Coulomb-gas representation of the two-dimensional XY model on a torus

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Superconducting networks and superfluid films in two dimensions are often described by a theoretical model in which the unique microscopic variables are phases. Among these models the XY model with Villain's interaction potential can be mapped exactly onto a lattice Coulomb gas. This is well known, but several questions still have no clear answers: First, what is the meaning of the charge of the Coulomb gas in terms of the original variables of the XY model? Second, how can the helicity modulus be expressed exactly in the Coulomb-gas representation on a finite torus? In this paper we answer these questions. The mapping onto a lattice Coulomb gas is done in a way that differs from the usual one. This mapping is applied to a phase model whose partition function has an identical mathematical structure as the one of the XY model with Villain's interaction. For this phase model, contrary to the XY model, the charges of the Coulomb gas describe indeed exactly the topological charges as we can define them in terms of the phase variables. However, this Coulomb gas contains an additional polarization energy and two additional fictitious variables accounting for the specific topological character of the torus. The helicity modulus is exactly the inverse of a dielectric constant which can be defined as the linear response to an external uniform electric field, even on a torus. The meaning of the Coulomb-gas representation is also discussed in terms of the original variables of the XY model.

I. INTRODUCTION

The two-dimensional (2D) XY model can describe various superconducting networks,[1] in particular Josephson junction arrays (e.g., Ref. 2) which correspond to a cosine interaction potential between the phase variables on neighboring sites. It seems also to be relevant for the theoretical understanding of superconductors with high critical temperature (e.g., Refs. 3 and 4).

In 1973 Kosterlitz and Thouless[5] decoupled the XY model into two subsystems, approximating the lattice by a continuum. Each state was given by a topological configuration (configuration of vortices) and small amplitude fluctuations around the vortices. The fluctuation part describes the linear spin waves of the system, and the topological part was mapped onto a Coulomb gas.

In 1975, Villain[6] proposed an approximation for the cosine interaction potential that has been used rather often since then. José, Kadanoff, Kirkpatrick, and Nelson[7] based an exact decoupling into a spin-wave part and a Coulomb-gas part on this approximation. However, in this case, the charges of the Coulomb gas do not correspond exactly to the topological charges as they are defined in the original variables. Moreover, the spin waves do not correspond to the fluctuations around a local minimum of the energy.

The crucial quantity for describing the behavior of the XY model system as a function of the temperature is the helicity modulus Γ, whereas for the Coulomb gas the relevant quantity is the inverse dielectric constant $1/\varepsilon$. The helicity modulus was introduced in 1973 by Fisher, Barber, and Jasnow[8] in order to define the superfluid density ρ_s of the superfluid helium film. The theoretical link between ρ_s and the dielectric constant $1/\varepsilon$ of a Coulomb gas has been studied, in a continuum version and in different ways, principally by Nelson and Kosterlitz,[9] Myerson,[10] and Minnhagen and Warren.[11] The definition of Γ has been extended to the XY model by Ohta and Jasnow[12] who consider the cosine potential at low temperature. They also gave a link between Γ and $1/\varepsilon$ at higher temperature using Villain's interaction instead of the cosine. Later, Shih, Ebner, and Stroud[13] expressed the helicity modulus of the XY model in terms of thermodynamical averages. Their expression has been extensively used in Monte Carlo simulations of the XY model with periodic boundary conditions imposed on the planar spin variables.

Several aspects are still unclear. First, the integers m_R introduced by José *et al.*[7] are arbitrary numbers (even if the angles of the spins are restricted to the interval $(-\pi,\pi]$). They are called "quantum number for a vortex excitation."[7] However, the topological charge q_R of a vortex as we can define it in terms of the original variables takes only values in the set $\{-1,0,1\}$ for the square lattice. Even if in practice the values of the m_R's are often restricted to this set, m_R and q_R do not have exactly the same meaning. This fact has been pointed out by Savit,[14] but the meaning of the charge of the Coulomb gas has not been given in terms of the original variables.

Second, the influence of different boundary conditions has not been carefully studied. Indeed, the periodic boundary conditions in terms of the original variables

have not been taken into account in the calculations of Ref. 7. However, the calculation of the helicity modulus as expressed in terms of thermodynamical averages depends on the boundary conditions, and it is crucial to calculate it with boundary conditions in the original variables. Fisher *et al.*[4] have considered an additional term in the Coulomb-gas Hamiltonian taking into account the periodic boundary conditions. Their Hamiltonian represents correctly the XY model (with Villain's interaction) for fixed boundary conditions, but their Coulomb gas misses some variables in the case of periodic ones.

Third, a well known result in statistical physics allows one to express the dielectric constant in terms of the quadratic average of the polarization. As the helicity modulus of the XY model can be defined on a torus, what is the polarization of a configuration of the associated Coulomb gas in such a manifold? Moreover, if the twist corresponds to a uniform electric field, how can we explain that the helicity modulus can be defined as a linear response to a twist since we know that the linear response of a Coulomb gas with periodic boundary conditions to a uniform electric field is not defined ($\varepsilon \to \infty$)?

The problem of the boundary conditions of the 2D Coulomb gas is well known and has been studied by Choquard, Piller, and Rentsch,[15] who found a dependence of the dielectric susceptibility on the boundary conditions even in the thermodynamic limit.

Thus, the aim of this paper is to clarify the Coulomb-gas representation of 2D XY models whose unique microscopic variables are phases. For this purpose, in Sec. II A we consider a particular version of the XY model where the planar spins are defined on a square wire network, not only at the nodes but at each point along the wires linking nearest neighbor nodes. We will call this version the "wire model" and we will keep the denomination XY model for that involving spins only at the nodes of a lattice. Experimentally, the wire model describes a superconducting wire network in the case where only the phase of the macroscopic wave function (in a Ginzburg-Landau theory) is relevant. It would also correspond to a superfluid film in a periodical porous medium as studied by Gallet and Williams.[16] We will show that the partition function of the wire model is mathematically identical to the one of the XY model with Villain's interaction potential. Then we map the wire model onto a Coulomb gas on a torus but, instead of following the way of Ref. 7, we shall use the same approach as Kosterlitz and Thouless,[5] but for the discrete case and with periodic boundary conditions. We shall see that in this model the charges of the Coulomb gas represent indeed the topological charges as we can define them in the original variables. We find that the Coulomb gas involves an additional polarization energy and two additional fictitious variables accounting for the periodic boundary conditions. In Sec. II B we turn to the XY model. We define a useful bond variable and the topological charge for the generalized XY model. Here, a generalized XY model is one in which an interaction potential different from the cosine is used. Then we compare it with the wire model, considering first Villain's interaction and second the piecewise parabolic potential. The interest of the second one lies in the fact that it is Villain's interaction in the limit of low temperatures and it permits one to split the Hamiltonian exactly into a topological part and a spin-wave part. In Sec. II C we generalize the results briefly to the frustrated case. (Experimentally the frustration represents a transverse external magnetic field applied to a superconducting network.[2]) The results of this generalization will be used in the following Secs. III and IV.

In Sec. III we give the meaning of the charge of the Coulomb gas associated with the XY model in terms of the original variables. For this purpose, we consider the second order variation of the free energy with respect to frustration terms around two given plaquettes. The derivation is performed in two different ways in Sec. III A in the original one and the Coulomb-gas representation, respectively. In Sec. III B, we consider the two different interaction potentials mentioned above and the usual cosine interaction corresponding experimentally to Josephson junction arrays. The calculations of Sec. III A will allow for an interpretation of the charge of the associated Coulomb gas in terms of the topological charge as it can be defined in the original variables.

In Sec. IV we first recall the notion of helicity modulus Γ. In Sec. IV A, for the wire model on a torus we transform Γ into the inverse dielectric constant $1/\varepsilon$ of a Coulomb gas whose charges represent exactly the topological charges. We will see that Γ can indeed be represented by the second order variation of the free energy of the associated Coulomb gas with respect to a uniform external electric field. In Sec. IV B we turn again to the XY model and compare Γ in the two different representations.

This work establishes in an exact way the formal analogy between the XY model and the Coulomb gas on a 2D torus.

II. MAPPING ONTO TWO-DIMENSIONAL COULOMB GAS

In Sec. II A we begin to define the wire model. Then we map this model onto a lattice Coulomb gas. In Sec. II B we turn to the XY model. We generalize it in the sense that we consider other interaction potentials than the cosine. We shall discuss how to apply the results of Sec. II A. In particular, we shall explain why we do not find the same result as in Ref. 7. In Sec. II C we briefly generalize to the frustrated case.

A. The wire model

In this model, a planar spin $\vec{S}_l$ determined by a phase $\theta_l \in (-\pi, \pi]$ is associated to each point indexed by l of a square wire network (cf. Fig. 1). More precisely, a state of the system is described by a continuous and differentiable function $\vec{S}_l$, defined on a regular network of horizontal and vertical straight lines. L will stand for the linear size and $N \equiv L^2$ for the number of nodes. For simplicity, the lattice constant is put equal to 1. The Hamiltonian of the system is

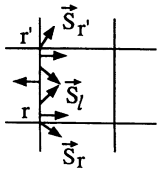

FIG. 1. In the wire model a state is described by a continuous function $\vec{S}_l$ defined on the sites (r and r') and on each point along the bonds linking them.

$$H = \frac{J}{2}\int dl \left(\frac{\partial \vec{S}_l}{\partial l}\right)^2 , \tag{1}$$

where J is the coupling constant.

For the description of a wire superconducting network, the phase variation is proportional and opposite to the supercurrent. However, it has to be noted that, for such description, Hamiltonian (1) should be completed by a term accounting for a charge energy related to the divergence of the supercurrent. But every computational step in this paper could be performed with such a term.

Let us consider an oriented bond rr', linking a node r of the lattice to a nearest neighbor r' and parametrized by $l \in [0,1]$. Then let us define the bond quantity $\phi_{rr'}$ as the phase variation from r' to its nearest neighbor r:

$$\phi_{rr'} = -\int_r^{r'} dl \left(\vec{S}_l \times \frac{\partial \vec{S}_l}{\partial l}\right)\hat{z}, \tag{2}$$

where $\hat{z}$ is a unit vector perpendicular to the plane of the system. Thus $J\phi_{rr'}$ represents the supercurrent from r to r'. We can also define integer bond variables $p_{rr'}$ such that

$$\phi_{rr'} = \theta_r - \theta_{r'} + 2\pi p_{rr'}, \tag{3}$$

where θ_r and $\theta_{r'}$ correspond to the spin angles on the nodes r and r' and belong to the interval $(-\pi,\pi]$. In fact $\vec{S}=(\cos\theta,\sin\theta)$ and $(\vec{S}_l \times \partial\vec{S}_l/\partial l)\hat{z}=\partial\theta_l/\partial l$ except where θ_l runs across the border of the interval.

The phase on this bond can also be expressed as a continuous scalar function $\vartheta_l \in (-\infty,\infty)$ (equal to $\theta_l + 2\pi p_l$ where the integers p_l ensure that ϑ_l is continuous) and can be written as follows:

$$\vartheta_l = \theta_{l=0} - l\phi_{rr'} + \zeta_l. \tag{4}$$

The second term on the right hand side represents the variation of ϑ_l along l that minimizes the energy for a given $\phi_{rr'}$ or for given θ_r, $\theta_{r'}$, and $p_{rr'}$. The quantity ζ_l is the fluctuation around this configuration leaving unchanged the given site and bond variables. It is a continuous and differentiable function in the interval $[0,1]$, such that $\zeta_0 = \zeta_1 = 0$.

The calculation of the energy $H_{rr'}$ for the bond rr' gives

$$H_{rr'} = \int_0^1 dl \frac{J}{2}\left(\frac{\partial\zeta_l}{\partial l}\right)^2 + \frac{J}{2}(\phi_{rr'})^2 . \tag{5}$$

The contribution of ζ_l to the partition function of the system is trivial (identical for each bond and independent of the site and bond integer variables). In the following, we shall no longer consider it.

It has to be noted that the equality $\vec{S}=(\cos\theta,\sin\theta)$ does not permit one to replace $\vec{S}$ by θ in Hamiltonian (1) because of the discontinuity occuring when θ runs across the border of the interval. Nor can we put $\vec{S}=(\cos\vartheta,\sin\vartheta)$, because that would impose $\vec{S}$ being a gradient of a continuous scalar function and, therefore, would not permit a nonzero winding number of the phase on a closed contour on the lattice:

$$\oint dl \frac{\partial\vartheta_l}{\partial l} = 0 .$$

Indeed ϑ cannot, in general, be defined continuously on a closed loop. We can express $\vec{S}$ by ϑ only on an open path on our lattice.

Since we can neglect the internal fluctuations in the bonds, a state of the system can be described either by the variables $\theta_r \in (-\pi,\pi]$ and $p_{rr'} \in Z$ or by the variables $\phi_{rr'} \in (-\infty,\infty)$ [definition (3)] but, in this last case, two types of constraints have to be imposed on these bond variables. The first of them is, for each plaquette R,

$$\sum_{\Box R} \phi_{rr'} = 2\pi q_R, \tag{6}$$

where the notation $\sum_{\Box R}$ means that the sum is taken in the clockwise direction over the bonds surrounding the plaquette R on the lattice. The quantity q_R is an integer representing the topological charge on that plaquette. In terms of the first kind of variables, q_R is the winding number of the phase around the plaquette R and is given by $q_R = \sum_{\Box R} p_{rr'}$. The definition of q_R yields also the following property:

$$2\pi \sum_{R\in D} q_R = \sum_{\partial D} \phi_{rr'}; \forall D, \tag{7}$$

where the sum on the left hand side is taken over the plaquettes inside any domain D on the lattice and the one on the right hand side over the bonds of its boundary ∂D in the clockwise direction.

The second type of constraint arises from the periodic boundary conditions imposed on the θ_r variables. To simplify their expressions we introduce the vector field $\vec{\phi}_r \equiv (\phi_{xr},\phi_{yr})$ representing the two $\phi_{rr'}$ variables associated to the bonds rr' pointing, respectively, in the positive horizontal and vertical directions from the site r. We can write them as

$$\sum_{r\in l_{xa}} \phi_{xr} = -2\pi q_{xa} , \quad \sum_{r\in l_{ya}} \phi_{yr} = -2\pi q_{ya} , \tag{8}$$

where q_{xa} and q_{ya} are integers and the two sums are taken over the sites of an arbitrary horizontal line ($r \in l_{xa}$) and a vertical line ($r \in l_{ya}$), respectively. The integer q_{xa} is the winding number (defined in the positive orientation) of the phase along the arbitrary chosen horizontal line. It is equal to the sum of the $p_{rr'}$'s over that line (keeping

r on the right hand side of r'). It has to be noted that the constraints imposed only on one horizontal and one vertical line are sufficient to satisfy the periodic boundary conditions on the θ_r's on all other lines. Indeed, from property (7), the sum of the ϕ_{xr}'s along another line l_{xj} is equal to $-2\pi q_{xa} + \sum_{R\in D} q_R$, where D is the domain bounded by the two lines, considering the line l_{xa} as the lower boundary of the domain D.

By definition, to each state $\{\theta_r, p_{rr'}\}$ corresponds one state $\{\phi_{rr'}\}$ (the latter notation implying that all the constraints are satisfied). But this mapping is not one to one because we have to know θ_r at one site to calculate the other θ_r's and the $p_{rr'}$ is starting from a state given by $\{\phi_{rr'}\}$. Therefore, there is a one-to-one correspondence between each set $\{\phi_{rr'}\}$ and an equivalent class of sets $\{\theta_r, p_{rr'}\}$, two of them being equivalent if they differ only by the same angle at each node. Then the partition function can be written either as

$$Z = \sum_{\{p_{rr'}\}} \int_{-\pi}^{\pi} [d\theta] \exp\left(-\beta \frac{J}{2} \sum_{\langle rr'\rangle} (\theta_r - \theta_{r'} + 2\pi p_{rr'})^2\right) \tag{9}$$

or as

$$Z = \sum_{\{q_R, q_{\alpha a}\}} \int_{-\infty}^{\infty} [d\phi_{rr'}] \prod_R \delta\left(2\pi q_R - \sum_{\square R} \phi_{rr'}\right) \times \prod_\alpha \delta\left(2\pi q_{\alpha a} + \sum_{r\in l_{\alpha a}} \phi_{\alpha r}\right) \times \exp\left(-\beta \frac{J}{2} \sum_{\langle rr'\rangle} \phi_{rr'}^2\right), \tag{10}$$

where the two products take into account the constraints (6) and (8) and the first sum is taken over all neutral topological configurations $\{q_R\}$. Indeed, the periodical boundary conditions in the $\phi_{rr'}$'s along with the property (7) imply the neutrality property of the topological configuration $\{q_R\}$:

$$\sum_R q_R = 0. \tag{11}$$

The $\phi_{rr'}$'s represent gauge invariant phase differences. For the superfluid films, the vector field $\vec{\phi}_r \equiv (\phi_{xr}, \phi_{yr})$ stands for the superfluid flow (see, for example, Refs. 11 and 16. It has to be noted that our boundary conditions restrict the gauge invariance as we shall see in Sec. II C. In order to map the wire model onto a lattice Coulomb gas, the partition function (10) in the bond variables $\phi_{rr'}$ will be used.

Let us fix a neutral topological configuration $\{q_R, q_{\alpha a}\}$. The variables $\phi_{rr'}$ can be expressed by their deviation $\varphi_r - \varphi_{r'}$ from the minimum energy configuration $\{\phi^0_{rr'}\}$ (because of the positive quadratic form of the Hamiltonian, the latter quantity exists and is unique):

$$\phi_{rr'} = \phi^0_{rr'} + \varphi_r - \varphi_{r'}. \tag{12}$$

So there is a one-to-one correspondence between equilibrium states $\{\phi^0_{rr'}\}$ and topological configurations $\{q_R, q_{\alpha a}\}$ satisfying the neutrality property. Moreover, for a given set $\{q_R, q_{\alpha a}\}$, we have a one-to-one correspondence between each state $\{\phi_{rr'}\}$ (satisfying the q_R's and the $q_{\alpha a}$'s) and an equivalent class of sets $\{\varphi_r\}$ (two of them being in the same class if they differ by the same quantity at each node).

The Hamiltonian of the system becomes

$$H = \frac{J}{2} \sum_{\langle rr'\rangle} (\phi^0_{rr'} + \varphi_r - \varphi_{r'})^2. \tag{13}$$

Since the set $\{\phi^0_{rr'}\}$ represents an equilibrium state, we have at each node r

$$\sum_{+r} \phi^0_{rr'} = 0, \tag{14}$$

where the notation $+r$ means that we sum over the four bonds linking r to its nearest neighbors. This condition corresponds to $\partial H/\partial\theta_r = 0$. Therefore, the sum of the cross terms in $\phi^0_{rr'}\varphi_r$ in Hamiltonian (13) vanishes and we have

$$H = \frac{J}{2} \sum_{\langle rr'\rangle} (\varphi_r - \varphi_{r'})^2 + \frac{J}{2} \sum_{\langle rr'\rangle} (\phi^0_{rr'})^2. \tag{15}$$

The system is decoupled into two subsystems. The first one represents the fluctuations around a topological configuration and has the form of a spin-wave Hamiltonian. The second one is the Hamiltonian $H_{\rm ch}$ depending only on the topological charges:

$$H_{\rm ch} = \frac{J}{2} \sum_{\langle rr'\rangle} (\phi^0_{rr'})^2. \tag{16}$$

We now want to express (16) explicitly in terms of q_R and $q_{\alpha a}$.

First of all we need to show the following property for any equilibrium state $\{\phi^0_{rr'}\}$:

$$\sum_{r\in l_{xj}} \phi^0_{yr} = \frac{\sigma_y}{L}, \tag{17}$$

where the sum is taken over the sites of any horizontal line l_{xj} [do not confuse Eq. (17) with Eq. (8), cf. Fig. 2] and $\sigma_y \equiv \sum_r \phi^0_{yr}$. Indeed, summing the left hand side of Eq. (14) over the sites r on the line l_{xj} gives

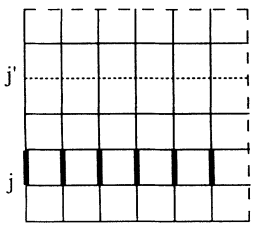

FIG. 2. The bonds marked in the dotted line correspond to those that are considered in Eq. (8) with l_{xa}=$l_{xj'}$. Those in the bold lines correspond to Eq. (17).

$$\sum_{r\in l_{xj}} \phi^0_{yr} - \sum_{r\in l_{xj-1}} \phi^0_{yr} = 0.$$

So the sum of Eq. (17) is the same for each line l_{xj}.

Then we can turn to the dual variables ψ_R defined on the plaquettes of the torus, with periodic boundary conditions, such that

$$\phi^0_{xr} = \psi_{R-\hat{y}} - \psi_R + \frac{\sigma_x}{L^2},$$
$$\phi^0_{yr} = \psi_R - \psi_{R-\hat{x}} + \frac{\sigma_y}{L^2}, \tag{18}$$

where R represents the upper right plaquette of the site r. Thus we can calculate all the ψ_R's (corresponding to any equilibrium state $\{\phi^0_{rr'}\}$) using definition (18) and choosing an arbitrary value ψ_R for one plaquette R. When we represent a state by its ψ_R's the equilibrium conditions (14) are automatically satisfied. Note that the introduction of σ_x and σ_y is necessary, otherwise the sum of the $\phi_{\alpha r}$'s over the whole lattice could only be equal to zero for any set $\{\psi_R\}$ satisfying the periodic boundary conditions.

The Hamiltonian $H_{\rm ch}$ becomes

$$H_{\rm ch} = H_{\rm ch1} + H_{\rm ch2} \tag{19}$$

with

$$H_{\rm ch1} = \frac{J}{2} \sum_{\langle RR'\rangle} (\psi_R - \psi_{R'})^2 \tag{20}$$

and

$$H_{\rm ch2} = \frac{J}{2L^2}(\sigma_x^2 + \sigma_y^2). \tag{21}$$

The sum of the cross terms $\sigma_\alpha(\psi_R - \psi_{R'})$ vanishes because of the periodic boundary conditions in the ψ_R's. Now we want to express the two terms in (20) and (21) explicitly by the topological charges. We begin with $H_{\rm ch1}$ which can be written as

$$H_{\rm ch1} = \frac{J}{2} \sum_{RR'} \psi_R M_{RR'} \psi_{R'}, \tag{22}$$

where

$$M_{RR'} = \begin{cases} 4 & \text{if } R = R', \\ -1 & \text{if } R \text{ and } R' \text{ are nearest neighbors,} \\ 0 & \text{elsewhere.} \end{cases}$$

Note that the matrix M represents the discrete Laplacian in two dimensions (with opposite sign). In the following ψ stands for the vector representing $\{\psi_R\}$ and q for $\{q_R\}$. We remark that

$$M\psi = 2\pi q, \tag{23}$$

$$\psi^t M = 2\pi q^t. \tag{24}$$

Replacing M by $MM^{-1}M$,

$$H_{\rm ch1} = 2\pi^2 J q^t M^{-1} q. \tag{25}$$

In order to link this calculation to the usual procedure in a continuum, let us consider the vector field $\vec{\phi}_r \equiv (\phi_{xr}, \phi_{yr})$ in a lattice with free boundary conditions. So the condition (14) reads $\vec{\nabla}\cdot\vec{\phi}_r = 0$. Therefore there exists a scalar field ψ_R such that $\vec{\phi}_r = \hat{z} \times \vec{\nabla}\psi_R$. Then writing the identity $\hat{z}\vec{\nabla}^2\psi_R = \vec{\nabla} \times (\hat{z} \times \vec{\nabla}\psi_R)$ and considering, in agreement with our definition of the q_R's, that $\vec{\nabla} \times \vec{\phi}_r = -\hat{z}2\pi q_R$, we obtain expression (23). Note that for periodic boundary conditions, we should also introduce quantities equivalent to σ_x and σ_y as in Eq. (18).

We shall call G the inverse matrix M^{-1}. G stands for the discrete Green function of the dual lattice. But M^{-1} does not exist because the normalized eigenvector $V_0 \equiv \{V_{0R}\}$ with $V_{0R} = 1/\sqrt{N}, \forall R$ has a zero eigenvalue [note that such an eigenvector is unique regarding (20)]. In order to solve this problem, we replace M by $M \equiv M_c - cP$ in (22), where P is the projector on V_0 ($P = V_0 V_0^t$) and c is any nonzero number. Then M_c is regular:

$$H_{\rm ch1} = 2\pi^2 J \psi^t M_c M_c^{-1} M_c \psi - \psi^t c P \psi. \tag{26}$$

Instead of Eq. (23), we have $M_c\psi = 2\pi q + cP\psi$ and $\psi^t M_c = 2\pi q^t + cP\psi^t$. Thus

$$H_{\rm ch1} = 2\pi^2 J q^t M_c^{-1} q + \pi J c P \psi^t M_c^{-1} q + \pi J q^t M_c^{-1} c P \psi + c P \psi^t M_c^{-1} c P \psi - \psi^t c P \psi. \tag{27}$$

The cross terms in ψ and q are equal to zero because the eigenvector $cP\psi$ is composed of equal components and m satisfies the neutrality property (11), and the two last terms are equal with opposite signs. Expression (27) becomes

$$H_{\rm ch1} = 2\pi^2 J q^t G_c q, \tag{28}$$

where $G_c = M_c^{-1}$ for any nonzero c. We can diagonalize the (symmetrical) matrix G_c by developing the vector ψ in plane waves. The normalized eigenvectors of M_c are identical to those of M and their corresponding eigenvalue is the same except for $\mathbf{k}=\mathbf{0}$, for which it becomes c. For $\mathbf{k} \neq \mathbf{0}$, the normalized eigenvectors and the eigenvalues are

$$V_{\mathbf{k}} = \left\{ \frac{1}{\sqrt{N}} e^{-i\mathbf{k}\cdot\mathbf{R}} \right\}, \tag{29}$$

$$\lambda_{\mathbf{k}} = 4 - 2\cos k_x - 2\cos k_y, \tag{30}$$

where $\mathbf{k} \equiv (k_x, k_y)$ is a reciprocal vector of the first Brillouin zone. Then we have

$$G_{cRR'} = c + \frac{1}{N} \sum_{\mathbf{k}\neq\mathbf{0}} e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')} \frac{1}{\lambda_{\mathbf{k}}}. \tag{31}$$

We can choose the value of c (that we call c_0) in order to let vanish the diagonal matrix element G_{cRR}:

$$c_0 \equiv -\frac{1}{N} \sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{\lambda_{\mathbf{k}}}. \tag{32}$$

Let us call G' the matrix associated with c_0:

$$G'_{RR'} = \frac{1}{N}\sum_{\mathbf{k}\neq 0}(e^{i\mathbf{k}\cdot(\mathbf{R}-\mathbf{R}')}-1)\frac{1}{\lambda_{\mathbf{k}}}. \quad (33)$$

Hamiltonian H_{ch1} alone corresponds to the charge Hamiltonian given by José *et al.*[7] for the XY model with Villain's interaction, but who found it in a different way. Their statement will be discussed in Sec. II B. In the limit of large systems, $G'_{RR'}$ can be well approximated by a logarithmic function for any couple (R,R') with $R\neq R'$ (exact in the limit of large distances $|\mathbf{R}-\mathbf{R}'|$):

$$G'_{RR'}(L\to\infty)\approx G''(\mathbf{R}-\mathbf{R}')$$

$$\equiv -\frac{1}{2\pi}(\ln|\mathbf{R}-\mathbf{R}'|+\tfrac{1}{2}\ln 8+\gamma), \quad (34)$$

γ being Euler's constant. This approximation [with $G''(\mathbf{0})\equiv 0$] can be used to estimate $G'_{RR'}(L)$ for finite L. For this purpose we consider an L-periodic neutral configuration of the infinite system. The energy H_{ch1} for an $L\times L$ cell becomes

$$H_{\mathrm{ch1}}\approx 2\pi^2 J\sum_{n_x n_y}\sum_{RR'} m_R[G''(\mathbf{R}-\mathbf{R}'+L\mathbf{n}) + c(\mathbf{n})]m_{R'}, \quad (35)$$

where R and R' run over the plaquettes of an $L\times L$ cell and $(n_x,n_y)=\mathbf{n}$ over all integer pairs. Since the configuration is neutral any function $c(\mathbf{n})$ can be added without changing the right hand side of Eq. (35). Thus we chose $c(\mathbf{n})=-G''(L\mathbf{n})$ and find the convergent estimation

$$G'_{RR'}(L)\approx\sum_{\mathbf{n}}[G''(\mathbf{R}-\mathbf{R}'+L\mathbf{n})-G''(L\mathbf{n})]. \quad (36)$$

Now we turn to H_{ch2}. In order to get σ_α in terms of the charges we denote $\phi_{\alpha r}$ and q_R by $\phi_{\alpha ij}$ and q_{ij} where i and j (running from 1 to L) indicate the coordinates of a site r or of the lower left site of a plaquette R (cf. Fig. 3). We write σ_x in terms of differences on consecutive lines:

$$\sigma_x\equiv\sum_{ij}\phi_{xij}=\sum_{j=1}^{L-1}j\left(\sum_{i=1}^{L}\phi_{xij}-\sum_{i=1}^{L}\phi_{xij+1}\right) +L\left(\sum_{i=1}^{L}\phi_{xiL}-\sum_{i=1}^{L}\phi_{xi1}\right)+L\sum_{i=1}^{L}\phi_{xi1}. \quad (37)$$

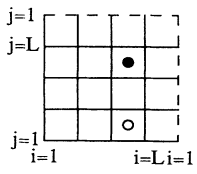

FIG. 3. Illustration of the coordinate system and of the notation. The full (or empty) circle indicates a positive (or negative) charge. It can be written $q_{33}=1$, $q_{31}=(1/2\pi)(\phi_{y31}+\phi_{x32}-\phi_{y41}-\phi_{x31})=-1$.

Then we use Eqs. (7) and (8) and choose the first horizontal line as the reference to define the topological charge $q_{x\alpha}$ [cf. Eq. (8)] associated with the periodic boundary conditions:

$$\sigma_x=-2\pi\left(\sum_{j=1}^{L}j\sum_{i=1}^{L}q_{ij}+Lq_{x1}\right). \quad (38)$$

In the same way in the other direction

$$\sigma_y=-2\pi\left(\sum_{i=1}^{L}i\sum_{j=1}^{L}q_{ij}+Lq_{y1}\right). \quad (39)$$

Since the system is neutral, the charges can be grouped by neutral pairs of unit charges. Considering each pair such that the positive charge is on a plaquette R_+ and the negative one on a plaquette R_-,

$$\sigma_x=-2\pi\left(\sum_{\mathrm{pairs}}[j(R_+)-j(R_-)]+Lq_{x1}\right). \quad (40)$$

Now we define the suitable topological polarization vector $\mathbf{P}=(P_x,P_y)$ for the case of periodic boundary conditions in the θ_r's:

$$P_x\equiv\sum_{\mathrm{pairs}}[i(R_+)-i(R_-)]+Lq_{y1},$$

$$P_y\equiv\sum_{\mathrm{pairs}}[j(R_+)-j(R_-)]+Lq_{x1}. \quad (41)$$

So we have

$$\sigma_x=-2\pi P_y,\quad \sigma_y=-2\pi P_x, \quad (42)$$

and

$$H_{\mathrm{ch2}}=2\pi^2 J\left[\left(\frac{P_x}{L}\right)^2+\left(\frac{P_y}{L}\right)^2\right]. \quad (43)$$

The above definition of the topological polarization removes the ambiguity of the polarization of a neutral pair of charges in a torus. The problem of the definition of the polarization of a Coulomb gas with periodic boundary conditions is well known. The problem arises as soon as a little pair is considered in the system: is it indeed a little pair or is it a very large pair comparable to the linear size of the lattice? However, the variables of the Coulomb gas associated with the wire model do not consist only of local charges, but also of two global charges corresponding to the periodic boundary conditions. And these additional charges permit one to define the polarization on a torus. Thus a definite neutral pair can correspond to different polarizations as it can correspond to a different topological configuration (not the same $q_{\alpha 1}$). For instance, the polarization of the pair in Fig. 4(a) is equal to $(1,3)$ whereas it is equal to $(1,-2)$ in Fig. 4(b).

Since $\mathbf{P}$ does not depend on the choice of the two reference lines defining the two $q_{\alpha 1}$'s, we can also define integer quantities q_{x0}, q_{y0}, $P_{\pi x}$, and $P_{\pi y}$ independent of

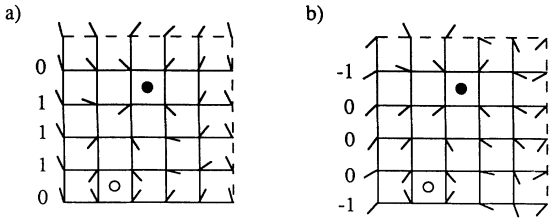

FIG. 4. Two configurations with same charges q_R, same q_{yi}'s (same $q_{y\alpha}$), and different q_{xj}'s (different $q_{x\alpha}$). (a) $\mathbf{P}=(1,3)$; (b) $\mathbf{P}=(1,-2)$. In this example, all the $\phi_{rr'}$'s are supposed to belong to $(-\pi,+\pi]$. The numbers on the left of each lattice are the values of the q_{xj}'s [cf. Eq. (8)].

reference choices using the following equations:

$$P_x = P_{\pi x} + Lq_{y0}\ , \quad P_y = P_{\pi y} + Lq_{x0}, \tag{44}$$

where $P_{\pi x}$ and $P_{\pi y}$ are restricted to the interval $(-L/2,+L/2]$. Thus the $P_{\pi\alpha}$'s are the contributions of the charges q_R to the polarization $\mathbf{P}$ and the $q_{\alpha 0}$'s can be considered as the global topological charges associated with the torus topology. Another advantage of using $q_{\alpha 0}$ instead of $q_{\alpha 1}$ is that for a given set $\{q_R\}$ the Hamiltonian value is always minimum for $(q_{x0},q_{y0})=(0,0)$.

In order to understand the link of the wire model on a torus to a lattice Coulomb gas, we can use approximation (36) and write

$$H_{\rm ch} = -\frac{1}{2}\sum_{R\neq R'} q_{eR}q_{eR'}\left(\ln\mid \mathbf{R}-\mathbf{R}'\mid + \sum_{\mathbf{n}\neq\mathbf{0}}(\ln\mid \mathbf{R}-\mathbf{R}'+L\mathbf{n}\mid -\ln\mid L\mathbf{n}\mid)\right) + \sum_R q_{eR}^2\mu + \pi\frac{\mathbf{P}_e^2}{L^2} \tag{45}$$

with $q_{eR}\equiv(2\pi J)^{1/2}q_R$, $\mathbf{P}_e\equiv(2\pi J)^{1/2}\mathbf{P}$, and $\mu\equiv(1/4)\ln 8+\gamma/2$.

Thus the wire model with periodic boundary conditions in the θ_r's and $p_{rr'}$'s can be represented by a lattice Coulomb gas with the two additional variables q_{x0} and q_{y0} and a polarization energy. The topological charge q_R is represented by the number of electrical elementary charges on that plaquette. The two additional charges, and therefore the polarization, have no static electric meaning. However, in a dynamic case, they can be interpreted. Indeed, assuming that charges can only jump (and be created or annihilated by pairs) on nearest neighbor plaquettes, we could define an electric polarization variation between two electric charge configurations by summing elementary polarization variations. So the polarization would be defined by its variation from a reference state. In order to give such a polarization at an arbitrary time the positions of the charges would not be sufficient. We must in addition keep in memory the two $q_{\alpha 0}$'s which ensure that the polarization in Eq. (44) is indeed increased by elementary quantities. Those two variables can be interpreted as the difference between the total winding numbers of the positive charges and the negative ones around each direction of the torus.

Finally, the partition function of the wire model with periodic boundary conditions can be expressed exactly as the product of two factors:

$$Z = Z_{\rm SW}Z_{\rm ch}, \tag{46}$$

with

$$Z_{\rm SW} = \int_{-\infty}^{\infty}[d\varphi_r]\exp\left(-\beta\frac{J}{2}\sum_{\langle rr'\rangle}(\varphi_r-\varphi_{r'})^2\right) \tag{47}$$

and

$$Z_{\rm ch} = \sum_{\{q_R,q_{\alpha 0}\}}\exp\left\{-\beta 2\pi^2 J\left[\sum_{RR'}q_RG'_{RR'}q_{R'} + \left(\frac{\mathbf{P}}{L}\right)^2\right]\right\}, \tag{48}$$

where the sum is taken over all topological configurations $\{q_R,q_{\alpha 0}\}$, and the q_R's and the $q_{\alpha 0}$'s can take any integer values subject to the constraint of neutrality, $\sum_R q_R=0$.

The first factor is the partition function of the spin waves representing the quadratic fluctuations around any equilibrium state and each of those states corresponds to a topological configuration. The second one can be represented by a neutral Coulomb gas on a lattice with two additional variables and a polarization energy. The topological charge q_R is exactly represented by the number of electric charges on the plaquette R. The polarization due to the two additional charges $q_{\alpha 0}$ could have an electrical meaning only in a dynamical version of the Coulomb gas.

An alternative way to map exactly the wire model onto a Coulomb gas without additional charges is to perform the sum over $q_{\alpha 0}$ in $Z_{\rm ch}$. But in that case we lose the possibility to represent completely the topology of a state by an electrical configuration.

A remark about fixed boundary conditions has to be made: in such a case, the Coulomb-gas Hamiltonian also contains an additional term involving the square of a polarization, but the boundary conditions and one set of q_R determine completely the polarization and the topology of a state.

B. Generalized XY model

We consider a generalized classical 2D XY model with Hamiltonian

$$H = \sum_{\langle rr'\rangle}V(\theta_r-\theta_{r'}), \tag{49}$$

where V is a 2π-periodic, even, and continuous function. The site variables θ_r are restricted to the interval $(-\pi,\pi]$. The partition function is

$$Z = \int_{-\pi}^{\pi}[d\theta_r]\exp\left(\beta\sum_{\langle rr'\rangle}V(\theta_r-\theta_{r'})\right). \tag{50}$$

Again it is useful to rewrite the Hamiltonian (49) in bond

variables $\phi_{rr'}$ defined as

$$\phi_{rr'} \equiv \theta_r - \theta_{r'} + 2\pi p_{rr'}, \tag{51}$$

where $p_{rr'}$ is the integer determined by $\theta_r - \theta_{r'}$ such that $\phi_{\alpha r}$ belongs to the interval $(-\pi, \pi]$. (For a consistent definition of the $\phi_{rr'}$'s, it has to be noted that if r' is on the left hand side or below r, definition (51) implies $\phi_{rr'} \in [-\pi, \pi)$.) Then the Hamiltonian expressed in these variables becomes

$$H = \sum_{\langle rr' \rangle} V(\phi_{rr'}). \tag{52}$$

In order that Hamiltonians (49) and (52) agree, and for periodic boundary conditions in θ_r, the following constraints have to be imposed on each plaquette R:

$$\sum_{\square R} \phi_{rr'} = 2\pi q_R \tag{53}$$

and

$$\sum_{r \in l_{\alpha a}} \phi_{\alpha r} = -2\pi q_{\alpha a}. \tag{54}$$

As for the wire model, the quantity q_R represents the topological charge of the plaquette R and $q_{\alpha a}$ represents the global topological charge due to the periodic boundary conditions in direction α and is defined on an arbitrary line $l_{\alpha a}$ of the lattice. But, contrary to the wire model, the q_R's and $q_{\alpha a}$'s are not arbitrary integers since the $\phi_{\alpha r}$'s belong to the interval $(-\pi, \pi]$ and the topological charges must satisfy the inequality

$$\left| \sum_{R \in D} q_R \right| < \pi L_D \ , \quad \forall D \tag{55}$$

where the sum is taken over the plaquettes inside any domain D on the lattice and L_D is the perimeter of D. For instance, for the square lattice, the q_R's take values in the set $\{-1, 0, 1\}$. q_R=1 (-1) means that a vortex (antivortex) is centered in that particular plaquette.

For periodic boundary conditions, we have a one-to-one correspondence between each class of states $\{\theta_r\}$ (where two states are in the same class if they differ only by a same rotation of all spins) and each state $\{\phi_{rr'}\}$ satisfying the constraints (53) and (54). It is known that the XY model can be mapped onto a Coulomb gas[7] using Villain's approximation.[6] Actually, we shall see below that the mapping has not been done correctly for periodic boundary conditions in the original variables.

1. *Villain's interaction*

The interaction potential proposed by Villain[6] is

$$V_v(\Delta\theta) = -T \ln \sum_{n=-\infty}^{\infty} \exp\left(-\beta \frac{J_v}{2}(\Delta\theta + 2\pi n)^2\right) + c_v. \tag{56}$$

In order that V_v approximates some other given potential V, the parameter J_v has to be fitted for each temperature. This can be done by evaluating the value of $J_v = J_v(T)$ minimizing the mean quadratic difference on a period $(-\pi, \pi]$ of the two Gibbs factors, namely,

$$\exp[-\beta V(\Delta\theta) + \bar{c}],$$

and

$$\exp[-\beta V_v(\Delta\theta)] = \sum_{n=-\infty}^{\infty} \exp\left(-\beta \frac{J_v}{2}(\Delta\theta + 2\pi n)^2\right) + \bar{c}_v, \tag{57}$$

where $\bar{c}$ and $\bar{c}_v$ are normalization constants. In the limit of low temperatures, the potential V_v is a piecewise parabolic potential. For example, if we consider a potential $V(\Delta\theta)$ with an absolute minimum at $\Delta\theta = 0$ and such that the two first derivatives at $\Delta\theta = 0$ are $V'(0) = 0$ and $V''(0) > 0$, then J_v takes the value of $V''(0)$. On the other hand, in the limit of high temperatures V_v becomes a cosine function.

Replacing V by V_v in the partition function (50), we obtain

$$Z = \sum_{\{n_{rr'}\}} \int_{-\pi}^{\pi} [d\theta_r] \times \exp\left(-\beta \frac{J_v}{2} \sum_{\langle rr' \rangle} (\theta_r - \theta_{r'} + 2\pi n_{rr'})^2\right). \tag{58}$$

Therefore the partition function of the XY model with Villain's interaction potential and periodic boundary conditions is mathematically strictly identical to the one of the wire model in Eq. (9).

However, in this case the $n_{rr'}$'s are only independent summation variables, giving the potential V_v [see Eq. (56)], whereas a statistical configuration of the system is determined by the θ_r's. We have defined $\phi_{rr'}$, q_R, and $q_{\alpha a}$ with Eqs. (51), (53), and (54) for the XY model in order to give them a precise meaning in terms of the original variables. In order to apply the same calculations as done for the wire model, we have to introduce nonphysical variables $\tilde{\phi}_{rr'}$, $n_{rr'}$, m_R, and $m_{\alpha a}$ correspondingly to Eqs. (3), (6) and (8):

$$\tilde{\phi}_{rr'} = \theta_r - \theta_{r'} + 2\pi n_{rr'} \ , \quad m_R \equiv \sum_{\square R} n_{rr'} \ ,$$

$$m_{\alpha a} = -\sum_{r \in l_{\alpha a}} n_{\alpha r}. \tag{59}$$

Thus Eqs. (46)–(48) are valid when replacing the q_R,$q_{\alpha 0}$ by m_R,$m_{\alpha 0}$ in these equations, as well as in expressions (38)–(41) for the polarization (replacing also $q_{\alpha 1}$ by $m_{\alpha 1}$), but now the m_R's do not have the physical meaning of topological charges, in the same way as the φ_r in Eq. (47) would not have a precise meaning in the original variables. We cannot define a set of states $\{\theta_r\}$ corresponding to one configuration $\{m_R, m_{\alpha 0}\}$. However,

we shall see in Sec. III how to interpret the m_R's by expressing the correlation function $\langle m_R m_{R'}\rangle$ in terms of $\langle q_R q_{R'}\rangle$.

Our Coulomb gas, containing a polarization energy, does not correspond exactly to that of Ref. 7 where the authors performed the Coulomb-gas mapping in a different way, namely, by integrating the $\theta_{rr'}$ of the partition function Z. Noting that the contribution of a set $\{n_{rr'}\}$ is nonzero only if $\sum_{+r} n_{rr'} = 0$ for all sites r, they expressed the $n_{rr'}$'s by dual variables. But their definition of the dual variables was too restrictive in the case of periodic boundary conditions in the $\theta_{rr'}$'s. Indeed, they implicitly imposed the additional restrictions

$$\sum_r n_{\alpha r} = 0. \tag{60}$$

They would have obtained the same results as ours if they had taken this effect into account in the same way as given by our Eq. (18). On the other hand, we would have obtained the same charge Hamiltonian if we had taken the periodic boundary conditions only in the $\phi_{rr'}$'s instead of the $\theta_{rr'}$'s. However in this case the system would be decoupled into a spin-wave system, a Coulomb gas [described by Hamiltonian (28)], and two additional modes. The latter would correspond to Hamiltonian H_{ch2} [as in Eq. (21)], but being independent of the set $\{m_R\}$ and where σ_x and σ_y would characterize the states of the modes and take any real value.

2. *Piecewise parabolic interaction*

The piecewise parabolic potential is interesting because it permits one to map the XY model exactly onto a spin-wave system coupled to a Coulomb gas where the charges are indeed topological charges. This potential can be defined by

$$V(x) = \frac{J}{2}(x + 2\pi p)^2, \tag{61}$$

where p is the integer such that $x + 2\pi p \in (-\pi, \pi]$. Thus the Hamiltonian expressed in bond variables is the same as the one of the wire model (see Sec. II A):

$$H = \frac{J}{2}\sum_{\langle rr'\rangle} \phi_{rr'}^2. \tag{62}$$

Although the $\phi_{rr'}$'s are now restricted to the interval $(-\pi, \pi]$, the same computational steps used for the wire model Hamiltonian can be applied to treat the XY model with a piecewise interaction potential. Therefore, we can still split the Hamiltonian into two parts and express the part in the variables $\phi^0_{rr'}$ in terms of topological charge:

$$H = \frac{J}{2}\sum_{\langle rr'\rangle}(\varphi_r - \varphi_{r'})^2 + 2\pi^2 J \sum_{RR'} q_R G'_{RR'} q_{R'} + \frac{2\pi^2 J}{L^2}(P_x^2 + P_y^2). \tag{63}$$

However, the fluctuation part in the φ_r's is now no longer independent of the topological part (expressed by the q_R's) because for a given set of $\{q_R\}$ (corresponding to a set $\{\phi^0_{rr'}\}$) the φ_r's have to be such that each $\phi_{\alpha r}$ defined by Eq. (51) belongs to the interval $(-\pi, \pi]$. This fact expresses the coupling between the topological charges and the fluctuations around them. Thus two different interesting approximations can be applied in the case of the piecewise parabolic potential in order to map the system onto a Coulomb gas. The first one is to neglect the above coupling and to release the restriction on the q_R's (and $q_{\alpha a}$'s) thus yielding the same Coulomb gas as the wire model. The second one is to apply Villain's approximation, calculating first J_v as a function of J/T and then decoupling the partition function. The advantage of the first one is to give the possibility of interpreting directly the charges of the Coulomb gas as topological charges. However, the use of Villain's approximation yields a Coulomb gas mathematically identical to the one obtained using the first approximation but with a coupling constant J_v instead of J. As J_v is larger than J at any temperature (for this particular potential) we can interpret J_v/J as a correction related to the fact that the Coulomb-gas charges run through all integers instead of being restricted as topological charges are.

C. Frustration

In the frustrated case, the Hamiltonian for the wire model (1) becomes

$$H = \frac{J}{2}\int \left(\frac{\partial \vec{S}_l}{\partial l} + A_l\right)^2, \tag{64}$$

with $A_l = \vec{A}_l \cdot \hat{l}$ where $\vec{A}$ is a vectorial field (experimentally, $\vec{A}$ is proportional to a vector potential of a transverse magnetic field) and $\hat{l}$ is the unit vector oriented in the direction of the wire. So the partition function (9) becomes

$$Z = \sum_{\{p_{rr'}\}} \int_{-\infty}^{\infty} [d\theta_r] \exp\left(-\beta \frac{J}{2}\sum_{\langle rr'\rangle}(\theta_r - \theta_{r'} - A_{rr'} + 2\pi p_{rr'})^2\right) \tag{65}$$

with $A_{rr'} = \int_r^{r'} d\vec{l}\vec{A}_l$.

Now the bond variables have to be defined as

$$\phi_{rr'} = \theta_r - \theta_{r'} - A_{rr'} + 2\pi p_{rr'}, \tag{66}$$

and therefore satisfy the constraints

$$\sum_{\square R} \phi_{rr'} = 2\pi(q_R - f_R), \tag{67}$$

where $f_R = (1/2\pi)\sum_{\square R} A_{rr'}$ $[=-(1/2\pi)\oint_R d\vec{l}\cdot\vec{A}_l]$.

In the frustrated case the bond variables are particularly convenient since the system is now described in terms of gauge invariant quantities. So we are interested in periodic boundary conditions in gauge invariant variables and that periodicity implies the condition of "neu-

trality" for the frustrated case: $\sum_R(q_R - f_R)=0$. Then, if we impose in addition the periodicity in the θ_r's, we have to satisfy an additional constraint for each direction α:

$$\sum_{r\in l_{\alpha a}} \phi_{\alpha r} = -2\pi(q_{\alpha a} - f_{\alpha a}) \tag{68}$$

with

$$f_{\alpha a} \equiv -\frac{1}{2\pi}\sum_{r\in l_{\alpha a}} A_{\alpha r}, \tag{69}$$

where the sum is taken as in Eq. (8).

Two remarks are in order. First, it has to be noted that, in the case of boundary conditions in the θ_r's, the gauge invariance is restricted. Indeed, the system is invariant under a gauge transformation provided that the closed integral of the vector potential around each of the two directions of the torus is conserved. This condition on two arbitrary orthogonal lines is sufficient for it to be satisfied as well on all the other lines.

Second, no periodicity has been imposed in the $A_{rr'}$'s (otherwise the sum of the f_R's over the whole lattice could only be equal to 0). So the usual uniform frustrated case (f_R has the same value on every plaquette) can be considered. However, in order to satisfy the condition $\sum_R(q_R - f)=0$, the size of the lattice must be adapted to any particular uniform frustration. For example, it is never possible for an irrational uniform frustration to have periodic boundary conditions in the $\phi_{rr'}$'s.

After decoupling, we find

$$H = H_{\mathrm{SW}} + H_{\mathrm{ch}}, \tag{70}$$

with

$$H_{\mathrm{ch}} = 2\pi^2 J\left[\sum_{RR'}(q_R - f_R)G'_{RR'}(q_{R'} - f_{R'}) + \left(\frac{\mathbf{P}}{L}\right)^2\right] \tag{71}$$

and

$$P_x = \sum_{i=1}^{L} i\sum_{j=1}^{L}(q_{ij} - f_{ij}) + L(q_{y1} - f_{y1}), \tag{72}$$

$$P_y = \sum_{j=1}^{L} j\sum_{i=1}^{L}(q_{ij} - f_{ij}) + L(q_{x1} - f_{x1}). \tag{73}$$

As for the partition function the sum over $\{q_R, q_{\alpha 0}\}$ is taken over all configurations such that $\sum_R(q_R - f_R) = 0$.

For the frustrated XY model the Hamiltonian becomes

$$H = \sum_{\langle rr'\rangle} V(\theta_r - \theta_{r'} - A_{rr'}) \tag{74}$$

and the $\phi_{rr'}$'s are defined as in Eq. (66). But in this case the $p_{rr'}$'s will be determined by the θ_r's in order to include the $\phi_{\alpha r}$'s in the interval $(-\pi, \pi]$. For Villain's interaction the partition function is again identical to the one of the frustrated wire model, but after decoupling the m_R's (that replace the q_R's) are the circulations of the $\tilde{\phi}_{rr'}$'s defined as

$$\tilde{\phi}_{rr'} = \theta_r - \theta_{r'} - A_{rr'} + 2\pi n_{rr'}. \tag{75}$$

For the piecewise interaction potential the Hamiltonian can also be split in terms of q_R's and spin waves. The remarks about the gauge invariant property and the uniform frustrated case for the wire model hold for the XY model.

III. MEANING OF THE CHARGE IN THE COULOMB GAS ASSOCIATED WITH AN XY MODEL

For this purpose, we consider a frustrated XY model with a nonzero frustration only around two plaquettes R_1 and R_2 without common nearest neighbors (see Fig. 5). We could also perform the same following calculations for the frustrated case and they do not depend on the type of boundary conditions. The bond frustration terms around these two plaquettes (R_k with k=1 or 2), turning in the clockwise direction, are ε_k. Therefore the plaquette frustrations are

$$f_R = \begin{cases} 4\varepsilon_k/2\pi & \text{if } R = R_k, \\ -\varepsilon_k/2\pi & \text{if } R \text{ is nearest neighbor of } R_k, \\ 0 & \text{else.} \end{cases}$$

In order to see the physical meaning of the m_R, we shall calculate in the following the second derivative

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\ln Z_\varepsilon \tag{76}$$

in two ways: first in terms of the original variables θ_r and second in terms of the m_R's. Of course the second way becomes possible only after performing Villain's approximation, therefore it will be exact only in the case of Villain's interaction.

A. Partition function derivatives

Before distinguishing the two ways, expression (76) can be expressed in terms of thermodynamical averages:

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\ln Z_\varepsilon = -\beta\left\langle\frac{\partial^2 H}{\partial\varepsilon_1\partial\varepsilon_2}\right\rangle + \beta^2\left\langle\frac{\partial H}{\partial\varepsilon_1}\frac{\partial H}{\partial\varepsilon_2}\right\rangle - \beta^2\left\langle\frac{\partial H}{\partial\varepsilon_1}\right\rangle\left\langle\frac{\partial H}{\partial\varepsilon_2}\right\rangle. \tag{77}$$

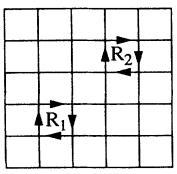

FIG. 5. The arrows indicate the frustrated bonds. $A_{rr'}=\varepsilon_k$ around R_k (k=1, 2) and rr' is to be considered with the same orientation as the arrow. $A_{rr'}=0$ elsewhere.

Let us consider Hamiltonian (74) in terms of the $\phi_{rr'}$'s. The partial derivatives, evaluated at $\varepsilon_1 = \varepsilon_2 = 0$ are

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\ln Z\Big|_{\varepsilon_1=\varepsilon_2=0} = \beta^2\left[\left\langle\left(\sum_{\square R_1}V'(\phi_{rr'})\right)\left(\sum_{\square R_2}V'(\phi_{rr'})\right)\right\rangle - \left\langle\sum_{\square R_1}V'(\phi_{rr'})\right\rangle\left\langle\sum_{\square R_2}V'(\phi_{rr'})\right\rangle\right]. \tag{78}$$

We shall call c_R the circulation of the derivative $V'(\phi_{rr'})$ around the plaquette R:

$$c_R \equiv \sum_{\square R} V'(\phi_{rr'}). \tag{79}$$

Now we note that the transformation $\theta_r \mapsto -\theta_r$ (or $\phi_{rr'} \mapsto -\phi_{rr'}$) on each node of the lattice does not change the Hamiltonian value (V is even by definition) but changes the sign of c_R. Thus we can write

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\ln Z\Big|_{\varepsilon_1=\varepsilon_2=0} = \beta^2\langle c_{R_1}c_{R_2}\rangle. \tag{80}$$

Let us now perform the same derivation, but considering the charge Hamiltonian $H_{\rm ch1}$ in terms of the quantities m_R (the frustration does not contribute to the spin-wave part in the φ_r's and $H_{\rm ch2}$ is invariant under the ε transformation) as obtained after using Villain's approximation:

$$H_{\rm ch1} = 2\pi^2 J_v \sum_{RR'}(m_R - f_{\varepsilon R})G'_{RR'}(m_{R'} - f_{\varepsilon R'}), \tag{81}$$

where $J_v{=}J_v(T)$ depends on the considered interaction potential. We obtain

$$\begin{aligned}\frac{\partial^2}{\partial\varepsilon_1\varepsilon_2}\ln Z_\varepsilon = &-\beta J_v\left\langle\sum_{RR'}h_{1R}G'_{RR'}h_{2R'}\right\rangle + \beta^2(2\pi J_v)^2\left\langle\left(\sum_{RR'}h_{1R}G'_{RR'}(m_{R'}-f_{R'})\right)\left(\sum_{RR'}h_{2R'}G'_{RR'}(m_R-f_R)\right)\right\rangle \\ &-\beta^2(2\pi J_v)^2\left\langle\sum_{RR'}h_{1R}G'_{RR'}(m_R-f_{R'})\right\rangle\left\langle\sum_{RR'}h_{2R}G'_{RR'}(m_R-f_{R'})\right\rangle,\end{aligned} \tag{82}$$

where

$$h_{kR} = \begin{cases} 4 & \text{if } R = R_k, \\ -1 & \text{if } R \text{ is nearest neighbor of } R_k, \\ 0 & \text{else.}\end{cases}$$

The functions h_{kR} appear in the derivatives of $f_{\varepsilon R}$ with respect to ε_k. Those functions correspond to the kth line of the matrix M, and therefore to those of the matrix $M_{c0} - c_0 P$, and G' is by definition the inverse of M_{c0} (cf. Sec. II A). Furthermore, considering the neutrality property $\sum_R(q_R - f_R){=}0$, the second variation at $\varepsilon_1 = \varepsilon_2 = 0$ becomes

$$\frac{\partial^2}{\partial\varepsilon_1\partial\varepsilon_2}\ln Z\Big|_{\varepsilon_1=\varepsilon_2=0} = \beta^2(2\pi J_v)^2\left(\langle m_{R_1}m_{R_2}\rangle - \langle m_{R_1}\rangle\langle m_{R_2}\rangle\right). \tag{83}$$

Now we note that the transformation $m_R \mapsto -m_R$ on each plaquette of the lattice does not change the Hamiltonian value but changes the sign of m_R. From Eqs. (80) and (83), we are now able to write

$$\langle c_R c_{R'}\rangle \approx [2\pi J_v(T)]^2\langle m_R m_{R'}\rangle. \tag{84}$$

This relation becomes exact for Villain's interaction. We have shown that the quantities m_R and c_R are correlated in the same way. This gives us an interpretation to the m_R's. Therefore, the meaning of m_R depends on the interaction potential of the XY model. Thus, in Villain's approximation, the circulation c_R around a plaquette is composed of a quantized contribution and fluctuations around this quantized value which are treated as decoupled spin waves.

Note that experimentally in Josephson junction arrays (corresponding to the cosine interaction), the quantity c_R corresponds to the supercurrent circulation around a plaquette R where the site R is a lattice node located at the center of four superconducting grains.

B. Examples of interaction potentials

Now we can interpret the m_R's in terms of the q_R's for the XY model and end this section with three examples.

1. *Villain's interaction*

The interaction potential is

$$V(\phi) = -T\ln\sum_n \exp\left(-\beta\frac{J_v}{2}(\phi+2\pi n)^2\right)$$

and the derivative becomes

$$V'(\phi) = \frac{J_v}{\sum_n \exp\left[-\beta\frac{J_v}{2}(\phi+2\pi n)^2\right]} \times\sum_n\left[\exp\left(-\beta\frac{J_v}{2}(\phi+2\pi n)^2\right)(\phi+2\pi n)\right]. \tag{85}$$

We shall denote $\langle\cdots\rangle_\phi$ such an average over all integers n (not to be considered as a thermodynamical average). Thus by definition we have

$$V'(\phi) = J_v \langle \phi + 2\pi n \rangle_\phi \tag{86}$$

and

$$\sum_{\square R} V'(\phi_{rr'}) = J_v 2\pi \left(q_R + \sum_{\square R} \langle n \rangle_{\phi rr'} \right). \tag{87}$$

Comparing the charge-charge correlation functions in q_R and m_R we find

$$\left\langle \left(q_R + \sum_{\square R} \langle n \rangle_{\phi rr'} \right) \left(q_{R'} + \sum_{\square R'} \langle n \rangle_{\phi rr'} \right) \right\rangle \tag{88}$$

$$= \langle m_R m_{R'} \rangle. \tag{89}$$

2. *Piecewise parabolic interaction*

Since the variables $\phi_{\alpha r}$ are defined in the interval $(-\pi, \pi]$ we can simply write

$$V(\phi) = \frac{J}{2}\phi^2. \tag{90}$$

Therefore

$$V'(\phi) = J\phi \tag{91}$$

and we simply find

$$J^2 \langle q_R q_{R'} \rangle \approx [J_v(T)]^2 \langle m_R m_{R'} \rangle. \tag{92}$$

So for the piecewise parabolic potential, the result (83) leads to a direct identification of the m_R's with the topological charges q_R.

3. *Cosine interaction*

The cosine interaction potential is

$$V(\phi) = -J\cos(\phi). \tag{93}$$

We can define the function $X(\phi)$ in order to write

$$\sin(\phi) = \phi + X(\phi) \tag{94}$$

which yields

$$J^2 \left\langle \left(q_R + \sum_{\square R} X(\phi_{rr'}) \right) \left(q_{R'} + \sum_{\square R'} X(\phi_{rr'}) \right) \right\rangle \tag{95}$$

$$\approx J_v^2 \langle m_R m_{R'} \rangle. \tag{96}$$

IV. HELICITY MODULUS AND INVERSE DIELECTRIC CONSTANT

The helicity modulus Γ for a spin system with periodic boundary conditions in the angles of the spins expressing the "rigidity" of the system with respect to an applied "twist" equal to $L\delta$ between two opposite boundaries is given by Ref. 13:

$$\Gamma = \frac{1}{N} \left. \frac{\partial^2 F}{\partial \delta^2} \right|_{\delta=0}. \tag{97}$$

Applying a twist implies that the periodic boundary conditions in the θ_r's in one direction have to be replaced by the condition that angles at opposite ends of the sample differ precisely by $L\delta$. Rather than changing the boundary conditions, it is, however, more convenient and strictly equivalent (it is simply a change of variables) to modify the frustration terms $A_{rr'}$ in the following way:

$$A'_{xr}(\delta) = A_{xr} + \delta\ , \quad A'_{yr}(\delta) = A_{yr}, \quad \forall r, \tag{98}$$

and to use again periodic boundary conditions. In fact, this corresponds to a gauge transformation leaving the frustration of all the plaquettes unchanged, but modifying the "frustration constraint" f_{xa} associated with the periodic boundary conditions.

Now the second derivative can be performed and expressed in terms of the thermodynamical average:

$$\Gamma = -\frac{T}{N}\frac{\partial^2}{\partial \delta^2} \ln Z$$
$$= \frac{1}{N} \left\{ \left\langle \frac{\partial^2 H}{\partial \delta^2} \right\rangle - \frac{1}{T} \left[\left\langle \left(\frac{\partial H}{\partial \delta} \right)^2 \right\rangle - \left\langle \frac{\partial H}{\partial \delta} \right\rangle^2 \right] \right\}. \tag{99}$$

In the nonfrustrated wire model or generalized XY model the term $\langle \partial H/\partial \delta \rangle^2$ vanishes at δ=0 because each state can be related to another one with the same energy and a derivative of opposite sign. In the frustrated case, we should adapt the gauge in order to fulfill the condition $\langle \partial H/\partial f \rangle = 0$ in $\delta = 0$. Thus we can write

$$\Gamma = \frac{1}{N} \left[\left\langle \left. \frac{\partial^2 H}{\partial \delta^2} \right|_{\delta=0} \right\rangle - \frac{1}{T} \left\langle \left(\left. \frac{\partial H}{\partial \delta} \right|_{\delta=0} \right)^2 \right\rangle \right]. \tag{100}$$

A. The wire model

Now we map the helicity modulus of the nonfrustrated wire model onto an inverse dielectric constant of a two-dimensional lattice Coulomb gas. The frustrated case could be treated in the same way. Since, in the charge representation, the frustration does not affect H_{SW} and the frustration variation δ does not change the f_R's, we can consider H_{ch2} instead of H in Eq. (100):

$$H_{\mathrm{ch2}} = 2\pi^2 J \left[\left(\frac{P_x}{L} \right)^2 + \left(\frac{P_y}{L} \right)^2 \right], \tag{101}$$

where P_x and P_y are given by Eqs. (72) and (73) with $f_R = 0$, $f_{y1} = 0$, $f_{x1} = -L\delta/(2\pi)$, so that we have $P_y(\delta) = P_y(\delta = 0) + L^2\delta/(2\pi)$ and we can write

$$H_{\mathrm{ch2}} = H_{\mathrm{ch2}}(\delta = 0) + \delta 2\pi J P_y(\delta = 0) + \frac{J}{2} L^2 \delta^2 \tag{102}$$

and

$$\Gamma = J\left(1 - \frac{4\pi^2 J}{TN}\langle P_y^2\rangle\right). \quad (103)$$

Since the two axis directions of the model are equivalent, we have $\langle P_x^2\rangle = \langle P_y^2\rangle$. And as we can find for any state of polarization (P_x, P_y) another one with the same energy and with a polarization $(P_x, -P_y)$, the quantity $\langle P_x P_y\rangle$ vanishes. Therefore, the helicity modulus becomes

$$\Gamma = J\left(1 - \frac{2\pi^2 J}{TN}\langle \mathbf{P}^2\rangle\right). \quad (104)$$

Now we can write $H_{\rm ch}$ and Γ in terms of a Coulomb gas in electric variables with an electric polarization $\mathbf{P}_e = e\mathbf{P}$ [$e = (2\pi J)^{1/2}$ is the elementary charge] and an external electric field $\mathbf{D} = (0, -e\delta)$:

$$H(\mathbf{D}) = H(\mathbf{D}=0) - \mathbf{D}\cdot\mathbf{P}_e(\mathbf{D}=0) + \frac{L^2}{4\pi}\mathbf{D}^2 \quad (105)$$

and

$$\Gamma = J\left(1 - \frac{\pi}{TN}\langle \mathbf{P}_e^2\rangle\right), \quad (106)$$

where $\mathbf{P}_e(\mathbf{D}=0)$ means [as in Eq. (102)] that $\mathbf{P}_e$ is defined with $f_{\alpha 1} = 0$.

Thus we can represent the twist δ in the direction x as an external electric field $\mathbf{D}$ that interacts linearly with the electric polarization $\mathbf{P}_e$. In order to establish the link between Γ and the dielectric constant ε, we define $1/\varepsilon$ as the response of the macroscopic electric field to a uniform external one:

$$\frac{1}{\varepsilon} \equiv \lim_{D_x\to 0}\frac{E_x}{D_x}. \quad (107)$$

By the use of the usual electrostatic relation

$$E_x = D_x - 2\pi\frac{\langle P_{ex}\rangle}{L^2}, \quad (108)$$

where P_{ex} corresponds to $P_{ex}(\mathbf{D}=0)$ as in Eq. (105) and expressing $\langle P_{ex}\rangle$ by the free energy derivative with respect to D_x,

$$\frac{1}{\varepsilon} = 1 - \frac{2\pi}{L^2}\lim_{D_x\to 0}\frac{\partial F/\partial D_x - \frac{L^2}{2\pi}D_x}{D_x} = \frac{2\pi}{L^2}\left.\frac{\partial^2 F}{\partial D_x^2}\right|_{D_x=0}. \quad (109)$$

Then the derivative is performed by Eq. (100) with Hamiltonian (105) and D_x replacing δ:

$$\frac{1}{\varepsilon} = 1 - \frac{2\pi}{TN}\langle P_{ex}^2\rangle = 1 - \frac{\pi}{TN}\langle \mathbf{P}_e^2\rangle. \quad (110)$$

Thus Γ/J is exactly represented by the dielectric constant of the associated Coulomb gas, $\Gamma/J = 1/\varepsilon$.

Furthermore, we can now correctly interpret our definition of the polarization in a dynamic Coulomb gas. When the charges are moving in an electric field, we have to take into account the winding number of the charges around the torus. Our definition of $\mathbf{P}$ corresponds then to the sum of the small displacements of the charges and therefore the term $-\mathbf{DP}$ is well represented by the coupling energy between the charges and the external electric field. But, in order to define such a displacement, we have to consider either a temporal succession of states (i.e., the dynamical version) or two additional fictitious variables that have no electric meaning in a static system.

Usually the susceptibility χ or the dielectric function ε of a Coulomb gas is defined in the Fourier representation. Then the dielectric constant is defined as the linear response to an oscillating field in the limit of large wavelengths. However, it has been known for a long time that in the usual Coulomb gas the linear response to a uniform electric field diverges ($\chi, \varepsilon \to \infty$) in the thermodynamical limit (even at zero temperature) for periodic or free boundary conditions. The reason is that for an arbitrary nonzero electric field there always exists a finite distance of separation of the pairs beyond which the energy of the system decreases as their distance increases. This problem has been studied by Choquard, Piller, and Rentsch,[15] who considered various boundary conditions and showed that, even in cases where the susceptibility remains finite, it depends on the boundary conditions even at the thermodynamical limit. In our case at low temperature the response to a uniform external electric field remains finite because the additional polarization energy stabilizes the system. In the case of fixed boundary conditions, the wire model also has this additional polarization term but the definition of the polarization would not involve additional variables. For example, if we impose $\phi_{rr'} = 0$ on the boundary the charge representation would correspond to a Coulomb gas stabilized by the use of the image method (where a charge and its four images all have the same sign). In the case of free boundary conditions the system does not contain the additional polarization energy and the two additional variables are replaced by two additional modes [representing σ_x and σ_y as defined below Eq. (17) with Hamiltonian $H_{\rm ch2}$ (21)] which are independent of $H_{\rm SW}$ and $H_{\rm ch1}$ and take any real value.

Alternatively, in order to map the wire model on a Coulomb gas whose only variables are charges on the sites of the dual lattice we can integrate the partition function (48) over the $q_{\alpha 0}$'s and the mean square polarization appearing in Eq. (103), with P_α expressed as

$$P_\alpha = P_{\pi\alpha} + q_{\alpha 0}. \quad (111)$$

We find

$$Z_{\rm ch} = \sum_{\{q_R\}}\exp\left\{-\beta\left[H_{\rm ch1} + V_v\left(2\pi\frac{P_{\pi x}}{L}\right) + V_v\left(2\pi\frac{P_{\pi y}}{L}\right)\right]\right\}, \quad (112)$$

$$\Gamma = J\left[1 - \frac{2J}{T}\left\langle\frac{\partial}{\partial J}V_v\left(\frac{2\pi P_{\pi y}}{L}\right)\right\rangle\right]. \quad (113)$$

Here the occurrence of Villain's function is not due to an approximation. Thus we can also interpret $Z_{\rm ch}$ as the partition function of a Coulomb gas. In that case, the

twist would be represented by an external field with a nonlinear coupling to the polarization. Moreover, in this case an electric state $\{q_R\}$ represents a whole family of topological configurations.

B. Generalized XY model

In order to have a good understanding of the helicity modulus of the XY model in its Coulomb-gas representation, we can express it by performing the derivation of the Hamiltonian in Eq. (100) in the two representations. In terms of the variables $\phi_{rr'}$,

$$\Gamma = \frac{1}{N}\left[\left\langle \sum_r V''(\phi_{xr})\right\rangle - \frac{1}{T}\left\langle \left(\sum_r V'(\phi_{xr})\right)^2\right\rangle\right], \tag{114}$$

and in terms of the quantities m_R and $m_{\alpha 1}$, after the use of Villain's approximation:

$$\Gamma \approx J_v - \frac{1}{TN}\langle (2\pi J_v P_y)^2\rangle \tag{115}$$

with

$$P_y = \sum_{ij} j m_{ij} + L m_{x1}. \tag{116}$$

In both expressions for Γ the two quantities whose mean square is taken are the first derivatives $\partial H/\partial\delta$ at $\delta = 0$. Like this we can interpret the polarization in terms of the original variables.

Note that this interpretation is completely consistent with our results of Sec. III. Indeed, we could also express exactly the sum of the derivatives $V'(\phi_{xr})$ in Eq. (114) as a polarization of the circulations c_R [definition (79)] following the same computational steps [Eqs. (37) and (38)] as for the wire model. We obtain

$$\sum_r V'(\phi_{xr}) = \sum_{i,j} j c_{ij} + L c_{x1}, \tag{117}$$

where c_{ij} is the circulation of the current around a plaquette located by i and j. The quantity c_{x1} would be defined as the sum of the derivative $V'(\phi_{xr})$ over the horizontal reference line l_{x1}. After the use of Villain's approximation we can of course apply the results of Sec. IV A with the m_R's and $m_{\alpha 0}$ replacing the q_R's and the $q_{\alpha 0}$'s.

We conclude this section by discussing once more the three examples of interaction potentials.

For Villain's interaction the two expressions for Γ, given in Eqs. (114) and (115), are, of course, equal and Γ/J is exactly represented by an inverse dielectric constant even on a torus.

For the piecewise parabolic potential, we have to pay attention to a δ function appearing for $V''(\phi_{rr'})$ in Eq. (114), coming from the point where two parabolas in the potential join each other.

As for the cosine interaction, corresponding to Josephson junction arrays, the polarization represents the macroscopic current in the array and this current can also be expressed exactly as a "polarization" of elementary supercurrent loops.

V. CONCLUSION

We have studied the Coulomb-gas representation of the 2D XY model on a torus. We started from a phase field defined on a square wire network and with a Hamiltonian depending on the square gradient of the phase along the wire. After integrating the phase fluctuations between the nodes of the lattice, the partition function of this model becomes mathematically identical to the XY model with Villain's interaction. We mapped the wire model onto a lattice Coulomb gas defined on a torus. The charges of the Coulomb gas represent the topological charges of the wire model. The Coulomb gas contains an additional polarization energy where the polarization has been defined by the use of two additional integer variables. The latter come from the topological constraints expressing the periodic boundary conditions. Indeed, the topological charges corresponding to the plaquettes are not sufficient to define completely the topology of a given state of the system. The two additional variables allow us to define a polarization and, in a dynamic case, they correspond to the difference between the winding numbers of the positive charges and the negative ones around each direction of the torus. The twist δ allowing one to define the helicity modulus can be represented as an external uniform electric field, and therefore the helicity modulus is exactly related to the inverse of a dielectric constant. In the thermodynamic limit with a nonzero twist, the system is stable and the Coulomb gas with a nonzero electric field is also stable because of the additional polarization energy in the mapped Coulomb gas.

Alternatively we integrated the partition function over the two additional variables. Thus we obtained a Coulomb-gas representation whose charge configuration represents a family of topological configurations, and twist no longer corresponds to an external electric field coupling linearly with the charges of this Coulomb gas.

Since the wire model and the XY model with Villain's interaction possess a partition function with an identical structure, we can perform exactly the mapping for this particular XY model. But then the charges of the Coulomb gas do not represent the topological charges as defined in terms of the original variables.

We have derived expressions giving the meaning of the charges of the Coulomb gas associated with the XY model with Villain's interaction in terms of the original variables. For XY models with other potentials, the charges of the Coulomb gas obtained after performing Villain's approximation correspond to the circulations of the interaction potential derivatives around the plaquettes.

We have emphasized the case of the piecewise parabolic potential since it permits an exact splitting of the Hamiltonian into two independent terms: one expressing the topological charges and the other one the fluctuations around them. But the two subsystems are not really independent because the fluctuations have to belong to a

phase space determined by the topological configuration.

Our results apply as well for the frustrated models. We have performed our calculations with gauge invariant bond variables but on the torus this gauge invariance is restricted. Indeed, the usual gauge invariance is broken by periodic or fixed boundary conditions (imposed on the original variables).

Thus we have clarified the Coulomb gas representation of the 2D XY model on a torus. In particular we have expressed exactly the helicity modulus of the XY model with Villain's interaction in this representation. It becomes the inverse of a dielectric constant that we can define as the response of the internal electric field to an external uniform electric field. This will permit one to perform Monte Carlo simulations in a charge representation to evaluate the helicity modulus corresponding exactly to its definition in the spin variables.

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