PHYSICAL REVIEW B VOLUME 50, NUMBER 1 1 JULY 1994-I

Cooper-pair current through ultrasmall Josephson junctions

Gert-Ludwig Ingold and Hermann Grabert*
Fachbereich Physik, Universität Essen, D-45117 Essen, Germany

Udo Eberhardt
Fachbereich Physik, Universität Duisburg, D-47048 Duisburg, Germany
(Received 21 December 1993)

The current-voltage characteristics of ultrasmall Josephson junctions sensitively depend on the electromagnetic environment of the junction. When the charging energy exceeds the Josephson coupling energy, the usual supercurrent at zero voltage is completely suppressed. However, for typical environmental impedances, which are small compared to the resistance quantum, stochastic Cooper-pair tunneling leads to a supercurrent peak at a small finite voltage which is proportional to the temperature and the low-frequency resistance of the external circuit. An analytic expression for the form of this universal peak, which is independent of the high-frequency behavior of the environment, is given. With increasing Josephson coupling the peak merges into the usual supercurrent of a Josephson junction. At larger voltages the Cooper-pair current depends on details of the environment. Current peaks are shown to result from resonances in the environmental impedance. Specifically, the case of an LC transmission line of finite length is discussed.

I. INTRODUCTION

Single charge tunneling phenomena are macroscopic manifestations of the microscopic charge quantization, which have attracted a great deal of interest recently.[1] These phenomena arise at low temperatures in ultrasmall tunnel junction systems with capacitances below 1 fF. It has been noted that for single junction systems the electromagnetic environment of the junction crucially affects the charge tunneling rates leading to a washout of the Coulomb blockade effects under standard conditions.[2–4] Only in the extreme case of high-resistance leads with impedances above 10 kΩ may effects originally predicted for isolated junctions[5–8] become observable. Experiments on single junctions in a high-impedance environment have been performed both for the case of normal junctions[9,10] and Josephson junctions[11] with results that are in fair agreement with theoretical predictions.[2–4,12–14]

The need for a high-impedance environment is unfortunately in conflict with the requirement of no heating and no electron localization in the resistor. Hence, manifestations of charge quantization in single tunnel junctions shunted by standard low-impedance environments with typical impedances of 100 Ω are of particular interest. For normal tunnel junctions such signatures of the Coulomb blockade effect are a high voltage shift of the current-voltage characteristic and a zero bias anomaly of the conductance.[2–4] Both effects have been seen in recent experiments[15–17] although quantitative confirmation of the theoretical predictions is still lacking.

Much of the earlier theoretical investigations on small capacitance Josephson junctions were based on the model of a current biased junction.[18] However, it has been pointed out by Devoret *et al.*[3] that for junction capacitances in the fF range or below, the parasitic capacitances of the external circuit always act as an effective voltage bias. Another important effect of the electromagnetic environment is the exchange of energy between the tunneling particle and the electromagnetic modes of the circuit. For normal tunnel junctions with a tunneling resistance that exceeds the resistance quantum, the influence of the electromagnetic environment on the tunneling rates can be calculated by treating the tunneling Hamiltonian as a perturbation.[3,4] This approach can be extended[12–14] to the case of a Josephson junction, provided the Josephson coupling energy is small compared with the charging energy.[19]

In Sec. II we first introduce a Hamiltonian describing the influence of the electromagnetic environment on ultrasmall Josephson junctions and review the derivation of Cooper-pair tunneling rates. We then study two pronounced effects occurring in the presence of a low-impedance environment. In Sec. III we show that at low temperatures a supercurrent peak appears in the current-voltage characteristic near zero bias. The peak is found to be independent of the detailed frequency dependence of the environmental impedance. We give an extended presentation of our earlier work[20,21] on the supercurrent peak. In Sec. IV we show that the peak is usually only weakly affected by the large environmental impedance in the kHz range observed in standard experimental setups. The relation between the peak and the usual supercurrent is clarified in Sec. V, where the phase diffusion in a voltage-biased Josephson junction is discussed. Additional peaks in the current-voltage characteristic near a larger voltage $\hbar\omega_0/2e$ are possible if the environmental impedance $Z(\omega)$ has a peak at the frequency ω_0.[12] This is due to the fact that the energy $2eV$ gained by a Cooper pair transferred through a junction biased at voltage V

has to be absorbed by the external circuit. In Sec. VI we study the position and form of peaks in the current-voltage characteristic for the specific model of a finite LC transmission line.[22]

II. INFLUENCE OF THE ENVIRONMENT

We start out from the Hamiltonian

$$H = \frac{Q^2}{2C} - E_J \cos(\varphi) + \sum_{n=1}^{N} \left[\frac{q_n^2}{2C_n} + \left(\frac{\hbar}{e}\right)^2 \frac{1}{2L_n} \left(\frac{\varphi}{2} - \frac{e}{\hbar} Vt - \varphi_n\right)^2 \right], \quad (1)$$

which describes a Josephson junction coupled to environmental modes. The junction is characterized by a capacitance C, carrying the charge Q, and a coupling term depending on the phase difference φ across the junction. The Josephson coupling energy E_J is related to the bare critical current I_c by $E_J = (\hbar/2e)I_c$. In the limit $N \to \infty$ the capacitances C_n and inductances L_n provide a Caldeira-Leggett model[23] for an arbitrary environmental impedance

$$Z(\omega) = \left[\int_0^\infty dt \exp(-i\omega t) \sum_{n=1}^{N} \frac{\cos(\omega_n t)}{L_n} \right]^{-1}, \quad (2)$$

where $\omega_n = (L_n C_n)^{-1/2}$. The phases φ_n are canonically conjugate to the charges q_n with the commutation relation $[\varphi_n, q_n] = ie$. In contrast, $[\varphi, Q] = 2ie$ to conform with the usual definition of the phase difference across the junction. Finally, V is the external voltage. The Hamiltonian (1) neglects quasiparticle excitations, which is adequate at temperatures well below the critical temperature of the superconductor and voltages below the gap voltage.

In view of $\exp(i\varphi)Q\exp(-i\varphi) = Q - 2e$, the Josephson coupling term $E_J\cos(\varphi) = (E_J/2)\exp(i\varphi) + \text{H.c.}$ is a Cooper-pair tunneling term changing the junction charge Q by $2e$. For $E_J \ll E_c = 2e^2/C$ this term can be treated perturbatively. After tracing out the environmental degrees of freedom one finds for the forward tunneling rate to second order in the Josephson coupling energy[19]

$$\vec{\Gamma}(V) = \frac{E_J^2}{4\hbar^2} \int_{-\infty}^{+\infty} dt \exp\left(i\frac{2e}{\hbar}Vt\right) \langle e^{i\varphi(t)} e^{-i\varphi(0)} \rangle. \quad (3)$$

The thermal average of the correlation function has to be taken with respect to a canonical distribution at inverse temperature $\beta = 1/k_BT$ defined by the Hamiltonian (1) for $V = 0$ and $E_J = 0$. Since this Hamiltonian is harmonic, the correlation function in (3) may be expressed as $\langle e^{i\varphi(t)}e^{-i\varphi(0)}\rangle = \exp[J(t)]$ with $J(t) = \langle[\varphi(t) - \varphi(0)]\varphi(0)\rangle$. The phase correlation function $J(t)$ depends on the total impedance

$$Z_t(\omega) = \frac{1}{i\omega C + Z^{-1}(\omega)} \quad (4)$$

of the junction capacitor in parallel with the external impedance according to

$$J(t) = 2\int_{-\infty}^{+\infty} \frac{d\omega}{\omega} \frac{\text{Re}Z_t(\omega)}{R_Q} \frac{e^{-i\omega t} - 1}{1 - e^{-\beta\hbar\omega}}, \quad (5)$$

where $R_Q = h/4e^2$ is the resistance quantum. In terms of the phase correlation function the tunneling rate (3) may be written as $\vec{\Gamma}(V) = (\pi E_J^2/2\hbar)P(2eV)$, where

$$P(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dt \exp\left[J(t) + \frac{i}{\hbar}Et\right] \quad (6)$$

gives the probability that a tunneling Cooper pair emits the energy E to the environment. Since Cooper pairs have no kinetic energy, tunneling is only possible if the energy $2eV$ provided by the voltage source is entirely transferred to the environment.

Clearly, from the symmetry of the circuit the backward tunneling rate is related to the forward tunneling rate by $\overleftarrow{\Gamma}(V) = \vec{\Gamma}(-V)$. Hence, the total Cooper pair current is given by[12–14,19]

$$\begin{aligned} I(V) &= 2e\left[\vec{\Gamma}(V) - \overleftarrow{\Gamma}(V)\right] \\ &= \frac{\pi e E_J^2}{\hbar}\left[P(2eV) - P(-2eV)\right]. \end{aligned} \quad (7)$$

III. COOPER-PAIR CURRENT PEAK AT LOW VOLTAGE

To determine the Cooper-pair current at low voltages it is convenient to shift the integration contour in (6) by $-i\hbar\beta/2$ yielding

$$P(E) = \frac{1}{2\pi\hbar}\exp(\beta E/2) \times \int_{-\infty}^{+\infty} dt \exp\left[J(t - i\hbar\beta/2) + \frac{i}{\hbar}Et\right]. \quad (8)$$

Since $J(t - i\hbar\beta/2)$ is real and symmetric the detailed balance relation $P(-E) = \exp(-\beta E)P(E)$ becomes apparent. From (5) we find that for long times and at low temperatures the phase correlation function may be written as

$$J(t - i\hbar\beta/2) = -2\rho\left\{\ln\left[\cosh\left(\frac{\pi t}{\hbar\beta}\right)\right] + \ln\left(\frac{\beta E_c}{\pi^2\rho}\right) + \zeta\right\}, \quad (9)$$

where

$$\rho = Z(0)/R_Q \quad (10)$$

and where we have omitted terms of order $\rho/\beta E_c$ as well as terms decaying exponentially fast in time. All terms in (9) except for the last one, depend only on the low-frequency behavior of the impedance determined by the parameter ρ. The details of the frequency dependence of the total impedance enter through the constant

$$\zeta = \gamma + \int_0^\infty \frac{d\omega}{\omega}\left[\frac{\text{Re}Z_t(\omega)}{\rho R_Q} - \frac{1}{1 + (\pi\rho\hbar\omega/E_c)^2}\right], \quad (11)$$

where $\gamma = 0.5772\ldots$ is Euler's constant. The integral in (11) vanishes for the special case of an ohmic external impedance $Z(\omega)/R_Q = \rho$, i.e., $\zeta_{\rm ohmic} = \gamma$. On the other hand, for an environmental impedance $Z(\omega) = i\omega L + R$ in series with the ultrasmall junction one has

$$\zeta_{\rm LCR} = \gamma - \ln(Q) - \frac{1-2Q^2}{(1-4Q^2)^{1/2}} \text{artanh}[(1-4Q^2)^{1/2}], \qquad (12)$$

where $Q = (L/C)^{1/2}/R$ is the quality factor of the LCR circuit.

The exponentially decaying terms omitted in (9) do not contribute to the low-energy behavior of $P(E)$ and are irrelevant for the current-voltage characteristics at low voltages. Inserting the result (9) into (6) with the shifted contour, one finds that the integral can be evaluated analytically in terms of the gamma function. Combining the result with (7), the current-voltage characteristic is found to read[20,21]

$$I(V) = \frac{\pi}{2} I_c \frac{E_J}{E_c} \rho^{2\rho} \left(\frac{\beta E_c}{2\pi^2}\right)^{1-2\rho} \times \exp[-2\zeta\rho] \frac{|\Gamma(\rho - i(\beta eV/\pi))|^2}{\Gamma(2\rho)} \sinh(\beta eV). \quad (13)$$

We note that in view of a duality between incoherent tunneling of charge and phase,[18,19] the supercurrent (13) through a voltage biased Josephson junction is dual to the finite voltage across a weakly current biased Josephson junction caused by macroscopic quantum tunneling.[24–26]

Although the current-voltage characteristic (13) is mainly determined by the low-frequency impedance $Z(0)$ it applies to arbitrary environmental impedances with a finite zero-frequency limit in the sense discussed above. The high-frequency behavior of the impedance enters only through ζ defined in (11). Due to this universal nature of the supercurrent peak it should be observable under feasible experimental conditions. Of course, the range of validity of the current-voltage characteristic (13) is limited by the approximations made. We have evaluated the tunneling rate for small E_J and have used an approximation of the phase correlation function valid for long times and low temperatures. The low-temperature approximation leads to the requirement $\beta E_c \gg \pi\rho$. For ultrasmall junctions and realistic values for ρ this condition may in practice be less stringent than $T \ll T_c$, where T_c is the critical temperature of the superconductor. The long-time approximation restricts the range of validity of the current-voltage characteristic to small voltages $V \ll \hbar\omega_Z/e$, where ω_Z is a characteristic frequency above which $Z_t(\omega)$ deviates significantly from $Z(0)$. For an ohmic impedance $\omega_Z = 1/RC$. Typically, $\hbar\omega_Z \gg E_c$, and the range of voltages where (13) holds is large on the voltage scale E_c/e of interest. Finally, from a study of higher-order terms in E_J, one finds that the maximum $P_{\rm max}$ of $P(E)$ should satisfy the condition $P_{\rm max}E_J \ll 1$. For small impedances ρ this gives the condition $\beta E_J \ll \rho$, which puts a lower bound on temperature. Together with the first requirement given above, one finds that (13) holds for a large range of temperatures if the condition $E_J \ll E_c$ is fulfilled.

We now take a closer look at the form of the current-voltage characteristic (13). As we have mentioned above it exhibits a peak at small voltages provided the zero frequency impedance ρ is sufficiently small. For $\rho \ll 1$ one finds that the current has a maximum at the voltage

$$V_{\rm max} = \frac{\pi\rho}{e\beta}[1 + 4\zeta(3)\rho^3 + \cdots], \qquad (14)$$

where $\zeta(3) = 1.202\ldots$ is a Riemann number. Note that $V_{\rm max}$ is proportional to T. Inserting $V_{\rm max} = \pi\rho/e\beta$ into (13) the maximum current is found to be[21]

$$I_{\rm max} = \frac{\pi}{2} I_c \frac{E_J}{E_c} \rho^{2\rho} \left(\frac{\beta E_c}{2\pi^2}\right)^{1-2\rho} \times \exp[-2\zeta\rho] \sinh(\pi\rho) \frac{|\Gamma(\rho(1-i))|^2}{\Gamma(2\rho)}, \qquad (15)$$

which is proportional to $T^{-1+2\rho}$. For $\rho \to 0$ (15) reduces to $I_{\rm max} = (eE_J^2/2\hbar)\beta$. The theory ceases to hold when $I_{\rm max}$ becomes comparable to ρI_c.

Another quantity of interest is the zero bias differential conductance. From (13) one immediately gets

$$R_Q \left.\frac{dI}{dV}\right|_{V=0} = \pi^4 \left(\frac{E_J}{E_c}\right)^2 \rho^{2\rho} \frac{\Gamma(\rho)^2}{\Gamma(2\rho)} \times \exp[-2\zeta\rho] \left(\frac{\beta E_c}{2\pi^2}\right)^{2-2\rho} \qquad (16)$$

which scales as $T^{-2+2\rho}$ for small temperatures. In the common case where ρ is much smaller than one, the zero bias differential conductance diverges for decreasing temperature. This is an indication of the zero bias anomaly of the current-voltage characteristics found at zero temperature where $I \sim V^{2\rho-1}$.[13,14]

The supercurrent peak at low voltages is shown in Fig. 1 for an environmental impedance $Z(0) = 100\Omega$ and different temperatures. For $\rho \ll 1$ there is a distinct peak at low temperatures, which becomes more pronounced as the temperature is lowered. We remark that, while Fig. 1 shows the analytical result (13), the appearance of a

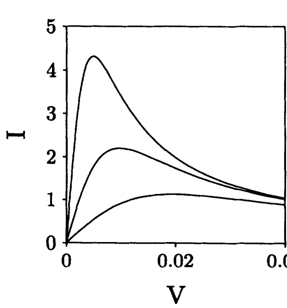

FIG. 1. Supercurrent-voltage characteristics, Eq. (13), for an ultrasmall Josephson junction coupled to an environment with a low-frequency resistance of 100 Ω. The temperature $k_BT/E_c = 0.05, 0.1, 0.2$ increases from the upper curve to the lower curve. The voltage is given in units of $E_c/2e$, while the current is given in units of $I_c(E_J/E_c)\exp[-2\zeta\rho]$.

peak in the Cooper-pair current has been seen in earlier theoretical work[13,14] based on a numerical evaluation of Eqs. (5)–(7).

IV. ROLE OF LOW-FREQUENCY IMPEDANCE

The results in the preceding section show that the supercurrent peak depends on the parameter $\rho = Z(0)/R_Q$. The question now arises what has to be inserted for $Z(0)$. We first note that the relevant frequency scale is of the order of $E_c/\hbar$, which for ultrasmall Josephson junctions with capacitances of 10^{-15} F yields 10^{11} Hz. As mentioned before, it is very difficult to place impedances, which at these frequencies are of the order of the resistance quantum, close to the junction. We therefore have assumed that the external impedance in the relevant frequency range is small compared to R_Q. However, at very low frequencies the environmental impedance can easily be in the MΩ range. Yet, such a high impedance at very low frequencies is not likely to affect the calculation given in the preceding section. This can be shown by considering an external impedance $Z(\omega) = R + (i\omega C_0 + 1/R_0)^{-1}$ consisting of a small resistance R in series with an RC circuit of large resistance R_0 and low cutoff frequency $\omega_\ell = (R_0C_0)^{-1}$. We may express the total impedance approximately as a sum of two terms, $Z_t(\omega) = Z_1(\omega) + Z_2(\omega)$, where $Z_1(\omega) = R/(i\omega/\omega_u + 1)$ with $\omega_u = (RC)^{-1}$ contains the junction capacitance and the relevant high-frequency resistance, while $Z_2(\omega) = R_0/(i\omega/\omega_\ell + 1)$ is the high-impedance contribution at very low frequencies. Now the probability (6) is given by the convolution

$$P(E) = \int_{-\infty}^{+\infty} dE' P_1(E - E')P_2(E') \tag{17}$$

of the two probabilities $P_1(E)$ and $P_2(E)$, which are determined by the total impedances $Z_1(\omega)$ and $Z_2(\omega)$, respectively, replacing $Z_t(\omega)$ in (5). Since $R_0/R_Q \gg 1$, we find[19]

$$P_2(E) = (4\pi E_0 k_B T)^{-1/2} \exp\left[-\frac{(E-E_0)^2}{4E_0k_BT}\right], \tag{18}$$

which is a Gaussian centered at an energy $E_0 = 2e^2/C_0 = E_c(R_0/R)(\omega_\ell/\omega_u)$. Since ω_ℓ/ω_u is small, we have $E_0 \ll E_c$ provided that R_0/R does not become too large. To get an estimate on the influence of the low-frequency impedance we insert (18) into (17) and replace $P_1(E)$ by a Gaussian with width σ_E centered at E_1. The changes in position and width of P_1 due to P_2 are negligible if $E_0 \ll E_1$ and $E_0k_BT \ll \sigma_E^2$. For the supercurrent peak, where both E_1 and σ_E are of order $\pi\rho/\beta$, an analysis of these conditions for the relevant range of temperatures yields $R_0\omega_\ell/R\omega_u \ll 4\pi\rho$. This requires that the area under the high impedance low-frequency part ($\propto R_0\omega_\ell$) is small compared to the area under the high-frequency total impedance $Z_t(\omega)$. Provided this condition is fulfilled the form of the supercurrent peak is little affected by the large low-frequency impedance. In the formulas of the preceding section the zero-frequency impedance $Z(0)$ should then in fact be replaced by the environmental impedance above the cutoff frequency ω_ℓ, typically the impedance in the upper MHz range.

V. RELATION TO CLASSICAL PHASE DIFFUSION

The results derived in Sec. III for a voltage biased ultrasmall Josephson junction in a low-impedance environment are related to the well-known phenomenon of phase diffusion in an overdamped Josephson junction.[27,28] For small impedances $Z(0)$ and large temperatures with $\beta eV \ll 1$, (13) reduces to

$$I(V) = \frac{1}{2}I_c^2 \frac{Z(0)V}{V^2 + [2eZ(0)k_BT/\hbar]^2}. \tag{19}$$

For the zero-bias differential resistance we obtain

$$R_0 = \left(\left.\frac{dI}{dV}\right|_{V=0}\right)^{-1} = 2Z(0)\left(\frac{k_BT}{E_J}\right)^2. \tag{20}$$

These results may be related to the current-voltage characteristic calculated from classical phase diffusion. For a voltage biased junction and small environmental impedance it is appropriate to consider the so-called Smoluchowski limit where the capacitance drops out and charging effects are no longer present. This is indeed the case in (19) and (20). The current-voltage characteristic in the phase diffusion regime may be expressed in terms of modified Bessel functions of complex order.[29] The result may be written in the compact form

$$I = I_c \mathrm{Im}\left(\frac{I_{1-i\nu}(E_J/k_BT)}{I_{-i\nu}(E_J/k_BT)}\right), \tag{21}$$

where $\nu = eV/\pi k_BT\rho$. In the limit of small Josephson coupling $E_J \ll k_BT$ one recovers the results (19) and (20). For arbitrary E_J one defines the zero bias differential resistance with respect to the voltage drop V_J across the junction and obtains[29,30]

$$R_0 = \left(\left.\frac{dI}{dV_J}\right|_{V_J=0}\right)^{-1} = \frac{Z(0)}{I_0^2(E_J/k_BT) - 1}. \tag{22}$$

For small E_J this leads again to (20), since in this limit it is not necessary to distinguish between the external voltage V and the voltage drop V_J across the junction. On the other hand, for $E_J \gg k_BT$ one finds

$$R_0 = 2\pi Z(0)\frac{E_J}{k_BT}\exp(-2E_J/k_BT), \tag{23}$$

which is associated with thermally activated jumps of the phase across the Josephson potential barrier. In view of the relation to classical phase diffusion the peak (13) in the Cooper-pair current-voltage characteristic may be viewed as a remnant of the usual supercurrent for ultrasmall junctions.

VI. CURRENT PEAKS FROM ENVIRONMENTAL RESONANCES

We now turn to the discussion of peaks in the current-voltage characteristic, which have their origin in resonances of the total impedance. To be definite, we consider a finite LC transmission line terminated by an ohmic load resistance R_L. Such an environment is of experimental interest.[31] The transmission line may be approximated by an LC ladder with specific inductance L_0 and specific capacitance C_0 as shown in Fig. 2. Here, we neglect resistances in series with the inductances as well as conductances in parallel with the capacitances. This is a good approximation for low resistive transmission lines. The infinite LC transmission line is characterized by its resistance $R_\infty = (L_0/C_0)^{1/2}$ and the velocity $u = (L_0C_0)^{-1/2}$ for wave propagation on the line. The load resistance terminating the finite transmission line is described by the dimensionless parameter $\rho = R_L/R_Q$, which coincides with $Z(0)/R_Q$ introduced previously. In addition, we introduce the ratio $r = R_L/R_\infty$ between the load resistance and the resistance of the infinite LC transmission line. The length of the finite line can be parametrized by the $\lambda/4$ frequency $\omega_0 = (\pi/2)(u/\ell)$, which is the frequency of a standing wave for which a quarter of the wave length fits on the transmission line of length ℓ. The external impedance of the finite LC transmission line is then given by

$$\frac{Z(\omega)}{R_Q} = \rho \frac{1 + \dfrac{i}{r}\tan\left(\dfrac{\pi}{2}\dfrac{\omega}{\omega_0}\right)}{1 + ir\tan\left(\dfrac{\pi}{2}\dfrac{\omega}{\omega_0}\right)}. \tag{24}$$

Defining the ratio $\kappa = \omega_0/\omega_R$ between the $\lambda/4$ frequency and the cutoff frequency $\omega_R = (R_\infty C)^{-1}$ due to the capacitance of the Josephson junction, we obtain for the total impedance seen by the junction

$$\frac{Z_t(\nu)}{R_Q} = \rho \frac{1 + \dfrac{i}{r}\tan\left(\dfrac{\pi}{2}\nu\right)}{\left[1 - \kappa\nu\tan\left(\dfrac{\pi}{2}\nu\right)\right] + ir\left[\kappa\nu + \tan\left(\dfrac{\pi}{2}\nu\right)\right]}. \tag{25}$$

Here, we have introduced the dimensionless frequency $\nu = \omega/\omega_0$. In the following we will mainly be interested in the case of small load resistances, i.e., $\rho \ll 1$. We further assume $r \ll 1$, which yields resonances with a high quality-factor in the total impedance.

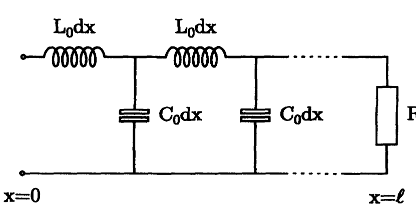

FIG. 2. Model circuit for a finite transmission line of length ℓ terminated by a load resistance R_L. The specific inductance and capacitance are L_0 and C_0, respectively.

In view of these assumptions it is useful to discuss first the case of a transmission line terminated by a short ($R_L = 0$). The environment then contains modes at frequencies $\nu_{n,0}$ determined by the condition $\kappa\nu_{n,0} = \cot(\pi\nu_{n,0}/2)$. For $\kappa \ll 1$, i.e., if the capacitive cutoff at ω_R is much larger than the $\lambda/4$ frequency ω_0, the frequencies of the low-lying environmental modes are given by

$$\nu_{n,0} = (2n+1)\left(1 - \frac{2}{\pi}\kappa\right) \tag{26}$$

for $n = 0, 1, \ldots$. In this case the excitation of an odd number of quanta of a low-frequency mode costs about the same energy as the excitation of a single quant of a higher-frequency mode. Therefore, it may be very difficult to distinguish experimentally between these two processes. The situation is quite different if $\omega_R \ll \omega_0$. Then the low-lying environmental modes have the frequencies

$$\nu_{n,0} = 2n + \frac{1}{\pi\kappa n} \tag{27}$$

for $n = 1, 2, \ldots$, which are again approximately multiples of each other. However, the lowest resonance now lies at $\nu_{0,0} = (2/\pi\kappa)^{1/2}$ and is therefore well separated from the other modes. Thus it becomes possible to study the influence of a single mode on the supercurrent. It is known[32] that at zero temperature the probability to excite several quanta of a single mode is given by a Poisson distribution in which the ratio between charging energy and mode energy appears as parameter. As a consequence, the excitation of two or more quanta will only be observable if $\pi\rho/\kappa r\nu_{n,0}$ is sufficiently large.

We now turn to the case of a finite but small load resistance R_L. Up to first order in r the real part $\nu_{n,0}$ of the resonance frequencies is unchanged, while there appears a small imaginary part

$$r\nu_{n,1} = r\frac{1 + (\kappa\nu_{n,0})^2}{\kappa + (\pi/2)[1 + (\kappa\nu_{n,0})^2]} \tag{28}$$

corresponding to a finite width of the resonance. Due to causality the poles $\nu_n = \nu_{n,0} + ir\nu_{n,1}$ and $-\nu_n^* = -\nu_{n,0} + ir\nu_{n,1}$ of the total impedance lie in the upper half of the complex plane.

Knowing the position of the poles in the complex plane we may decompose the total impedance in terms of these poles according to[22]

$$Z_t(\nu) = \sum_{n=0}^{\infty}\left[\frac{z_n}{\nu - \nu_n} - \frac{z_n^*}{\nu + \nu_n^*}\right]. \tag{29}$$

The coefficients z_n are given by

$$z_n = -\frac{i}{r}\frac{R_L}{\kappa + (\pi/2)(1 + \kappa^2\nu_n^2)}. \tag{30}$$

We note that for $r \ll 1$ the real part of z_n is smaller than the imaginary part by a factor of $1/r$.

As a first step in calculating the Cooper-pair current we have to determine the phase correlation function $J(t)$. For simplicity, we restrict ourselves to the case of zero temperature, where (5) becomes

$$J(t) = 2\int_0^\infty \frac{d\omega}{\omega}\frac{\mathrm{Re}Z_t(\omega)}{R_Q}(e^{-i\omega t}-1). \tag{31}$$

Introducing $g(z) = [e^{iz}E_1(iz) + e^{-iz}E_1(-iz)]/2$, where $E_1(z)$ is an exponential integral,[33] we find

$$J(t) = \frac{1}{R_Q}\sum_{n=0}^{\infty}\left[2\frac{z_n}{\nu_n}[g(\nu_n\omega_0 t)+\ln(\nu_n\omega_0|t|)+\gamma] + 2\frac{z_n^*}{\nu_n^*}[g(\nu_n^*\omega_0 t)+\ln(\nu_n^*\omega_0|t|)+\gamma] \right.$$
$$\left. -2\pi i\frac{z_n^*}{\nu_n^*}(e^{-i\nu_n^*\omega_0 t}-1)\Theta(t) + 2\pi i\frac{z_n}{\nu_n}(e^{-i\nu_n\omega_0 t}-1)\Theta(-t)\right]. \tag{32}$$

Here, $\Theta(t)$ is the unit step function. Due to the g function the result (32) does not allow for an analytic calculation of the Cooper-pair current. However, since $g(z)$ decays like $1/z^2$ for large arguments, it does not affect the long-time behavior of the phase correlation function. Since we are interested in small load resistances leading to rather sharp peaks in the Cooper-pair current, we may neglect the terms containing the g function. We emphasize that thereby we still retain the physically important terms. The oscillating exponential functions describe resonances due to the excitation of modes in the transmission line, while the logarithmically diverging terms lead to the broadening of the resonances.

The approximate phase correlation function, which we distinguish from the exact expression by a tilde, may be written in the form

$$\tilde{J}(t) = -2\rho\zeta - 2\rho\left[\ln\left(\frac{\omega_0 t}{r\kappa}\right)+i\frac{\pi}{2}\right] + \frac{1}{R_Q}\sum_{n=0}^{\infty}\left(-2\pi i\frac{z_n^*}{\nu_n^*}e^{-i\nu_n^*\omega_0 t}\Theta(t) + 2\pi i\frac{z_n}{\nu_n}e^{-i\nu_n\omega_0 t}\Theta(-t)\right). \tag{33}$$

Here, we have made use of

$$\zeta = \gamma + \ln(r\kappa) - \frac{1}{2\rho R_Q}\sum_{n=0}^{\infty}\left[\frac{z_n}{\nu_n}[\ln(-\nu_n)+\ln(\nu_n)] + \frac{z_n^*}{\nu_n^*}[\ln(-\nu_n^*)+\ln(\nu_n^*)]\right], \tag{34}$$

which follows from (11), (29), and (30) and contains information about the total impedance at all frequencies.

Within our approximation the Cooper-pair current at zero temperature is proportional to

$$\tilde{P}(E) = \frac{1}{\pi\hbar}\int_0^\infty dt \exp\left[\tilde{J}(t) + \frac{i}{\hbar}Et\right]. \tag{35}$$

This expression may be evaluated analytically by expanding $\exp[\tilde{J}(t)]$. The resulting sum over products of exponential functions corresponds to all different possibilities to excite modes of the transmission line. From (33) and (35) we then obtain

$$\tilde{P}(E) = \frac{\rho}{E_c}(\pi\rho)^{2\rho-1}\Gamma(1-2\rho)e^{-2\rho\zeta}$$
$$\times\mathrm{Re}\left\{ie^{-2i\pi\rho}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n_1,\ldots,n_k}^{\infty}\left[\prod_{j=1}^{k}\left(-\frac{2\pi i}{R_Q}\frac{z_{n_j}^*}{\nu_{n_j}^*}\right)\left(\frac{E-\hbar\omega_0\sum_{j=1}^{k}\nu_{n_j}^*}{E_c}\right)^{2\rho-1}\right]\right\}. \tag{36}$$

The first sum in this result runs over the total number k of mode excitations due to the tunneling Cooper pair. The k indices n_j in the second sum denote which of the infinitely many modes are excited. Of course, there is a possibility that a certain mode is excited more than once, so that some of the indices n_j may coincide.

If the voltage applied to the ultrasmall Josephson junction is smaller than $\hbar\omega_0\nu_{0,0}/2e$ where $\nu_{0,0}$ is the smallest dimensionless mode frequency, (7) and (36) give for the Cooper-pair current at zero temperature

$$I(V) = \frac{\pi}{2}I_c\frac{E_J}{E_c}(\pi\rho)^{2\rho}\frac{e^{-2\rho\zeta}}{\Gamma(2\rho)}\left(\frac{2eV}{E_c}\right)^{2\rho-1} \tag{37}$$

for positive voltages. This result displays the zero bias anomaly mentioned in Sec. III.

For sufficiently large voltages, modes in the transmission line may be excited and the corresponding peaks in $P(E)$ and the current-voltage characteristic are given by (36). For small ρ we may simplify (36) and write $\tilde{P}(E)$ as a sum over Lorentzians. The peaks are centered at $E = \hbar\omega_0\sum_{j=1}^{k}\nu_{n_j,0}$ and have a width of $2r\hbar\omega_0\sum_{j=1}^{k}\nu_{n_j,1}$. According to (28) the imaginary part of ν_n increases with n. Hence, the sharpest resonance will correspond to a single excitation of the lowest frequency mode. Depending on the value of κ double excitation of the lowest-frequency mode may lead to a peak that is considerably broader than the peak corresponding to a single excitation of the next mode. By dividing peak position and width by $2e$ we immediately get the corresponding quantities for peaks in the current-voltage characteristics. The prefactor of a peak in (36) contains a factor $(\rho/r\kappa)^k/k!$, which has the consequence that multiexcitations are only observable if $\rho/r\kappa$ is sufficiently large. This is in agreement with our previous considerations.

In the current-voltage characteristic shown in Fig. 3 peaks associated with multiexcitations are clearly visible. This is a consequence of the transmission line parameters $\rho = 0.01$, $r = 0.1$, and $\kappa = 1$, which also lead to rather sharp peaks. The parameter κ is large enough for multiexcitations of the lowest-frequency mode to be well separated from single excitations of higher-frequency modes. Figure 3(a) presents the approximate current-voltage characteristic based on (36). For comparison, Fig. 3(b) shows the corresponding result obtained from the numerical solution of an exact integral equation for $P(E)$ at zero temperature, which was derived in Ref. 4. The peaks of the current-voltage characteristic may be labeled by the number N_k of quanta of the kth mode excited by the tunneling Cooper pair. The identification of the peaks is given in the figure caption, e.g., peak e corresponds to the excitation of two quanta of the lowest mode and one quant of the second mode.

Comparing Figs. 3(a) and 3(b) we find qualitative agreement. The approximate result (36) gives a good estimate for peak positions and heights. However, it generally gives a Cooper-pair current which is somewhat too large. This is related to the neglect of the g function in (32), which leads to a violation of the sum rules for $P(E)$.[19] For a quantitative comparison with experimental results finite-temperature effects have to be taken into account too. Then one may either employ the integral equation developed in Ref. 34 or evaluate $P(E)$ from (6) by means of Fourier transform techniques.

On the other hand, if the circuit parameters are such that only single excitations are important, one may expand the exponent in (6) to first order in $J(t)$. This yields for the current-voltage characteristic[35,20]

$$I(V) = \frac{\pi E_J^2}{\hbar} \frac{1}{V} \frac{\mathrm{Re}Z_t(2eV/\hbar)}{R_Q}. \quad (38)$$

For $\rho \ll 1$, this result, which completely neglects multiexcitations, may also be obtained from (36) and (7).

VII. DISCUSSION

We have studied two types of structures appearing in the current-voltage characteristics of ultrasmall Josephson junctions with $E_J \ll E_c$ embedded in a standard low-impedance environment. The first structure, a peak at low voltages has recently been seen in experiments on lithographically fabricated junctions[31] as well as break junctions.[36] The second structure, peaks in the current-voltage characteristics due to resonances in the environmental impedance have also been seen in experiments.[37,31] Recently, a well-defined environment consisting of two transmission line segments has been employed[31] to allow for a quantitative test of the theoretical predictions and good agreement was found.

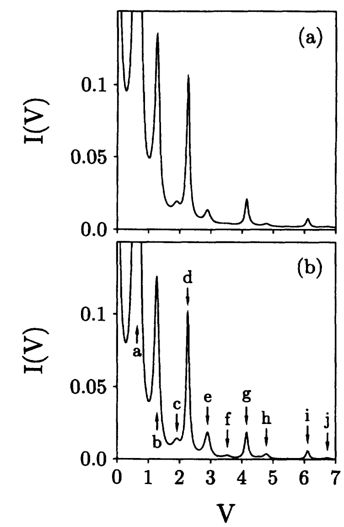

FIG. 3. Cooper-pair current-voltage characteristic for an ultrasmall tunnel junction coupled to a finite transmission line with $\rho = 0.01$, $r = 0.1$, and $\kappa = 1$. (a) shows the approximate result based on (36), while (b) was calculated from an exact integral equation. The voltage is given in units of $\hbar\omega_0/2e$ and the current is given in units of $(\pi E_J/2\hbar\omega_0)I_c$. The peaks correspond to excitations denoted by $(N_1N_2N_3N_4)$, where N_k is the number of quanta of the kth mode excited. a: (1000), b: (2000), c: (3000), d: (0100), e: (1100), f: (2100), g: (0010), h: (1010), i: (0001), and j: (1001).

ACKNOWLEDGMENTS

The authors would like to thank M. H. Devoret, D. Esteve, T. Holst, and C. Urbina for inspiring discussions, which motivated this work. Part of this work was carried out at the Centre d'Etudes de Saclay whose hospitality is gratefully acknowledged. Partial financial support was provided by the European Community under Contract No. SC1*-CT91-0631.

* Present address: Fakultät für Physik, Universität Freiburg, Hermann-Herder-Str. 3, D-79104 Freiberg, Germany.

1 For a survey, see *Single Charge Tunneling*, Vol. 294 of *NATO Advanced Study Institute Series B: Physics*, edited by H. Grabert and M. H. Devoret (Plenum, New York, 1991).

2 Yu. V. Nazarov, Zh. Eksp. Teor. Fiz. **95**, 975 (1989) [Sov. Phys. JETP **68**, 561 (1989)].

3 M. H. Devoret, D. Esteve, H. Grabert, G.-L. Ingold, H. Pothier, and C. Urbina, Phys. Rev. Lett. **64**, 1824 (1990).

4 S. M. Girvin, L. I. Glazman, M. Jonson, D. R. Penn, and M. D. Stiles, Phys. Rev. Lett. **64**, 3183 (1990).

5 A. Widom, G. Megaloudis, T. D. Clark, H. Prance, and R. J. Prance, J. Phys. A **15**, 3877 (1982).

[6] K. K. Likharev and A. B. Zorin, J. Low Temp. Phys. **59**, 347 (1985).
[7] E. Ben-Jacob and Y. Gefen, Phys. Lett. A **108**, 289 (1985).
[8] D. V. Averin and K. K. Likharev, J. Low Temp. Phys. **62**, 345 (1986).
[9] A. N. Cleland, J. M. Schmidt, and J. Clarke, Phys. Rev. Lett. **64**, 1565 (1990).
[10] A. N. Cleland, J. M. Schmidt, and J. Clarke, Phys. Rev. B **45**, 2950 (1992).
[11] L. S. Kuzmin, Yu. V. Nazarov, D. B. Haviland, P. Delsing, and T. Claeson, Phys. Rev. Lett. **67**, 1161 (1991).
[12] D. V. Averin, Yu. V. Nazarov, and A. A. Odintsov, Physica B **165&166**, 945 (1990).
[13] G. Falci, V. Bubanja, and G. Schön, Europhys. Lett. **16**, 109 (1991).
[14] G. Falci, V. Bubanja, and G. Schön, Z. Phys. B **85**, 451 (1991).
[15] P. Delsing, K. K. Likharev, L. S. Kuzmin, and T. Claeson, Phys. Rev. Lett. **63**, 1180 (1989).
[16] L. J. Geerligs, V. F. Anderegg, C. A. van der Jeugd, J. Romijn, and J. E. Mooij, Europhys. Lett. **10**, 79 (1989).
[17] S. Gregory, Phys. Rev. B **44**, 12 868 (1991).
[18] For a review, see G. Schön and A. D. Zaikin, Phys. Rep. **198**, 237 (1990).
[19] G.-L. Ingold and Yu. V. Nazarov, in *Single Charge Tunneling* (Ref. 1).
[20] H. Grabert and G.-L. Ingold, in *Computations for the Nano-Scale*, Vol. 240 of *NATO Advanced Study Institute Series E: Applied Sciences*, edited by P. E. Blöchl *et al.* (Kluwer, Dordrecht, 1993).
[21] G.-L. Ingold and H. Grabert, in *Proceedings of the LT-20*, [Physica B **194–196**, 1025 (1994)].
[22] U. Eberhardt, diploma thesis, University of Essen, 1993.
[23] A. O. Caldeira and A. J. Leggett, Ann. Phys. (N.Y.) **149**, 374 (1983).
[24] U. Weiss and H. Grabert, Phys. Lett. A **108**, 63 (1985).
[25] H. Grabert and U. Weiss, Phys. Rev. Lett. **54**, 1605 (1985).
[26] M. P. A. Fisher and A. T. Dorsey, Phys. Rev. Lett. **54**, 1609 (1985).
[27] A. Barone and G. Paterno, *Physics and Applications of the Josephson Effect* (Wiley, New York, 1982).
[28] R. L. Kautz and J. M. Martinis, Phys. Rev. B **42**, 9903 (1990).
[29] Yu. M. Ivanchenko and L. A. Zil'berman, Zh. Eksp. Teor. Fiz. **55**, 2395 (1968) [Sov. Phys. JETP **28**, 1272 (1969)].
[30] V. Ambegaokar and B. I. Halperin, Phys. Rev. Lett. **22**, 1364 (1969).
[31] T. Holst, D. Esteve, C. Urbina, and M. H. Devoret (unpublished).
[32] M. H. Devoret, D. Esteve, H. Grabert, G.-L. Ingold, H. Pothier, and C. Urbina, Physica B **165&166**, 977 (1990).
[33] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1971).
[34] G.-L. Ingold and H. Grabert, Europhys. Lett. **14**, 371 (1991).
[35] D. V. Averin and Yu. V. Nazarov, Physica B **162**, 309 (1990).
[36] C. J. Muller and R. de Bruyn Ouboter, in *Proceedings of the LT-20* (Ref. 21), p. 1043.
[37] L. J. Geerligs, Ph.D. thesis, Delft University of Technology, 1990.