Investigation of different classes of variational functions for the triangular and *kagomé* spin-$\frac{1}{2}$ Heisenberg antiferromagnets

P. Sindzingre, P. Lecheminant,* and C. Lhuillier

Laboratoire de Physique Théorique des Liquides, 4, Place Jussieu, 75232 Paris Cedex 05, France

(Received 8 March 1994)

We investigate a class of resonating-valence-bond wave functions on the triangular lattice which interpolate between the $\sqrt{3}\times\sqrt{3}$ Néel state and a dimer state according to the range of the bonds and the similar two classes of resonating-valence-bond wave functions on the *kagomé* lattice constructed from the $\sqrt{3}\times\sqrt{3}$ and $\mathbf{q}=\mathbf{0}$ Néel states. Numerical calculations show that a $\sqrt{3}\times\sqrt{3}$ wave function gives for the triangular lattice a variational energy and spin-spin correlations in very good agreement with diagonalization results on 12- and 36-site systems. Rather low variational energies are also obtained with trial functions of the $\sqrt{3}\times\sqrt{3}$ and $\mathbf{q}=\mathbf{0}$ type for the *kagomé* lattice but spin-spin correlations beyond first neighbors are not in good agreement with diagonalization results on 12- and 36-site systems. For the 12-site system, the spin-spin correlations of the best $\mathbf{q}=\mathbf{0}$ wave function most resemble those of the first excited state. The $\mathbf{q}=\mathbf{0}$ and perhaps the $\sqrt{3}\times\sqrt{3}$ wave functions may describe excited states close to the ground state in the case of larger systems.

I. INTRODUCTION

The two dimensional spin 1/2 antiferromagnetic Heisenberg model (AFHM) has generated much interest in recent years. Its Hamiltonian reads

$$\mathcal{H} = \sum_{\langle i,j\rangle} \mathbf{S}_i \cdot \mathbf{S}_j \tag{1}$$

where the sum is over all nearest neighbor pairs. A variety of numerical techniques have demonstrated that the ground state of the AFHM on the square lattice has Néel long-range order (LRO) with a staggered magnetization reduced by quantum fluctuations to about 60% of its classical value.[1,2] The nature of the ground state, especially the existence of LRO, for the AFHM on triangular and *kagomé* lattices, which are both systems with geometric frustration, is still controversial.

The classical ($S=\infty$) ground state on the triangular lattice has LRO. It is a three-sublattice Néel state where neighboring spins (labeled A, B, C in the following) are oriented 120° relative to each other ($\sqrt{3}\times\sqrt{3}$ order) but it was not clear if Néel order survives quantum fluctuations which could be larger than in the square lattice case. Quite some time ago, Anderson[3] postulated that the ground state of the triangular AFHM is a resonating-valence-bond (RVB) state of short-range singlet bonds, which is a disordered spin liquid. Large-n expansions predict a disordered ground state for sufficiently small S.[4] Several numerical studies by means of exact diagonalizations on finite clusters[5–8] or quantum Monte Carlo calculations at finite temperature[9,10] were interpreted in favor of this picture. However, spin-wave[11] and Schwinger boson mean field calculations,[12] an analysis of the low-lying levels obtained from exact diagonalizations on finite clusters,[13] and most recent high temperature series expansion[14] support the existence of Néel LRO. Contrary to the finite size scaling analysis of the exact diagonalizations data, which leads to a sublattice magnetization of about 50% of the classical value (in agreement with spin-wave calculations), numerical series expansions[14,15] suggest that antiferromagnetic order may be small. A particular kind of RVB state, without magnetic LRO but with chiral symmetry breaking was also proposed by Kalmeyer and Laughlin[16] as the ground state of the triangular AFHM. Huse and Elser[17] found variational states displaying Néel LRO with a significantly lower energy than the Kalmeyer and Laughlin state and no evidence of chiral symmetry breaking has been obtained in exact diagonalizations.[8,18]

Unlike the triangular lattice, the classical Heisenberg model on the *kagomé* lattice has an infinite number of continuously degenerate ground states, continuous local distortions of the spin configuration from a ground state being possible keeping the energy constant, but such a degeneracy may be lifted by thermal or quantum fluctuations.[19] The nature of the ground state of the quantum model is highly speculative. The two simplest Néel-like classical ground states are the three-sublattice planar states $\mathbf{q}=\mathbf{0}$ and $\sqrt{3}\times\sqrt{3}$ displayed in Fig. 1. It has been argued that quantum fluctuations could select one of these planar states.[20] Other kinds of ground state have also been proposed: spin nematic,[21,22] chiral spin liquid,[23] or dimerized.[24,25] For the classical model, thermal fluctuations select all the coplanar solutions[22] while high temperature expansions favor the $\sqrt{3}\times\sqrt{3}$ structure.[26] Large-n expansions based on the group $Sp(n)$ predict the $\sqrt{3}\times\sqrt{3}$ state at large S (Ref. 4) (but a disordered ground state when S is small). For half-odd integer S, this state may survive down to a smaller value of S than that for integer spin.[27] However, most recent work concludes to the absence of Néel order.[15,25,28,29] In particular, series expansion indicate the magnetization of such a state should vanish[15] and no evidence of a Néel type of symmetry breaking has been found in the low-lying states computed from exact diago-

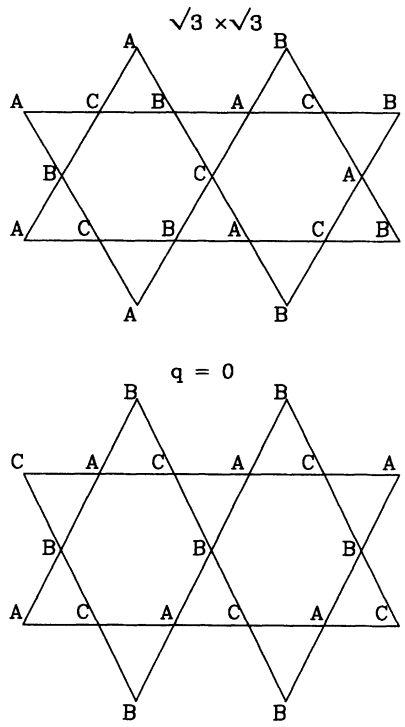

FIG. 1. The two classical planar Néel states ($\mathbf{q} = \mathbf{0}$ and $\sqrt{3} \times \sqrt{3}$) on the *kagomé* lattice.

nalizations on small clusters, which show that the spectra of excited states of the *kagomé* and triangular AFHM are qualitatively different.[29]

In this article, we do not address the question of the existence of LRO, but try to assess from variational Monte Carlo calculations, whether states with the $\sqrt{3} \times \sqrt{3}$ structure in the case of the triangular AFHM or states with either the $\mathbf{q} = \mathbf{0}$ and $\sqrt{3} \times \sqrt{3}$ structure in the case of the *kagomé* AFHM are viable candidates to describe the short-range correlations of the ground state of these AFHM. For this purpose we consider RVB wave functions[30,31] which interpolate between a three-sublattices planar Néel state and a disordered state with the same three-sublattices pattern according to the range of the bonds taken as variational parameters and compare the results obtained for systems of 12 and 36 sites with those of exact diagonalizations.

This paper is organized as follows. Section I describes the classes of trial wave functions and the Monte Carlo method used. Section II presents the results for the triangular lattice and Sec. III those for the *kagomé* net. Section IV summarizes our results and conclusions.

II. TRIAL WAVE FUNCTIONS

The RVB trial wave functions we study, are weighted combinations of valence-bond states $|c\rangle$:

$$|\psi\rangle = \sum_c \omega(c)|c\rangle, \tag{2}$$

with

$$\begin{aligned} |c\rangle = {} & (i_1, j_{N/2+1}) \dots (i_{N/6}, j_{2N/3}) \\ & (i_{N/6+1}, k_{2N/3+1}) \dots (i_{N/3}, k_{5N/6}) \\ & (j_{N/3+1}, k_{5N/6+1}) \dots (j_{N/2}, k_N), \end{aligned}$$

where N is the number of lattice sites, while i_α, j_β, and k_γ label the A, B, and C sites, respectively, and $(ij) = |\uparrow_i\downarrow_j - \downarrow_i\uparrow_j\rangle$ denote a singlet bond between the two sites i and j. The singlet bonds connect sites on different sublattices, one half of the A (i_α) sites are connected to B (j_β) sites and the other half to C (k_γ) sites, the B sites left being connected to the remaining C sites (this requires N to be a multiple of 6). In (2) the sum is over all valence-bond states obtained from distinct permutations of the i_α, j_β, and k_γ among themselves. For the triangular lattice, we consider the assignment of the sites to the A, B, and C sublattices which leads to the $\sqrt{3} \times \sqrt{3}$ Néel state. For the *kagomé* lattice, we focus on the two assignments which give the $\mathbf{q} = \mathbf{0}$ and $\sqrt{3} \times \sqrt{3}$ Néel states. The weights of the $|c\rangle$ states are given by the product of the weights of each bond

$$\begin{aligned} \omega(c) = {} & h(i_1, j_{N/2+1}) \dots h(i_{N/6}, j_{2N/3}) \\ & h(i_{N/6+1}, k_{2N/3+1}) \dots h(i_{N/3}, k_{5N/6}) \\ & h(j_{N/3+1}, k_{5N/6+1}) \dots h(j_{N/2}, k_N), \end{aligned}$$

where the weight of a bond connecting two sites is a positive function $h(i,j) = h(\mathbf{r}_{ij})$ of the vector $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ joining the sites, having the symmetries of the lattice. As shown by Ma,[31] the wave function $|\psi\rangle$ describes a three-sublattice Néel state projected into the singlet subspace when $h(i,j)$ is a constant independent of the relative position of sites i and j. In the other limit where $h(i,j)$ is only nonzero between nearest neighbor sites, $|\psi\rangle$ represents a disordered dimer state. As the energy is most sensitive to the short-range behavior of h, the weights $h(\mathbf{r})$ of bonds between sites at separation $r \leq \sqrt{7}$ (in unit of the nearest neighbor distance) are taken to be free variational parameters. For the *kagomé* lattice, in the case of the wave function with the $\sqrt{3} \times \sqrt{3}$ pattern, we allow the weights of the bonds between third neighbor sites along a full row $h(2_1)$ and across an empty hexagon $h(2_2)$ to be different (third neighbor sites are on the same sublattice in the case of the $\mathbf{q} = \mathbf{0}$ pattern). At larger distance we choose the power law parametrization $h(r) = h(\sqrt{7})(\sqrt{7}/r)^p$ with an integer exponent p being a variational parameter. The wave functions will describe either a state with LRO if $h(i,j)$ decays sufficiently slowly with distance, or a disordered state. Two-sublattice RVB wave functions for the AFHM on the square lattice, similar to (2) have been investigated by Liang, Douçot, and Anderson[30] who found variational energies extremely close to the ground state energy either with wave functions having LRO or wave functions describing disordered states.

We use a Monte Carlo method similar to the one adopted by Liang, Douçot, and Anderson[30] to calculate the spin-spin correlations:

$$\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle = \frac{\langle \psi | \mathbf{S}_i \cdot \mathbf{S}_j | \psi \rangle}{\langle \psi | \psi \rangle}.$$

Upon substitution of the wave function (2), one has

$$\langle\psi|\mathbf{S}_i\cdot\mathbf{S}_j|\psi\rangle = \sum_{c_1,c_2}\omega(c_1)\omega(c_2)\langle c_1|\mathbf{S}_i\cdot\mathbf{S}_j|c_2\rangle$$

and

$$\langle\psi|\psi\rangle = \sum_{c_1,c_2}\omega(c_1)\omega(c_2)\langle c_1|c_2\rangle.$$

As with the RVB wave functions for the square lattice, investigated by Liang, Douçot, and Anderson,[30] a pair of valence-bond states $|c_1\rangle$, $|c_2\rangle$ with a nonzero overlap $\langle c_1|c_2\rangle$ may be associated with a covering of the triangular or *kagomé* lattice with loops of alternating bonds where each site is connected to a bond from $|c_1\rangle$ and a bond from $|c_2\rangle$, so that the evaluation of matrix elements is reduced to simple rules. One has

$$\langle c_1|\mathbf{S}_i\cdot\mathbf{S}_j|c_2\rangle = \pm\tfrac{3}{4}\langle c_1|c_2\rangle,$$

when i and j are on the same loop with the plus (minus) sign if there is an even (odd) number of bonds between i and j, whereas $\langle c_1|\mathbf{S}_i\cdot\mathbf{S}_j|c_2\rangle = 0$ when i and j are on different loops. However, there is a notable difference with the wave functions considered for the square lattice. Here all the overlaps between valence-bond states are not positive. One has instead

$$\langle c_1|c_2\rangle = \pm 2^{N(c_1,c_2)},$$

where $N(c_1,c_2)$ is the number of loops in the bond covering of the lattice obtained from $|c_1\rangle$ and $|c_2\rangle$ with the sign of the overlap determined by the relative sequence of sublattice encountered along the loops. For this reason we used a Monte Carlo process which samples the pair of states (c_1,c_2) according to a probability distribution proportional to the weights ω and the absolute value of the overlap:

$$P(c_1,c_2) = \frac{\omega(c_1)\omega(c_2)|\langle c_1|c_2\rangle|}{\sum_{c_1,c_2}\omega(c_1)\omega(c_2)|\langle c_1|c_2\rangle|}$$

and evaluated the spin-spin correlations from the ratios:

$$\langle\mathbf{S}_i\cdot\mathbf{S}_j\rangle = \frac{C_{ij}}{D}$$

of

$$C_{ij} = \sum_{c_1,c_2}P(c_1,c_2)\,\frac{\langle c_1|c_2\rangle}{|\langle c_1|c_2\rangle|}\,\frac{\langle c_1|\mathbf{S}_i\cdot\mathbf{S}_j|c_2\rangle}{\langle c_1|c_2\rangle}$$

and

$$D = \sum_{c_1,c_2}P(c_1,c_2)\,\frac{\langle c_1|c_2\rangle}{|\langle c_1|c_2\rangle|}.$$

The basic Monte Carlo step to generate a new pair (c'_1,c_2) [or (c_1,c'_2)] from a pair (c_1,c_2) is to choose randomly two nearest neighbor sites on the same sublattice, then exchange the bonds from $|c_1\rangle$ (or $|c_2\rangle$) connected to these sites and accept the new pair with probability $P(c'_1,c_2)/P(c_1,c_2)$ [or $P(c_1,c'_2)/P(c_1,c_2)$]. As the value of D decreases rather rapidly with increasing lattice sizes, the Monte Carlo method used here can only deal with rather small systems. We consider clusters of hexagonal shape with periodic boundary conditions.

III. TRIANGULAR AFHM

For the triangular AFHM, we report results for 12-, 36-, and 48-site systems. The most extensive search for the optimum RVB wave functions with the $\sqrt{3}\times\sqrt{3}$ structure was carried out on $N = 12$ and $N = 36$ systems for which diagonalization results are available. For the $N = 48$ system we only search for the best RVB wave function with an exponent $p = 5$ (corresponding to the optimum value of the $\sqrt{3}\times\sqrt{3}$ trial function for the *kagomé lattice*) in the bond weight function h as calculations become rather time consuming for this size. In Table I, we display the values of the parameters of the bond weight function h which optimize the trial wave functions. For $N = 12$ there is only one variational parameter, $h(2)$, while for $N = 36$ and $N = 48$ there are two: $h(2)$ and $h(\sqrt{7})$. Not unsurprisingly, the best wave functions have vanishing weights for bonds connecting third nearest neighbor sites, $h(2) = 0$ (one would expect such bonds to decrease antiferromagnetic correlations between nearest neighbors). Consequently for $N = 12$ the optimum $|\psi\rangle$ is a dimer wave function. However the energy is only rather weakly changed by a moderate amount of third nearest neighbor bonds (but not spin-spin correlations beyond nearest neighbor). The lowest trial energies obtained for $N = 12$ and $N = 36$ are $E(12) = -0.6096(3)$, $E(36) = -0.5579(3)$. They are very close (respectively, within 0.1% and 0.4%) to the ground state energies obtained from exact diagonalizations: $E_0(12) = -0.6103$ and $E_0(36) = -0.5604$ (Ref. 8) and significantly lower than those previously reported with other variational functions.[5,16,17,32] As shown in Table II, very good agreement is also found for the spin-spin correlations. This suggests that the RVB wave functions considered here have a very strong overlap with the ground state wave function and shows that the factorized form of the weight $\omega(c)$ is an excellent approximation for the AFHM on the triangular lattice. Nevertheless, this form is not exact since the magnitude of the variational results for the spin-spin correlations beyond first nearest neighbors sites are slightly higher than the diagonalization results. Also displayed in Table II are our variational results for the 48-site lattice. Unfortunately, because of the sign problem we have been unable to obtain accurate results for larger systems which precludes discrimination between wave functions with or without LRO. Nevertheless, as the energy is most sensitive to short-range behavior of the bond weight function h, it is not unlikely that

TABLE I. Best variational parameters for the $\sqrt{3}\times\sqrt{3}$ state on triangular lattices of N sites.

N	$h(2)$	$h(\sqrt{7})$
12	0.0	
36	0.0	0.0921
48	0.0	0.0928

TABLE II. Spin-spin correlations $\langle \mathbf{S}_0 \cdot \mathbf{S}_r \rangle$ at various separations r in the triangular AFHM. The numbers in brackets are the estimated errors. Exact results are from Ref. 8.

r	$N = 12$ RVB	$N = 12$ exact	$N = 36$ RVB	$N = 36$ exact	$N = 48$ RVB
1	-0.2032 (1)	-0.2034	-0.1860 (1)	-0.1868	-0.185 (1)
$\sqrt{3}$	0.2065 (4)	0.1930	0.166 (1)	0.1535	0.163 (4)
2	-0.075 (1)	-0.0511	-0.069 (1)	-0.0548	-0.065 (4)
$\sqrt{7}$			-0.073 (1)	-0.0664	-0.070 (4)
3			0.132 (1)	0.1136	0.118 (4)
$2\sqrt{3}$			0.131 (1)	0.1174	0.120 (4)
$\sqrt{13}$					-0.054 (4)
4					-0.060 (4)

one would have found wave functions with or without LRO very close in energy as for the square lattice.[30]

IV. *KAGOMÉ* AFHM

For the *kagomé* AFHM, we carried out calculations on a $N = 12$ system with RVB wave functions having the $\mathbf{q} = \mathbf{0}$ structure and on a $N = 36$ system with RVB wave functions displaying either the $\mathbf{q} = \mathbf{0}$ or the $\sqrt{3} \times \sqrt{3}$ pattern (see Fig. 2). Both are allowed for $N = 36$, but only the structure $\mathbf{q} = \mathbf{0}$ for $N = 12$. Diagonalization results are available up to $N = 36$.[25,29,33] We did not carry out calculations on the $N = 18$ system which allow the $\mathbf{q} = \mathbf{0}$ pattern (but not the $\sqrt{3} \times \sqrt{3}$ structure). This system is less interesting since it lacks the $\frac{2\pi}{3}$ rotation symmetry of the infinite lattice. For $N = 12$, the lowest energy obtained with a RVB wave function with the $\mathbf{q} = \mathbf{0}$ structure $E(\mathbf{q} = \mathbf{0}, 12) = -0.4395(2)$ is 3% above the ground state energy $E_0(12) = -0.4537$.[33] For $N = 36$ the best variational energies $E(\mathbf{q} = \mathbf{0}, 36) = -0.4199(2)$ and $E(\sqrt{3} \times \sqrt{3}, 36) = -0.4181(2)$ are 4.1% and 4.5% above the ground state energy $E_0(36) = -0.4384$.[25] Note that the $N = 36$ results do not provide evidence of the conjectured selection of the $\sqrt{3} \times \sqrt{3}$ state by quantum fluctuations as $E(\mathbf{q} = \mathbf{0}, 36)$ is lower than $E(\sqrt{3} \times \sqrt{3}, 36)$. The optimum parameters in the trial wave functions are listed in Table III. Rather surprisingly, the best $\mathbf{q} = \mathbf{0}$ trial wave function for $N = 12$ is not a dimer wave function since it contains bonds between second nearest neighbors (longer bonds are not allowed for this size). In fact, the best $\mathbf{q} = \mathbf{0}$ trial wave function for $N = 36$ has no such bonds. On the other hand, the best $\sqrt{3} \times \sqrt{3}$ trial wave function for $N = 36$ has no bonds to third nearest neighbors. The absence of third nearest neighbor bonds along a full row is not unexpected. But the lack of third nearest neighbor bonds across empty hexagons is more surprising. Our best variational energies are not much above the ground state energies computed by means of exact diagonalizations but there is not an agreement comparable to the one seen for the triangular AFHM. In addition, as shown in Table IV and Table V our trial wave functions yield spin-spin correlations rather different from those given for the ground state by exact diagonalizations. A striking feature of the spin-spin correlations in the ground state of the *kagomé* AFHM is their very rapid decay with increasing separation. By contrast the spin-spin correlations in our trial wave functions decay much more slowly [note that the spin-spin correlations of the $\sqrt{3} \times \sqrt{3}$ states on the *kagomé* and triangular lattices (see Table II) are somewhat similar]. This suggests that neither of our trial wave functions are good choices to describe the ground state of the *kagomé* lattice. In fact, the

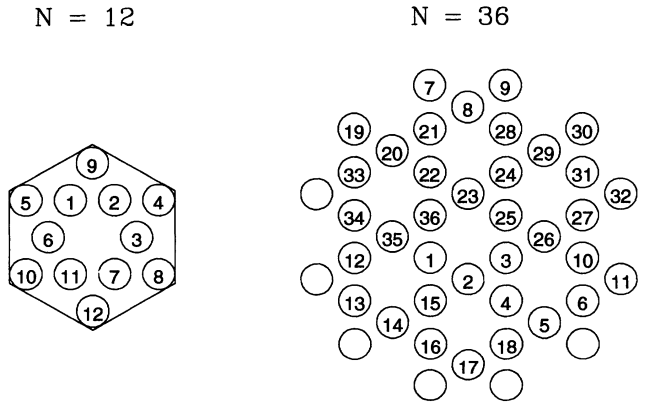

FIG. 2. The $N = 12$ and $N = 36$ *kagomé* systems considered showing the site labeling used in Table IV and Table V. The hexagonal shape of each cell is shown for $N = 12$. For $N = 36$ unlabeled sites which are redundant and related to labeled ones via cell translation vectors are included to illustrate the hexagonal shape of each cell.

TABLE III. Best variational parameters for the $\mathbf{q} = \mathbf{0}$ and $\sqrt{3} \times \sqrt{3}$ states on *kagomé* lattices of N sites. For the $\sqrt{3} \times \sqrt{3}$ state, $h(2_1)$ and $h(2_2)$ are, respectively, the bonds' weights between third neighbors sites along a full row (sites 1 and 4 in Fig. 2) and across an empty hexagon (sites 1 and 25 in the $N = 36$ system shown in Fig. 2).

		$\mathbf{q} = \mathbf{0}$		
N	$h(\sqrt{3})$	$h(\sqrt{7})$	p	
12	0.0577			
36	0.0	0.0668	5	
		$\sqrt{3} \times \sqrt{3}$		
N	$h(2_1)$	$h(2_2)$	$h(\sqrt{7})$	p
36	0.0	0.0	0.0876	5

TABLE IV. Spin-spin correlations $\langle \mathbf{S}_0 \cdot \mathbf{S}_r \rangle$ between site 1 and site n in Fig. 2 at distance r in the 12 sites *kagomé* AFHM. The numbers in brackets are the estimated errors. Exact results are from Ref. 29.

n	r	RVB ($\mathbf{q} = \mathbf{0}$)	Exact first excited state	Exact ground state
2	1	−0.2197 (1)	−0.2220	−0.2269
3	$\sqrt{3}$	−0.109 (1)	−0.0949	0.0887
4	2	0.194 (1)	0.1781	−0.0037
7	2	0.177 (1)	0.1614	−0.1235

exact ground state of the $N = 12$ sample is odd under inversion whereas our variational wave functions are even. However, they may have strong overlap with the excited states. Indeed, we see in Table IV that the spin-spin correlations obtained from the RVB wave function with a $\mathbf{q} = \mathbf{0}$ pattern on the $N = 12$ system are quite similar to those of the first excited state which is also a singlet and has the same symmetry. Low-lying excited states with the symmetry of a $\mathbf{q} = \mathbf{0}$ state which are singlets for N even or doublets for N odd have been found in diagonalizations studies on systems up to $N = 21$. Unfortunately, no data on excited states are available for $N = 36$ while a singlet state with $\sqrt{3} \times \sqrt{3}$ symmetry requires at least a $N = 36$ system. But it would not be unexpected if the spin-spin correlations found for $N = 36$ with either the RVB wave function with a $\mathbf{q} = \mathbf{0}$ pattern, or the one with the $\sqrt{3} \times \sqrt{3}$ structure, were similar to the spin-spin correlations of low-lying excited states. Hsu and Schofield[34] have studied projected fermionic trial wave functions for the *kagomé* AFHM and report spin-spin correlations which decay faster than ours, but still are not in very good agreement with those found for the ground state. However, their use of an approximate cluster method to evaluate these quantities makes difficult a comparison of the trial energies of these states with those of our trial wave functions. Yang *et al.*[23] studied a wave function of the Kalmeyer and Laughlin type[16] on $N = 12$ and $N = 18$ systems. Their variational energy for $N = 12$, −0.420, is somewhat higher than the energy of our $\mathbf{q} = \mathbf{0}$ trial wave function for this size. Note that their wave function is reported to have zero overlap with the ground state for both sizes.

V. CONCLUSIONS

In summary, we have constructed a RVB wave function based on the $\sqrt{3} \times \sqrt{3}$ structure which is seen to approximate very well the ground state of the triangular AFHM on small systems. A $\sqrt{3} \times \sqrt{3}$ like state is a good candidate for the ground state of the triangular AFHM. The analysis of the low-lying levels obtained from exact diagonalization[13] and recent high temperature series expansion[14] support the existence of Néel LRO for such a state. It would be very interesting if one could, as was done for the square lattice AFHM,[2] investigate the ground state of larger systems using a projection Monte Carlo method starting from this wave function, overcoming the computational difficulties due to the negative valence-bond overlaps, in order to provide further evidence in favor of the existence of Néel order in the triangular AFHM. By contrast, the trial wave functions based on the $\mathbf{q} = \mathbf{0}$ or $\sqrt{3} \times \sqrt{3}$ structure are not adequate to describe the ground state of the *kagomé* AFHM. But the $\mathbf{q} = \mathbf{0}$ wave function and probably the $\sqrt{3} \times \sqrt{3}$ wave function are good candidates to represent low-lying excited states very close in energy to the ground state in the case of small systems. Further study of the other low-lying singlet excited states may shed light on the properties of the *kagomé* AFHM. As this work was completed, we became aware of an unpublished manuscript by Yong-Cong Chen which reports results of Monte Carlo investigation of the triangular AFHM with a RVB wave function derived from a Schwinger boson mean field approach describing a $\sqrt{3} \times \sqrt{3}$ state, also showing excellent agreement with diagonalization results.

TABLE V. Same as Table IV for the 36 sites *kagomé* AFHM. Exact results are from Ref. 25.

n	r	RVB ($\sqrt{3} \times \sqrt{3}$)	RVB ($\mathbf{q} = \mathbf{0}$)	Exact
2	1	−0.2090 (2)	−0.2100 (2)	−0.2192
3	$\sqrt{3}$	0.138 (1)	−0.077 (1)	0.0116
4	2	0.077 (1)	0.133 (1)	0.0527
25	2	−0.053 (1)	0.118 (1)	−0.0090
18	$\sqrt{7}$	−0.063 (1)	−0.057 (1)	−0.0048
5	3	0.108 (1)	−0.050 (1)	−0.0230
19	$2\sqrt{3}$	0.111 (1)	0.101 (1)	0.0063
10	$2\sqrt{3}$	0.101 (1)	0.093 (1)	0.0032
6	$\sqrt{13}$	−0.050 (1)	−0.046 (1)	−0.0098
7	4	−0.050 (1)	0.089 (1)	0.0222

ACKNOWLEDGMENTS

We thank B. Bernu and L. Pierre for useful discussions.

*Also at Groupe de Physique Statistique, Université de Cergy-Pontoise, 95806 Cergy-Pontoise Cedex, France.

[1] E. Manousakis, Rev. Mod. Phys. **63**, 1 (1991), and references therein; see K. J. Runge, Phys. Rev. B **45**, 12 292 (1992) for most recent references.
[2] S. Liang, Phys. Rev. B **42**, 6555 (1990).
[3] P. W. Anderson, Mater. Res. Bull. **8**, 153 (1973); P. Fazekas and P. W. Anderson, Philos. Mag. **30**, 423 (1973).
[4] S. Sachdev, Phys. Rev. B **45**, 12 377 (1992).
[5] T. Oguchi, H. Nishimori, and Y. Taguchi, J. Phys. Soc. Jpn. **55**, 323 (1985).
[6] H. Nishimori and H. Nakanishi, J. Phys. Soc. Jpn. **57**, 626 (1988); **58**, 2607 (1989); **57**, 3433 (1989).
[7] S. Fujiki, Can. J. Phys. **65**, 489 (1987).
[8] P. W. Leung and K. J. Runge, Phys. Rev. B **47**, 5861 (1993).
[9] M. Takasu, S. Miyashita, and M. Suzuki, Prog. Theor. Phys. **75**, 1254 (1986).
[10] M. J. Imada, J. Phys. Soc. Jpn. **56**, 311 (1987).
[11] T. Jolicoeur and J. C. Le Guillou, Phys. Rev. B **40**, 2727

(1989).
[12]D. Yoshioka and J Miyazaki, J. Phys. Soc. Jpn. **60**, 614 (1991).
[13]B. Bernu, C. Lhuillier, and L. Pierre, Phys. Rev. Lett. **69**, 2590 (1992).
[14]N. Elstner, R. R. P. Singh, and A. P. Young, Phys. Rev. Lett. **71**, 1629 (1993).
[15]R. R. P. Singh and D. A. Huse, Phys. Rev. Lett. **68**, 1766 (1992).
[16]V. Kalmeyer and R. B. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987); Phys. Rev. B **39**, 11 879 (1989).
[17]D. A. Huse and V. Elser, Phys. Rev. Lett. **60**, 2531 (1988).
[18]P. Lecheminant, B. Bernu, C. Lhuillier, and L. Pierre (unpublished).
[19]J. Villain, R. Bidaux, J. P. Carton, and R. Conte, J. Phys. (Paris) **41**, 1263 (1980).
[20]A. Chubukov, Phys. Rev. Lett. **69**, 832 (1992).
[21]P. Chandra, P. Coleman, and I. Ritchey, Int. J. Mod. Phys. B **5**, 171 (1991).
[22]J. T. Chalker, P. C. Holdsworth, and E. F. Shender, Phys. Rev. Lett. **68**, 855 (1992).
[23]K. Yang, L. K. Warman, and S. M. Girvin, Phys. Rev. Lett. **70**, 2641 (1993).
[24]J. B. Marston and C. Zeng, J. Appl. Phys. **69**, 5962 (1991).
[25]P. W. Leung and V. Elser, Phys. Rev. B **47**, 5459 (1993).
[26]A. B. Harris, C. Kallin, and A. J. Berlinsky, Phys. Rev. B **45**, 2899 (1992).
[27]J. A. Delft and C. Henley, Phys. Rev. B **48**, 965 (1993).
[28]J. T. Chalker and J. F. G. Eastmond, Phys. Rev. B **46**, 14 201 (1992).
[29]B. Bernu, P. Lecheminant, C. Lhuillier, and L. Pierre, Phys. Scr. **T49**, 192 (1993).
[30]S. Liang, B. Douçot, and P. W. Anderson, Phys. Rev. Lett. **61**, 365 (1988).
[31]M. Ma, Phys. Rev. B **38**, 6813 (1988).
[32]T. K. Lee and S. Feng, Phys. Rev. B **41**, 11 110 (1990).
[33]C. Zeng and V. Elser, Phys. Rev. B **42**, 8436 (1990).
[34]T. C. Hsu and A. Schofield, J. Phys. Condens. Mater. **3**, 8067 (1991).