a one-dimensional disordered system reflects the nature of the eigenstates for a disordered linear chain connected at both ends to semi-infinite nondisordered chains, rather than those of a free disordered chain. In a first approximation (for a sufficiently long disordered system) the relevant eigenstates may be taken to be those for a semi-infinite disordered chain joined to a semi-infinite nondisordered chain at its free end. We are thus led to study eigenstates of an infinite linear chain described by the following Schrödinger equation:

$$\Psi_{n+1}+\Psi_{n-1}+\varepsilon_n\Psi_n=E\Psi_n,\quad n=0,\pm1,\pm2,\ldots;$$
$$\varepsilon_n=0\ \text{ for } n=0,-1,-2\ldots, \tag{8}$$

where the atomic energies ε_n take on nonzero values at sites $n=1,2,3,\ldots$ of the semi-infinite disordered chain, while being zero at the sites $n=0,-1,-2,\ldots$ of the adjacent infinite nondisordered chain. We choose ε_n's to be identically disturbed independent Gaussian variables with zero mean ($\langle\varepsilon_n\rangle=0$) and a correlation

$$\langle\varepsilon_m\varepsilon_n\rangle=\varepsilon_0^2\delta_{m,n} . \tag{9}$$

Contact with the original Anderson model, where the ε_n's are defined by a rectangular distribution of width W, may be established by making the substitution $\varepsilon_0^2=W/12$.

In Sec. II, after discussing general aspects of the DOL junction, we formulate our perturbation expansion of Eq. (8) for weak disorder and obtain convenient recursion relations for the corrections in the wave functions up to fourth order, at finite length scales. The fourth-order corrections in the wave functions determine the next to leading correction in the exponential growth rates defined by the real part of

$$\gamma_N=N^{-1}\ln\Psi_N . \tag{10}$$

As we now briefly discuss, the study of the next to leading correction in the averaged growth rates leads us to introduce a characteristic length scale for the energy domains in the vicinity of the band center and in the vicinity of the band edges of a nondisordered system. While for intermediate energies the successive contributions in (10) are approximately stationary for finite N

($N \gg 1$), it turns out that near both the band center and band edges they vary strongly with N. This variation is more rapid the higher the order of the considered contribution, so that there exists a length scale $N = N_0 \equiv \xi_1$ beyond which the weak disorder expansion ceases to converge. The detailed form of ξ_1 indicates that near the band center and band edges the localization of eigenstates takes place at scales $N \gg \xi_1$, where the disorder acts as a large perturbation (which can no longer be treated perturbatively). In contrast, at intermediate energies the growth rate of wave functions is defined perturbatively all the way up to the asymptotic scales $N \gg \xi_0$, where exponential localization is obtained for weak disorder.

The explicit results for the average growth rate $\langle \mathrm{Re}\gamma_N \rangle$ and for the characteristic length ξ_1, as well as for the variance $\Delta(\mathrm{Re}\gamma_N) = \langle (\mathrm{Re}\gamma_N - \langle \mathrm{Re}\gamma_N \rangle)^2 \rangle$, are presented in Sec. III, successively for energies corresponding to the vicinity of the band center, the vicinity of the band edges, and the domain intermediate between these two limits in a nondisordered system. For completeness' sake we also present in Sec. III the detailed results for $\langle \mathrm{Im}\gamma_N \rangle$ and $\Delta(\mathrm{Im}\gamma_N)$, which describe the statistics of the phase of Ψ_N. Some concluding remarks, including further analysis of ξ_1, as well as a qualitative discussion of resistance in the quasimetallic regime, are presented in Sec. IV. For studying the resistance we use a simple ansatz for the transmission coefficient of a quasimetallic sample of length $L \equiv N$ in terms of the average growth rate $\langle \mathrm{Re}\gamma_N \rangle$ of wave-function amplitudes at scales $N \ll \xi_1$ in the infinite DOL junction. This allows us to predict length-scale dependencies for the mean resistance in the various energy domains above, using the Landauer formula (1).

II. WEAK DISORDER EXPANSION

A. Generalities

Our treatment will be restricted to the study of eigenstate solutions of (8) which correspond to Bloch wave propagation in the ordered region of the system. Thus in the ordered region of the DOL junction the eigenstates of interest have the usual form,

$$\Psi_n = e^{iqn}, \quad n = 0, -1, -2, \ldots , \tag{11}$$

whose quasicontinuous spectrum is confined to the energy band

$$E = 2\cos q \ . \tag{12}$$

The eigenstate amplitudes of energy E within the band (12) for the DOL junction are then obtained by choosing

$$\Psi_1 = e^{iq} , \tag{13}$$

together with $\Psi_0 = 1$, as the starting amplitude for determining successively the amplitudes Ψ_n at the sites $n = 2, 3, \ldots$ by iterating Eqs. (8) for a given realization of the disorder. Choice (13) is imposed by the form of the solution (11)–(12) for the site $n = 0$.

In general we expect that, besides the states corresponding to propagation in the nondisordered region, the DOL system also has eigenstates which are localized in the disordered region and exponentially decrease in the ordered one, with energies outside the band (12). Such states will occur in realizations of site energy levels that correspond to sufficiently deep potential wells. Their existence is also suggested by the well-known shift of the boundaries of the energy spectrum of a disordered chain with free ends with respect to the band (12) for a pure chain.[14]

The cumulative density of states associated with the eigenstates which are propagating in the nondisordered region is clearly given by [with $dN(E) = \pi^{-1} dq$]

$$N(E) = 1 - q/\pi = \pi^{-1} \arccos(-E/2) \ . \tag{14}$$

On the other hand, the density of states in a long chain with free ends is generally defined in terms of the phase $\mathrm{Im}\gamma^0$ as a result of the free end boundary conditions [such as (3)] which determine the eigenvalue equation for a free Anderson chain. In contrast, the phases $\mathrm{Im}\gamma_N$ [Eq. (10)] for the case of a DOL junction have no simple interpretation beyond describing oscillations of the wave function about exponentially growing (or decreasing) amplitudes. As such, they provide complementary information about wave functions in the DOL junction, which has prompted us to discuss the real and imaginary parts of γ_N on the same footing in the following.

B. Expansion of wave functions

For weak disorder, it is convenient to look for solutions in the disordered region of the form

$$\Psi_n = e^{iqn + if_n} , \tag{15}$$

and to convert Eq. (8) into an exact two-point recursion relation for local wave-function exponent increments:

$$g_n = f_n - f_{n-1} , \tag{16}$$

namely

$$e^{iq}(e^{ig_{n+1}} - 1) + e^{-iq}(e^{-ig_n} - 1) = -\varepsilon_n , \quad n = 1, 2, \ldots . \tag{17}$$

For weak disorder we shall determine g_n in the form of a perturbation expansion:

$$g_n = \sum_{p=1,2,3,\ldots} g_n^{(p)}, \quad g_n^{(p)} = f_n^{(p)} - f_{n-1}^{(p)} , \tag{18}$$

where $g_n^{(1)}(f_n^{(1)})$ is a linear functional of the site energies ε_n, $g_n^{(2)}(f_n^{(2)})$ is a quadratic functional of the ε_n, etc. By inserting (18) into (17) and equating the two sides of the equation order by order in the site energies, we obtain the following exact recursion relations for successive order contributions: $g_n^{(p)}, n = 1, 2, \ldots, N$, up to fourth order:

$$g_{n+1}^{(1)} - a g_n^{(1)} = i\varepsilon_n b, \quad a = e^{-2iq}, \quad b = e^{-iq} , \tag{19a}$$

$$g_{n+1}^{(2)} - a g_n^{(2)} = \Gamma_n , \tag{19b}$$

$$g_{n+1}^{(3)} - a g_n^{(3)} = \Delta_n , \tag{19c}$$

$$g_{n+1}^{(4)} - a g_n^{(4)} = \Lambda_n , \tag{19d}$$

where

$$\Gamma_n=\frac{i\varepsilon_n^2 a}{2}+\varepsilon_n b^3 g_n^{(1)}-\frac{ia}{2}(1+a)g_n^{(1)2}\ , \tag{20}$$

$$\Delta_n=\frac{i\varepsilon_n^3 b^3}{3}+\varepsilon_n^2 a^2 g_n^{(1)}-i\varepsilon_n b^3(a+\tfrac{1}{2})g_n^{(1)2}+\varepsilon_n b^3 g_n^{(2)}-ia(a+1)g_n^{(1)}g_n^{(2)}-\frac{\alpha}{6}(1+3a+2a^2)g_n^{(1)3}\ , \tag{21}$$

$$\Lambda_n=\frac{ia^2\varepsilon_n^4}{4}+ba^2\varepsilon_n^3 g_n^{(1)}-\frac{ia^2\varepsilon_n^2}{2}(3a+1)g_n^{(1)2}+a^2\varepsilon_n^2 g_n^{(2)}-iba(2a+1)\varepsilon_n g_n^{(1)}g_n^{(2)}-\frac{ba\varepsilon_n}{6}(1+6a+6a^2)g_n^{(1)3}$$
$$+ba\varepsilon_n g_n^{(3)}+\frac{ia^2}{2}\left[\tfrac{7}{12}+a+\frac{a^2}{2}+\frac{1}{12a}\right]g_n^{(1)4}-\frac{a}{2}(2a^2+3a+1)g_n^{(1)2}g_n^{(2)}-ia(a+1)g_n^{(1)}g_n^{(3)}-\frac{ia}{2}(a+1)g_n^{(2)2}\ . \tag{22}$$

The right-hand sides [(20)–(22)] of Eqs. (19b)–(19d) are obtained after replacing the quantities $g_{n+11}^{(q)}q\le p-1$ occurring in the equation for the pth-order perturbation, in terms of $g_n^{(q)}$, using the equation for the qth-order perturbation. Some further algebra is involved in grouping various terms in the form displayed, e.g., in (21) and (22).

We are interested in the solution of Eqs. (19a)–(19d) for energies within the energy band (12). Since these solutions are not asymptotically stationary in general, we shall study the complex site-dependent exponent (10) describing the form of the wave function $[\Psi_N\sim\exp(\gamma_N N)]N$ steps away from the edge ($n=1$), where the iteration of (19a)–(19d) is started for a given realization of site energies. From (10) and (15), we obtain

$$\gamma_N=iq+\frac{\mathrm{if}_N}{N}=iq+\frac{i}{N}\sum_{n=1}^{N}g_n\ , \tag{23}$$

where we expressed f_N in terms of the increments g_n by iterating (16), starting from the values

$$f_1=g_1=0\ , \tag{24}$$

which follow from the expressions (11) (for $n=0$) and (13).

On the other hand, by iterating Eqs. (19a)–(19d), starting from value (24), we obtain the exact solutions

$$g_n^{(1)}=ib\sum_{m=1}^{n-1}\varepsilon_m a^{n-m-1}\ , \tag{24a}$$

$$g_n^{(2)}=\sum_{m=1}^{n-1}\Gamma_m a^{n-m-1}\ , \tag{24b}$$

$$g_n^{(3)}=\sum_{m=1}^{n-1}\Delta_m a^{n-m-1}\ , \tag{24c}$$

$$g_n^{(4)}=\sum_{m=1}^{n-1}\Lambda_m a^{n-m-1},\quad n=2,3,4,\ldots\ , \tag{24d}$$

As discussed in Sec. I, we expect the complex exponential rate γ_N at finite distances to be a strongly fluctuating quantity. Therefore we can only study the behavior of some typical values of $\mathrm{Re}\gamma_N$ and $\mathrm{Im}\gamma_N$ (unless it would be possible to find the full probability distribution), such as the mean, the root-mean-squared values, etc. In Sec. III we analyze in detail the mean values $\langle\mathrm{Re}\gamma_N\rangle$ and $\langle\mathrm{Im}\gamma_N\rangle$ [given by the mean of (23)] and the variances

$$\Delta(\mathrm{Re}\gamma_N)=\langle(\mathrm{Re}\gamma_N-\langle\mathrm{Re}\gamma_N\rangle)^2\rangle$$
$$=N^{-2}\sum_{n,m=1}^{N}(\langle\mathrm{Im}g_n\mathrm{Im}g_m\rangle-\langle\mathrm{Im}g_n\rangle\langle\mathrm{Im}g_m\rangle)\ , \tag{25a}$$

$$\Delta(\mathrm{Im}\gamma_N)=\langle(\mathrm{Im}\gamma_N-\langle\mathrm{Im}\gamma_N\rangle)^2\rangle$$
$$=N^{-2}\sum_{n,m=1}^{N}(\langle\mathrm{Re}g_n\mathrm{Re}g_m\rangle-\langle\mathrm{Re}g_n\rangle\langle\mathrm{Re}g_m\rangle) \tag{25b}$$

which measure the fluctuations about the mean values.

III. DETAILED RESULTS

The pth-order contribution in the perturbation series (18) is a linear functional of products of p-site energies. Since, for the Gaussian randomness (9), averages of products of odd numbers of site energies vanish, we require the g_n to fourth order [Eqs. (24a)–(24d)] for obtaining the first two leading contributions to γ_N, i.e.,

$$\langle\gamma_N\rangle\equiv iq+\langle\gamma_N^{(2)}\rangle+\langle\gamma_N^{(4)}\rangle+\cdots$$
$$=iq+\frac{i}{N}\sum_{n=1}^{N}(\langle g_n^{(2)}\rangle+\langle g_n^{(4)}\rangle)+O(\varepsilon_0^6)\ . \tag{26}$$

As discussed at the end of Sec. I, for energies close to the band center and the band edges, we define a characteristic length ξ_1 as the length $N=N_0\equiv\xi_1$ at which the perturbation series (26) for $\langle\mathrm{Re}\gamma_N\rangle$ ceases to converge, i.e.,

$$|\langle\mathrm{Re}\gamma_{\xi_1}^{(4)}\rangle|=|\langle\mathrm{Re}\gamma_{\xi_1}^{(2)}\rangle|\ . \tag{27}$$

Averages of products of four Gaussian site energies in $\langle g_n^{(4)}\rangle$ factorize in the form

$$\langle\varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_L\rangle$$
$$=\langle\varepsilon_i\varepsilon_j\rangle\langle\varepsilon_k\varepsilon_L\rangle+\langle\varepsilon_i\varepsilon_k\rangle\langle\varepsilon_j\varepsilon_L\rangle+\langle\varepsilon_i\varepsilon_L\rangle\langle\varepsilon_j\varepsilon_k\rangle$$
$$=\varepsilon_0^4(\delta_{ij}\delta_{kL}+\delta_{ik}\delta_{jL}+\delta_{iL}\delta_{jk})\ , \tag{28a}$$

where we have used Eq. (9). Furthermore, in averaging Γ_m and Λ_m in (24b) and (24d), using (20) and (22), it is useful to note that since $g_n^{(p)}$ depends on ε_m for $1\le m\le n-1$ but not on ε_m for $m\ge n$, one has

$$\langle\varepsilon_n g_n^{(p)}\rangle=\langle\varepsilon_n g_n^{(p)}g_n^{(q)}\rangle=0\ . \tag{28b}$$

The averaging of $g_n^{(2)}$, using (9), (20), (24a), and (24b)

yields the simple closed form expression $\langle g_n^{(2)} \rangle = i2^{-1}\varepsilon_0^2 a(a-1)^{-2}(a^{n-1}-1)^2$, whose substitution in (26) leads to the exact result

$$\langle \gamma_N^{(2)} \rangle = -\frac{\varepsilon_0^2}{2}\frac{a}{(a-1)^2} \times \left[1+\frac{1}{N}\left[\frac{1-a^{2N}}{1-a^2}-\frac{2(1-a^N)}{1-a}\right]\right] . \quad (29)$$

In a similar way, the evaluation of (25a) and (25b) to order ε_0^2, using (9) and (24a), yields exactly

$$\Delta(\mathrm{Re}\gamma_N)=\frac{\varepsilon_0^2}{4N^2}[(A_N+B_N)+\mathrm{c.c.}] , \quad (30a)$$

$$\Delta(\mathrm{Im}\gamma_N)=\frac{\varepsilon_0^2}{4N^2}[(A_N-B_N)+\mathrm{c.c.}] , \quad (30b)$$

where

$$A_N=\sum_{m=1}^{N} a^m\left[\sum_{n=1}^{N}(n-1)a^{-n}+\sum_{n=1}^{m}(n-m)a^{-n}\right] , \quad (31)$$

$$B_N=\frac{1}{a^2-1}\sum_{m=1}^{N} a^m\left[\sum_{n=1}^{N} a^{n-1}(1-a^{-2(n-1)}) + \sum_{n=1}^{m} a^{n+1}(a^{-2m}-a^{-2n})\right] , \quad (32)$$

For our purposes it is not particularly useful to perform the remaining summations in (31) and (32) at this stage.

On the other hand, the averaging of (22) after inserting (24a)–(24c) is straightforward using (9) and (28a) and (28b). For the clarity of later discussions of $\langle g_n^{(4)} \rangle$, it is convenient to present the explicit expression of $\langle \Lambda_m \rangle$ in terms of simple partial geometric sums labeled $S_k^{(p)}, p=1,2,3$, in the following. After a fair amount of algebra, we obtain

$$\begin{aligned}\frac{\langle \Lambda_m \rangle}{i\varepsilon_0^4} &= \frac{3a^2}{4}+\frac{1}{2a}(3a+1)a^{2m}S_m^{(2)}+\frac{a^{m+1}}{2}\left[S_m^{(1)}+\frac{1+a}{a^2}\sum_{k=1}^{m-1}a^k S_k^{(2)}\right]+\frac{3}{2}\left[\tfrac{7}{12}+a+\frac{a^2}{2}+\frac{1}{12a}\right]a^{4(m-1)}S_m^{(2)2} \\ &+\tfrac{1}{4}(2a^2+3a+1)a^{3(m-1)}\left[S_m^{(1)}S_m^{(2)}+2S_m^{(3)}+4a\sum_{k=1}^{m-1}a^{-k}S_k^{(2)}+a^{-2}(a+1)\sum_{k=1}^{m-1}a^k S_k^{(2)}(S_m^{(2)}+2S_k^{(2)})\right] \\ &+\tfrac{1}{2}(a+1)a^{2m-1}\left\{\tfrac{5}{2}S_m^{(2)}+(5+a^{-1})\sum_{k=1}^{m-1}S_k^{(2)}+a\sum_{k=1}^{m-1}a^{-k}S_k^{(1)}+a^{-1}(1+a)\sum_{j=1}^{m-1}a^{-j}\sum_{k=1}^{j-1}a^k S_k^{(2)}\right. \\ &\quad +a^{-2}(1+a)\sum_{j=1}^{m-1}a^j\left[S_j^{(1)}S_j^{(2)}+2S_j^{(3)}+4a\sum_{k=1}^{j-1}a^{-k}S_k^{(2)}\right. \\ &\quad\quad \left.\left.+a^{-2}(a+1)\sum_{k=1}^{j-1}a^k S_k^{(2)}(S_j^{(2)}+2S_k^{(2)})\right]\right. \\ &+a^{-4}(2+5a+3a^2)\sum_{k=1}^{m-1}a^{2k}S_k^{(2)2} \\ &+\tfrac{1}{2}S_m^{(1)2}+\frac{(1+a)^2}{4a^4}\left[\left[\sum_{k=1}^{m-1}a^k S_k^{(2)}\right]^2+4\sum_{k=1}^{m-1}\sum_{l=1}^{k-1}a^{k+l}S_l^{(2)2}\right] \\ &\left.+(1+a)\sum_{k=1}^{m-1}a^k\left[\frac{1}{2a^2}S_m^{(1)}S_k^{(2)}+\frac{1}{a^2}S_k^{(3)}+\frac{2}{a}\sum_{j=1}^{k-1}a^{-j}S_j^{(2)}\right]\right\} , \end{aligned} \quad (33)$$

where

$$S_k^{(p)}=\frac{1-a^{-p(k-1)}}{1-a^{-p}}, \quad p=1,2,3 . \quad (34)$$

Although $\langle \Lambda_m \rangle$ could be given in a fully closed form by performing the various site summations in (33), which would further enable one to obtain the general forms of $\langle g_n^{(4)} \rangle$ and $\langle \gamma_N \rangle$, the resulting expressions are far too complicated for transparent analysis in various domains of the energy band (12). In contrast, (33) is well suited for this purpose. Equations (29)–(33) considerably simplify for energies in the vicinity of the band center and in the vicinity of the edges of the band (12), respectively. In addition, at these energies the dependencies of $\langle \mathrm{Re}\gamma_N \rangle$ and $\langle \mathrm{Im}\gamma_N \rangle$ on the disorder parameter and/or on N are quite anomalous, as compared to the results for the intermediate domain of (12), i.e., energies sufficiently far from both the band center and the band edges. For these reasons we present separate discussions below for the neighborhood of the band center, the neighborhood of the band edges, and, finally, for intermediate energies between these two limits.

Let us emphasize that the anomalies of the average exponential growth rates near the band center and band edges are quite distinct from the so-called Kappus-Wegner anomalies in the inverse localization length ξ_0^{-1} at these energies. The Kappus-Wegner anomalies, which

have been extensively studied,[8–13] arise from the breakdown of a perturbative expansion for weak disorder, within the stationarity domain ($N\to\infty$). In the present treatment of exponential rates at finite length scales N, the perturbation expansion is well behaved for any energy, yet the forms of $\langle \mathrm{Re}\gamma_N\rangle$ and $\langle \mathrm{Im}\gamma_N\rangle$ and the variances $\Delta(\mathrm{Re}\gamma_N)$ and $\Delta(\mathrm{Im}\gamma_N)$ are found to differ drastically for the three energy domains above. Note, however, that in addition to the smallness of the correlation parameter ε_0^2, the validity of our weak-disorder expansion is restricted to short length scales, $N\ll\xi_1$, near the band center and band edges.

A. The vicinity of the band center

Near the center of the energy band (12), we expand the parameter a in (19a) as

$$a=-1+z\ , \tag{35a}$$

where, to second order in E,

$$z=x+iy,\quad y=-E,\quad x=\frac{E^2}{2}=\frac{y^2}{2}\ . \tag{35b}$$

By inserting (35a) and (35b) into (29) and expanding to order E^2, we obtain for $N\gg 1$,

$$\langle \mathrm{Re}\gamma_N^{(2)}\rangle=\frac{\varepsilon_0^2}{4}\left\{1-E^2\left[\frac{N^2}{3}-\frac{3N}{4}-\tfrac{1}{12}+\tfrac{1}{4}(N-\tfrac{1}{2})(-1)^N\right]+O\left[\frac{1}{N}\right]\right\}, \tag{36a}$$

and

$$\langle \mathrm{Im}\gamma_N^{(2)}\rangle=-\frac{\varepsilon_0^2 E}{8}\left[1-\tfrac{3}{2}(-1)^N\right]\ . \tag{36b}$$

The precise sense in which (36a) is viewed as being anomalous (Sec. I) is discussed in Sec. III C.

Next we consider the fourth-order contribution in (26) near the band center, for which we shall restrict ourselves to the lowest (linear) order in $E\to 0$. It is obtained by expanding (24d), and (33) and (34) to linear order in E, using (35a) and (35b) and performing various summations over lattice sites. Note that for $E=0$ only the first three terms in (33) are nonzero, while to order E several terms in this expression can be ignored since, by inspection, they are seen to be of order E^2. On the other hand, we recall that in all explicit calculations of this section the site summations reduce to either simple geometric progressions or sums of integer powers of consecutive integers, which are found in standard mathematical tables.

We obtain successively

$$\langle \Lambda_m\rangle=i\varepsilon_0^4\{m-\tfrac{1}{2}-\tfrac{1}{4}(-1)^m+\frac{iE}{8}[17m^2-10m+1-(3m+1)(-1)^m]\}\ , \tag{37}$$

$$\langle g_n^{(4)}\rangle=-\frac{i\varepsilon_0^4}{2}\{(n-1)[1+\tfrac{1}{2}(-1)^n]+\frac{iE}{8}[17n^2-31n+12+n(5n-7)(-1)^n]\}\ , \tag{38}$$

and, from (26),

$$\langle \gamma_N^{(4)}\rangle=-\frac{\varepsilon_0^4}{4}\left\{N-1-\tfrac{1}{2}(-1)^N+\frac{iE}{4}\left[\tfrac{17}{3}N^2-7N-\tfrac{2}{3}-\left[\frac{5N}{2}-1\right](-1)^N\right]+O\left[\frac{1}{N}\right]+O(E^2)\right\}. \tag{39}$$

Equations (39) and (36a) enable us to determine the characteristic length ξ_1, defined by (27). At the band center $E=0$, we obtain

$$\xi_1\simeq\frac{1}{\varepsilon_0^2}+1+O(\varepsilon_0^2)\ . \tag{40}$$

Finally, we consider the fluctuations of the real and imaginary parts of the complex rate γ_N near the band center. For simplicity we restrict ourselves to the lowest order in ε_0^2. Rather than starting from Eqs. (30)–(32) for the variances at an arbitrary energy (12), it is simpler to expand $g_n^{(1)}$ [Eq. (24a)] itself for $E\to 0$, and to substitute the result in (25a) and (25b). Using Eq. (9) we thus obtain, to order E^2,

$$\Delta(\mathrm{Re}\gamma_N)=\frac{E^2\varepsilon_0^2}{N^2}\sum_{m=1}^{N}(-1)^m\left[\sum_{n=1}^{m}(-1)^n U_{m,n}+\sum_{n=m+1}^{N}(-1)^n U_{n,m}\right], \tag{41a}$$

and

$$\Delta(\mathrm{Im}\gamma_N)=\frac{\varepsilon_0^2}{N^2}\sum_{m=1}^{N}(-1)^m\left[\sum_{n=1}^{m}(-1)^n V_{m,n}+\sum_{n=m+1}^{N}(-1)^n V_{n,m}\right] \tag{41b}$$

where

$$U_{m,n}=\sum_{k=1}^{n-1}(m-k-\tfrac{1}{2})(n-k-\tfrac{1}{2})\ , \tag{42}$$

and

$$V_{m,n}=\sum_{k=1}^{n-1}\left[1-\frac{E^2}{4}+(m-k-4)(m-k-1)\frac{E^2}{2}+(n-k-4)(n-k-1)\frac{E^2}{2}\right]. \tag{43}$$

Performing the site summations in (41a) and (41b), we obtain, after somewhat tedious algebra,

$$\Delta(\mathrm{Re}\gamma_N)=\frac{E^2\varepsilon_0^2}{12}\left[N-\tfrac{3}{2}+O\left[\frac{1}{N}\right]\right], \tag{44a}$$

$$\Delta(\mathrm{Im}\gamma_N)=\varepsilon_0^2\left\{O\left[\frac{1}{N}\right]+\frac{E^2}{12}\left[N[1+(-1)^N]-\tfrac{3}{2}[5-(-1)^N]+O\left[\frac{1}{N}\right]\right]\right\}. \tag{44b}$$

Since $\Delta(\mathrm{Re}\gamma_N)$ vanishes to order ε_0^2 at the band center, we conclude that at this energy the ensemble of random growth rates $\mathrm{Re}\gamma_N^{(2)}$ is well described by the mean value (36a) for finite N, at least if the third and higher moments are small. Away from the band center $\Delta(\mathrm{Re}\gamma_N)$ varies less rapidly with N than $\langle \mathrm{Re}\gamma_N^{(2)}\rangle$ [Eq. (36a)], and so $\mathrm{Re}\gamma_N^{(2)}$ may be regarded as self-averaging for large N, within the domain of validity of the weak-disorder expansion.

B. The vicinity of the band edges

In terms of the deviation from the value at the band edges we have

$$a=1+z\ , \tag{45a}$$

where

$$z=iy-\frac{y^2}{2}+O(y^3),\quad y=-2\arccos\frac{|E|}{2}\ . \tag{45b}$$

By substituting this expression into (29), we obtain, to order y^2,

$$\langle \mathrm{Re}\gamma_N^{(2)}\rangle=-\frac{\varepsilon_0^2}{2}\left[\frac{N^2}{3}-\frac{N}{2}+\frac{1}{6}-\frac{y^2}{6}\left[\frac{7N^4}{10}+\frac{13N^3}{4}-4N^2+\frac{5N}{4}+\frac{1}{5}\right]+O(y^4)\right], \tag{46a}$$

$$\langle \mathrm{Im}\gamma_N^{(2)}\rangle=-\frac{\varepsilon_0^2}{8}N(N-1)^2y+O(y^3)\ . \tag{46b}$$

The remarkable (or surprising) features of the average exponential rate (46a) are its negative sign for large N and its strong variation with N. This implies, in particular, that at short distances the wave functions in the disordered region of the DOL junction are weakly localized near the interface site (rather than in the interior of this region, as in the asymptotic, strong-localization limit $N\to\infty$).

Next we consider the fourth-order contribution $\langle \gamma_N^{(4)}\rangle$ to the average rate. For simplicity, we shall give the final expression at the band edges, as the latter is typical of the behavior at neighboring energies as well. At the band edges Eqs. (45a) and (45b) yield $S_k^{(p)}=k-1$, and all other powers of a in (33) reduce to unity. After summing the remaining series in (33) as indicated above, and collecting terms, we obtain successively from (33), (24d), and (26)

$$\langle \Lambda_m\rangle=i\varepsilon_0^4\left(\tfrac{7}{6}m^4+m^3-\tfrac{23}{3}m^2+\tfrac{23}{2}m-\tfrac{21}{4}\right), \tag{47}$$

$$\langle g_n^{(4)}\rangle=i\varepsilon_0^4\sum_{m=1}^{n-1}\langle \Lambda_m\rangle=i\varepsilon_0^4\left[\frac{7n^4}{30}-\frac{n^4}{3}-\tfrac{8}{3}n^3+\tfrac{59}{6}n^2-\tfrac{109}{15}n-\frac{21}{4}\right], \tag{48}$$

and

$$\langle \gamma_N^{(4)}\rangle=-\frac{\varepsilon_0^4}{3}\left(\tfrac{7}{60}N^5+\tfrac{3}{20}N^4-\tfrac{53}{24}N^3+\tfrac{33}{6}N^2+\tfrac{71}{120}N-\tfrac{321}{5}\right). \tag{49}$$

We note that, like $\langle \gamma_N^{(2)}\rangle$, $\langle \gamma_N^{(4)}\rangle$ is real at the band edges. Again we may use (49) and (46a) to obtain the characteristic length ξ_1, at the band edges, using definition (27). This yields

$$\xi_1=\left[\frac{30}{7\varepsilon_0^2}\right]^{1/3}-\tfrac{13}{4}+O(\varepsilon_0^{2/3})\ , \tag{50}$$

whose nontrivial $\frac{1}{3}$-power dependence on ε_0^2 is the same as that obtained for ξ_0 at the band edges by Derrida and Gardner, from a completely different approach.[10]

Turning to the analysis of the mean-squared fluctuations of γ_N, it is again convenient to expand (24a) to order y^2 from the beginning (which requires going to order z^2 in the z expansion). We thus obtain, for energies E close to the band edges,

$$g_n^{(1)}=\pm i\sum_{k=1}^{n-1}\varepsilon_k\left[1+\frac{iy}{2}-\frac{y^2}{8}+(n-k-1)(iy-y^2)-(n-k-1)(n-k-2)\frac{y^2}{2}\right], \tag{51}$$

whose substitution in (25a) and (25b) yields the variances to order ε_0^2 in the form

$$\Delta(\mathrm{Re}\gamma_N)=\frac{\varepsilon_0^2}{N^2}\sum_{m=1}^{N}\left[\sum_{n=1}^{m}S_{m,n}+\sum_{n=m+1}^{N}S_{n,m}\right], \tag{52a}$$

$$\Delta(\mathrm{Im}\gamma_N)=\frac{y^2\varepsilon_0^2}{N^2}\sum_{m=1}^{N}\left[\sum_{n=1}^{m}U_{m,n}+\sum_{n=m+1}^{N}U_{n,m}\right], \tag{52b}$$

where

$$S_{m,n}=\sum_{k=1}^{n-1}\left[1-\frac{y^2}{4}-(m-k)(m-k-1)\frac{y^2}{2}-(n-k)(n-k-1)\frac{y^2}{2}\right]. \tag{53}$$

The explicit evaluation of (52a) and (52b), using (42) and (53), is straightforward, though somewhat tedious. We obtain

$$\Delta(\mathrm{Re}\gamma_N)=\frac{\varepsilon_0^2}{3}\left[N-\tfrac{3}{2}+\frac{1}{4}\left[\frac{N^3}{5}+N^2-2N+\frac{1}{2}\right]y^2+O\left[\frac{1}{N}\right]\right], \tag{54a}$$

$$\Delta(\mathrm{Im}\gamma_N)=\frac{\varepsilon_0^2}{24}\left[\tfrac{11}{5}N^3-N^2-N+1+O\left[\frac{1}{N}\right]\right]y^2. \tag{54b}$$

Thus we find that $\Delta(\mathrm{Re}\gamma_N)$ at the band edges ($y=0$) grows less rapidly with N than the mean value $\langle \mathrm{Re}\gamma_N^{(2)}\rangle$ [Eq. (46a)], while $\Delta(\mathrm{Im}\gamma_N)$ grows with the same (cube) power of N as $\langle \mathrm{Im}\gamma_N^{(2)}\rangle$ [Eq. (46b)]. In this sense both $\mathrm{Re}\gamma_N^{(2)}$ and $\mathrm{Im}\gamma_N^{(2)}$ are self-averaging quantities at the band edges.

C. Intermediate energies

For energies sufficiently far from both the band edges and the band center (i.e., energies for which $|a^2-1|$ is not small) Eq. (29) reduces, for $N\gg 1$, to the real expression

$$\langle\gamma_N^{(2)}\rangle=-\frac{\varepsilon_0^2}{2}\frac{a}{(a-1)^2}+O\left[\frac{1}{N}\right]=\frac{\varepsilon_0^2}{2}\frac{1}{4-E^2}. \tag{55}$$

This is the familiar Thouless result[7] for the inverse localization length for weak disorder. It shows, somewhat surprisingly, that at intermediate energies the growth rate of the wave function in the metallic regime, $1\ll N\ll\xi_0$, has already reached its asymptotic value at large distances corresponding to the insulating regime (i.e., for $N\gg\xi_0$).

Similarly, we obtain the asymptotic forms of variances (30a) and (30b) at these energies by retaining only the leading terms proportional to N in (31) and (32). The results are $A_N=-Na(1-a)^{-2}[a^N(1-a)^{-1}-a^{-N}]$ and $B_N=Na(1-a)^{-2}$, from which we obtain

$$\Delta(\mathrm{Re}\gamma_N)=\frac{\varepsilon_0^2}{4N}\frac{a}{(1-a)^3}[2(1-a)+a^{-N}-a^{N+1}]+O\left[\frac{\varepsilon_0^2}{N^2}\right], \tag{56a}$$

$$\Delta(\mathrm{Im}\gamma_N)=-\frac{\varepsilon_0^2}{4N}\frac{a}{(1-a)^3}[2(1-a)-a^{-N}+a^{N+1}]+O\left[\frac{\varepsilon_0^2}{N^2}\right]. \tag{56b}$$

The stationarity of the mean value of $\gamma_N^{(2)}$ [Eq. (55)] and the $1/N$ decay of the variances imply that the distributions of the real and imaginary parts of $g_n^{(2)}$ in Eq. (23) are asymptotically stationary (i.e., independent of n), with mean values given by (55). This follows from the familiar discussion of the central limit theorem.[19] This shows that the mean value (55) is actually a central limit for the exponential growth rate of wave functions at intermediate energies, for finite distances $N\gg 1$. It is then quite natural to refer to strong departures from the simple behavior described by (55), (56a), and (56b), which are found near the center and near the edges of the energy band, as anomalies. As shown by the sets of equations (36a) and (36b) and (44a) and (44b), on the one hand, and (46a) and (46b) and (54a) and (54b), on the other, the anomalies consist basically of nonstationary variations of the averages and of the variances, particularly in the vicinity of the band edges. Some more specific anomalies include, e.g. the factor of two differences between Eq. (36a) for $E=0$ and Eq. (55) when extrapolated to $E=0$, as well as the negative sign of $\langle \mathrm{Re}\gamma_N^{(2)}\rangle$ near the band edges noted above.

We now analyze the next higher correction to the mean value of γ_N. The dominant contribution to $\langle g_n^{(4)}\rangle$ for large n at intermediate energies comes from terms proportional to n in (24d). There are two types of terms in (33) which lead to contributions proportional to n in (24d): the first type are terms proportional to $m-1$ in (33), which arise from various summations in which the summands are independent of the summation index (an example of such terms are the terms proportional to $\sum_{k=1}^{m-1}S_k^{(2)}$); the second type are terms proportional to a^m with m-independent coefficients [e.g., the term proportional to $a^m S_m^{(1)}$ in (33)]. By including these two types of dominant contributions, we obtain, after some reductions,

$$\langle\Lambda_m\rangle=\frac{i\varepsilon_0^4 a^m}{(1-a^{-1})^2}\left\{(m-1)a^m\left[-\tfrac{3}{2}(1+3a^{-1})+\frac{1}{1-a^{-2}}(a^{-3}+5a^{-2}+5a^{-1}+1)\right]+\frac{1}{4}\left[-a-5+\frac{a^{-1}}{1-a^{-2}}(a^2+5a+5+a^{-1})\right]\right\}, \tag{57}$$

where the first square bracket gives the contribution of the terms of the first type, and the second square bracket represents the contribution of the terms of the second type. Next we combine (57), (24d), and (26) to obtain the final result:

$$\langle \gamma_N^{(4)} \rangle = -\frac{\varepsilon_0^4 a^{N+2}}{(a-1)^3}\left\{\frac{a^{N+1}}{a^2-1}\left[-\tfrac{3}{2}(1+3a^{-1})+\frac{1}{1-a^{-2}}(a^{-3}+5a^{-2}+5a^{-1}+1)\right]\right.$$
$$\left.+\frac{1}{4}\left[-a-5+\frac{a^{-1}}{1-a^{-2}}(a^2+5a+5+a^{-1})\right]+O\left[\frac{1}{N}\right]\right\}, \tag{58}$$

which shows that to this order $\langle \gamma_N \rangle$ is stationary (i.e., independent of N) up to oscillatory terms. This supports the approximate validity of the central limit property of the exponential rate γ_N beyond lowest order in our weak-disorder expansion. The stationarity of (55) and (58) for finite $N \gg 1$ shows that this expansion is stable for large N, in particular for $N \gg \xi_0$. The localization of eigenstates at intermediate energies is thus a truly perturbative effect for weak disorder: the disorder which causes the localization acts as a small perturbation at any length scale.

IV. CONCLUDING REMARKS

A more precise interpretation of the length scale ξ_1, defined by (27), may be obtained by comparing it with the localization lengths derived by Derrida and Gardner[10] for energies near the center and edges of the energy band (12). Such a comparison is justified in the present case, since the localization length in a long disordered chain is independent of the boundary conditions at the edges [such as Eq. (24) for the DOL junction, or Eq. (3) for an open chain]. This is demonstrated, e.g. by its definition in terms of the trace of a product of a large number of transfer matrices.[3,10,14] From the Lyapunov exponents given in Ref. 10, we obtain

$$\xi_0 \simeq \frac{8.77}{\varepsilon_0^2} \quad \text{for } E=0 , \tag{59}$$

$$\xi_0 \simeq \frac{3.46}{\varepsilon_0^{2/3}} \quad \text{for } E=\pm 2 . \tag{60}$$

By comparing, respectively, Eqs. (40) and (59) and (50) and (60), one sees that in both cases ξ_1 scales with the disorder parameter (ε_0^2) in the same way as does the localization length. It follows that, at the scale ξ_1, the (identical) successive order contributions in the average growth rates of the wave functions also scale as ξ_0 with disorder. This indicates that ξ_1 can be viewed as a characteristic length scale at which strong (nonperturbative) localization effects set in for weak disorder near the band center and band edges. The length ξ_1 which separates the small perturbation (weak localization) and large perturbation (strong localization) domains is between two and nine times smaller than ξ_0.

In the strong-localization region, $N \gg \xi_1$, the effects of the disorder are nonperturbative near the band center and band edges. While this is obvious in the present work, it is also demonstrated by the studies of the localization length for weak disorder.[8,10] Indeed, at these energies the inverse localization length cannot be expressed as a power series expansion in the strength of the disorder, for weak disorder.[8,10] This may be the origin of the qualitative differences between the form of $\langle \mathrm{Re}\gamma_N \rangle$ at short and long scales (inverse localization length) and, in particular, their sign difference near the band edges [see (46a)].

We now wish to relate the average exponential growth rates of the wave functions in Sec. III to the resistance of a metallic sample of length $L \equiv N$, using the Landauer formula (1). In the metallic regime the transmission coefficient is close to unity, so that

$$\rho_N \simeq |r_N|^2 = 1-|t_N|^2 . \tag{61}$$

Here we argue that the averaged transmission coefficient for a disordered sample of length n may be represented by the physically plausible expression

$$\langle |t_N|^2 \rangle \simeq \exp(-2|\langle \mathrm{Re}\gamma_N \rangle| N) , \tag{62}$$

which interpolates between the exact expression for the transmission coefficient of a long sample[3] in terms of the localization length ξ_0,

$$|t_{N\to\infty}|^2 = \exp(-2N/\xi_0) , \tag{63}$$

$$\frac{1}{\xi_0} = \lim_{N\to\infty}\frac{1}{N}\langle \ln|\Psi_N^0| \rangle = \lim_{N\to\infty}\frac{1}{N}\langle \ln|\Psi_N| \rangle \tag{64}$$

[where the last equality follows from the independence of the localization length of the boundary conditions at the edges of the sample where the iterations of Eqs. (2) and (8) are started], and the limiting value $|t_{N\to 0}|^2 = 1$ for $N \to 0$. Equation (62) assumes that the rate of exponential decay of the transmission coefficient with sample size N is determined by the exponential growth rate of wave functions at scale N in a DOL junction. The approximate validity of (62) for short samples is further supported by the fact that at intermediate energies it leads to the same expression for the mean resistance [Eq. (69) below] as the one derived by Pichard[18] from a first-principles treatment for weak disorder.[20] Here our aim is merely to analyze the mean resistance in terms of the mean growth rates, $|\langle \mathrm{Re}\gamma_N \rangle|$, of the wave functions and to discuss the resistance in the domains of anomalous behavior of these wave functions (i.e., near the band center and band edges). We note that it has not been possible to discuss the resistance in the latter energy domains by means of a first-principles treatment for the transmission coefficient of tight-binding systems.[3,18] This is due to the formal divergencies of transfer matrices in Ishii's formalism at the band edges,[3] as well as at higher orders in a weak-disorder expansion at the band center.

For lengths $N \ll \xi_1$, where the perturbation results for $\langle \mathrm{Re}\gamma_N \rangle$ discussed in Sec. III are valid in the domains $E \to 0$ and $E \to \pm 2$, we have

$$2|\langle \mathrm{Re}\gamma_N \rangle| N \ll 1 , \tag{65}$$

and since $\xi_1 < \xi_0$ these length scales are comprised within the metallic domain. On the other hand, for intermediate energies, (65) is nothing but the definition of the metallic regime since the growth rate for finite $N(\gg 1)$ [Eq. (55)] coincides with the inverse localization length. We then obtain, from (61) and (62),

$$\langle \rho_N \rangle \simeq 2|\langle \mathrm{Re}\gamma_N \rangle| N , \tag{66}$$

and by inserting successively (36a), (46a) and (55), we have (for $N \gg 1$)

$$\langle \rho_N \rangle \simeq \frac{\varepsilon_0^2 N}{2}\left\{1-E^2\left[\frac{N^2}{3}+\frac{N}{8}\left[1-(-1)^N\right]-\frac{7}{12}\right]\right\}, \quad E\to 0 , \tag{67}$$

$$\langle \rho_N \rangle \simeq \varepsilon_0^2 N\left\{\frac{N^2}{3}-\frac{N}{2}+\frac{1}{6}\left[\tfrac{8}{15}(N^4-1)-\frac{N}{2}(N^2-N+1)\right]\frac{y^2}{4}\right\}, \quad E\to\pm 2 , \tag{68}$$

$$\langle \rho_N \rangle \simeq \frac{2N}{\xi_0} \quad \text{for intermediate energies} , \tag{69}$$

to lowest order in the disorder. Here Eq. (69) coincides with Pichard's result for the Ohmic resistance for weak disorder at intermediate energies. Equation (67) displays Ohmic resistance at the band center with an enhanced classical resistivity,[2,18] $N^{-1}\langle \rho_N \rangle \sim 4/\xi_0$ [compare Eq. (59)]. Finally, Eq. (68) demonstrates a strongly non-Ohmic behavior for the resistance near the band edges. We emphasize that Eqs. (67)–(69) are valid for lengths $N \ll \xi_1$ which are significantly smaller than the localization length, which is generally regarded as the upper limit for metallic behavior.

The resistance of a disordered sample is a random quantity for which the average discussed above may describe only typical behavior. The study of resistance fluctuations at any energy, starting from the Landauer formula, would require a detailed derivation of the transmission (or reflection) coefficient similar to that discussed by Pichard[18] for intermediate energies, for weak disorder. We recall that in the scaling theory of Anderson *et al.*,[21] which treated a one-dimensional disordered system as a sequence of random barriers, the relative rms fluctuation of the resistance in the metallic regime is independent of sample length; i.e., the resistance is marginally non-self-averaging in this case.

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