Exact results of the one-dimensional $1/r^2$ supersymmetric t-J model without translational invariance

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In this work, we continue the study of the supersymmetric t-J model with $1/r^2$ hopping and exchange without translational invariance. A set of Jastrow eigenfunctions are obtained for the system, with eigenenergies explicitly calculated. The ground state of the t-J model is included in this set of wave functions. The spectrum of this t-J model consists of equidistant energy levels which are highly degenerate.

In recent years, there have been considerable interests in study of low-dimensional electronic models of strong correlation, due to the possibility that the normal state of the two-dimensional (2D) novel superconductivity[1] may share some interesting feature of a 1D interacting electron system (non-Fermi-liquid behavior). The one-band two-dimensional Hubbard model reduces to the t-J model in the large on-site energy limit. The Hubbard model and the t-J model have been under intense study through various approaches. For these strongly correlated electron models, few exact results may be obtained in two dimensions. The high-dimensional versions are much harder to study than their one-dimensional ones. In one dimension, however, the Bethe-ansatz technique may allow us to exactly solve Hamiltonians in some special cases, such as the Lieb-Wu solution[2] and the ordinary t-J model at its supersymmetric point.[3] In particular, the 1D long-range exactly solvable electronic models have attracted a lot of attention, since they display interesting physics with solutions of simple mathematical structure.[4–21]

Recently, we have been able to explicitly construct all the constants of motion for the translational invariant long-range supersymmetric t-J model, by mapping the system to a mixture of fermions and bosons, with the superalgebra representation for the electron fields.[12] Moreover, we have introduced a one-dimensional supersymmetric t-J model with $1/r^2$ hopping and exchange without translational invariance. This system has also been shown by us to be completely integrable, with infinite number of conserved quantities explicitly constructed.[12] In this work, we continue the study of this t-J model. A set of Jastrow eigenfunctions, as well as their eigenenergies, are obtained explicitly. The ground state of the t-J model is included in this set of wave functions. We also briefly discuss the structure of the full spectrum for the system.

The Hamiltonian for the one-dimensional t-J model is given by[12]

$$H_{t\text{-}J} = (1/2)P_G \left[-\sum_{1\le i\ne j\le L}\sum_{\sigma=1}^{N} t_{ij}(c^{\dagger}_{i\sigma}c_{j\sigma}) + \sum_{1\le i\ne j\le L} J_{ij}\left[P_{ij} - (1-n_i)(1-n_j)\right] \right] P_G, \tag{1}$$

where we take the hopping matrix and the spin exchange interaction to be $t_{ij}/2 = J_{ij} = 1/(r_i - r_j)^2$. L is the number of sites on the chain. In this model, the positions of the sites $\{r_i\}$ are given by the roots of the Hermite polynomial $H_L(x)$, and the spin component σ takes values from 1 to N. The operator $c^{\dagger}_{i\sigma}$ is the the creation operator for an electron at site i with spin σ; $c_{i\sigma}$ is the corresponding annihilation operator. Their anticommutation relations are given by $\{c_{i\sigma_i}, c^{\dagger}_{j\sigma_j}\}_+ = \delta_{ij}\delta_{\sigma_i\sigma_j}, \{c_{i\sigma_i}, c_{j\sigma_j}\}_+ = 0, \{c^{\dagger}_{i\sigma_i}, c^{\dagger}_{j\sigma_j}\}_+ = 0$. The Gutzwiller projection operator P_G projects out all the double or multiple occupancies, $P_G = \prod_{i=1}^{L} P_G(i)$, and $P_G(i) = \delta_{0,n_i} + \delta_{1,n_i}$, with $n_i = \sum_{\sigma=1}^{N} c^{\dagger}_{i\sigma}c_{i\sigma}$. The operator $P_{ij} = \sum_{\sigma=1}^{N}\sum_{\sigma'=1}^{N} c^{\dagger}_{i\sigma}c_{i\sigma'}c^{\dagger}_{j\sigma'}c_{j\sigma}$ exchanges the spins of the electrons at sites i and j, if both sites are occupied, and is zero otherwise. At half-filling, our t-J model becomes the long-range spin model, introduced first by Polychronakos on such a nontranslational-invariant lattice.[14]

In terms of the b and f fields, the eigenequation of the t-J model can be written as[12]

$$-\sum_{1\le i<j\le L} (q_i - q_j)^{-2} M_{ij}\phi(\{q\},\{\sigma\}) = E\phi(\{q\},\{\sigma\}), \tag{2}$$

where $\phi(\{q\},\{\sigma\}) = \phi(q_1\sigma_1, q_2\sigma_2, \ldots, q_{N_e}\sigma_{N_e} | q_{N_e+1}, q_{N_e+2}, \ldots, q_L)$ is the amplitude for the f fermions to be at $q_1, q_2, \ldots, q_{N_e}$, while the spinless b bosons are at $q_{N_e+1}, q_{N_e+2}, \ldots, q_L$. Here, $\{\sigma\} = (\sigma_1, \sigma_2, \ldots, \sigma_{N_e})$ and $\{q\} = (q_1, q_2, \ldots, q_L) = (x_1, x_2, \ldots, x_{N_e}, y_1, y_2, \ldots, y_Q)$. The wave function ϕ is symmetric in the b boson positions $\{y\}$, while antisymmetric when exchanging $x_i\sigma_i$ and $x_j\sigma_j$. The operator M_{ij} exchanges only the position variables q_i and q_j, that is, $M_{ij}\phi(\{q\},\{\sigma\}) = \phi(\{q'\},\{\sigma\})$, with $\{q\} = (q_1, q_2, \ldots, q_i, \ldots, q_j, \ldots, q_L)$ and $\{q'\} = (q_1, q_2, \ldots, q_j, \ldots, q_i, \ldots, q_L)$. In this approach, the f

fermions and the b bosons occupy the whole chain; i.e., $\{q\}$ and $\{q'\}$ are permutations of the sites $\{r_1, r_2, \ldots, r_L\}$, and we work in the Hilbert space where at each site there is exactly one particle. Q, the number of the b bosons, is also the number of holes in the original problem; N_e, the number of the f fermions, is also the number of the c electrons on the lattice. $\tilde{N}_\alpha$, with $\alpha = 1, 2, \ldots, N$, the number of the f fermions with spin component α, is also the number of the c electrons with spin component α on the chain.

The Hamiltonian in the first quantization, as given by Eq. (2), is

$$H = -\sum_{1\le i<j\le L} (q_i - q_j)^{-2} M_{ij}. \tag{3}$$

It commutes with the permutation operator $T_{ij} = P^\sigma_{ij} M_{ij}$ exchanging the f fermion spin and position simultaneously. Let us work in the Hilbert space where the number of fermions of each flavor is fixed, i.e., $\tilde{N}_\sigma$, $\sigma = 1, 2, \ldots, N$, is fixed. Consider the following wave function in Jastrow product form:

$$\phi(x_1\sigma_1, x_2\sigma_2, \ldots, x_{N_e}\sigma_{N_e} | y_1, y_2, \ldots, y_Q) = \prod_{i<j} (x_i - x_j)^{\delta_{\sigma_i\sigma_j}} e^{i\frac{\pi}{2}\mathrm{sgn}(\sigma_i - \sigma_j)}, \tag{4}$$

where $\{x\}$ and $\{y\}$ span the whole lattice. We would like to show that this wave function is an eigenstate of the system. The Hamiltonian in Eq. (2) can be broken up into three parts: The first part H_1 exchanges the f fermions, the second H_2 exchanges the b bosons, and the third H_3 exchanges the bosons and the fermions:

$$H_1 = (-1) \sum_{1\le i<j\le N_e} (q_i - q_j)^{-2} M_{ij},$$

$$H_2 = (-1) \sum_{1+N_e\le\alpha<\beta\le L} (q_\alpha - q_\beta)^{-2} M_{\alpha\beta},$$

$$H_3 = (-1) \sum_{N_e+1\le\alpha\le L} \sum_{1\le j\le N_e} (q_\alpha - q_j)^{-2} M_{\alpha j}. \tag{5}$$

We then calculate the effects of these three parts when acting the Jastrow wave function given by Eq. (4). The contribution from H_2 is immediate:

$$H_2\phi = -\sum_{\alpha<\beta} (y_\alpha - y_\beta)^{-2}\phi. \tag{6}$$

The contributions from H_1 and H_3 are harder to deal with since many particle terms are involved. Using a similar trick introduced in Refs. 8, 19, we have

$$\begin{aligned} H_3\phi &= -\sum_i \sum_\alpha (x_i - y_\alpha)^{-2} \prod_{j(\neq i)} \left(\frac{y_\alpha - x_j}{x_i - x_j}\right)^{\delta_{\sigma_i\sigma_j}} \phi \\ &= -\sum_i \sum_\alpha (x_i - y_\alpha)^{-2} \prod_{j(\neq i)} \left(1 + \delta_{\sigma_i\sigma_j} \frac{y_\alpha - x_i}{x_i - x_j}\right) \phi \\ &= -\sum_i \sum_\alpha \left[(x_i - y_\alpha)^{-2} + \sum_{j(\neq i)} \frac{\delta_{\sigma_i\sigma_j}}{(y_\alpha - x_i)(x_i - x_j)}\right] \phi - \text{rest}, \end{aligned} \tag{7}$$

where

$$\text{rest} = \sum_\alpha \sum_{\sigma=1}^{N} \sum_{r=3}^{N_e} \sum_{\substack{J\subset\wp^\sigma \\ |J|=r}} \left[\sum_{i\in J} (y_\alpha - x_i)^{r-3} \prod_{x_j\in X_J/x_i} \frac{1}{(x_i - x_j)}\phi\right], \tag{8}$$

with $\wp^\sigma = \{k \in \{1, 2, \ldots, N_e\}; \sigma_k = \sigma\}, X_J = \{x_j; j \in J\}$. Then using the fact that for any set $X = (x_1, x_2, \ldots, x_n)$, we have the identity (see Appendix A)

$$\sum_{i=1}^{n} x_i^t \prod_{x_j\in X/x_i} \frac{1}{(x_i - x_j)} = 0, \tag{9}$$

for all $t = 0, 1, 2, \ldots, n-2$, we conclude that rest $= 0$.

The contribution from H_1 is calculated in a similar manner:

$$\begin{aligned} H_1\phi &= -\sum_{i<j} (x_i - x_j)^{-2}(1 - 2\delta_{\sigma_i\sigma_j}) \prod_{k(\neq i,j)} \left(\frac{x_k - x_j}{x_k - x_i}\right)^{\delta_{\sigma_i\sigma_k}} \left(\frac{x_k - x_i}{x_k - x_j}\right)^{\delta_{\sigma_j\sigma_k}} \phi \\ &= \sum_{i<j} \delta_{\sigma_i\sigma_j} (x_i - x_j)^{-2}\phi - \sum_{i<j} (x_i - x_j)^{-2}(1 - \delta_{\sigma_i\sigma_j}) \prod_{k(\neq i,j)} \left(1 + \delta_{\sigma_i\sigma_k}\frac{x_i - x_j}{x_k - x_i} - \delta_{\sigma_j\sigma_k}\frac{x_i - x_j}{x_k - x_j}\right) \\ &= \sum_{i\neq j} \delta_{\sigma_i\sigma_j} (x_i - x_j)^{-2}\phi - \sum_{i<j} (x_i - x_j)^{-2}\phi \\ &\quad - \sum_{i<j} (x_i - x_j)^{-2}(1 - \delta_{\sigma_i\sigma_j}) \sum_{\substack{K_1\subset\{1,2,\ldots,N_e\}/ij \\ K_2\subset\{1,2,\ldots,N_e\}/ij \\ K_1\cup K_2\neq 0}} \prod_{k\in K_1} \delta_{\sigma_i\sigma_k}\frac{x_i - x_j}{x_k - x_i} \cdot \prod_{k\in K_2} \frac{x_i - x_j}{x_k - x_j}(-\delta_{\sigma_j\sigma_k})\phi. \end{aligned} \tag{10}$$

Using the sum rule Eq. (9), and the fact that $\sum_{i\neq j\neq k}(x_k-x_i)^{-1}(x_k-x_j)^{-1}\delta_{\sigma_i\sigma_j}\delta_{\sigma_i\sigma_k}=0$, the last term in the above equation becomes

$$-\phi\sum_{i<j}(1-\delta_{\sigma_i\sigma_j})\sum_{k(\neq i,j)}\left[(x_i-x_j)^{-1}(x_k-x_i)^{-1}\delta_{\sigma_i\sigma_k}-(x_i-x_j)^{-1}(x_k-x_j)^{-1}\delta_{\sigma_j\sigma_k}\right]$$

$$=-\phi\sum_{i\neq j}\sum_{k(\neq i,j)}(x_i-x_j)^{-1}(x_k-x_i)^{-1}\delta_{\sigma_i\sigma_k}. \quad (11)$$

In the end, we have

$$H\phi=-\left[\sum_{1\leq i<j\leq L}(q_i-q_j)^{-2}\right]\phi+\sum_{i\neq j}(x_i-x_j)^{-1}\delta_{\sigma_i\sigma_j}\left[\sum_{k(\neq i)}(x_i-x_k)^{-1}+\sum_{\alpha}(x_i-y_\alpha)^{-1}\right]\phi. \quad (12)$$

Using the properties of the roots of the Hermite polynomial

$$r_i=\sum_{j(\neq i)}(r_i-r_j)^{-1},\qquad \sum_{1\leq i<j\leq L}(r_i-r_j)^{-2}=L(L-1)/4, \quad (13)$$

we thus conclude that the wave function ϕ is an eigenstate with eigenvalue

$$E=-L(L-1)/4+(1/2)\sum_{\sigma=1}^{N}(\tilde{N}_\sigma-1)\tilde{N}_\sigma. \quad (14)$$

Although it is expected that this wave function is the lowest-energy state in the subspace of fixed $\tilde{N}_1,\tilde{N}_2,\ldots,\tilde{N}_N$, we were not able to prove it. However, in the case of SU(2), the small lattice diagonalization up to eight sites confirms this conjecture. Moreover, the discussion below will also confirm this idea for the general case. For fixed number N_e of the electron number, the minimum of the energy is obtained when $|\tilde{N}_\sigma-\tilde{N}_{\sigma'}|$ is as small as possible for each pair $\sigma\neq\sigma'$.

In the SU(2) case, the above result becomes

$$E=(-1)L(L-1)/4+(1/2)\tilde{N}_\uparrow(\tilde{N}_\uparrow-1)+(1/2)\tilde{N}_\downarrow(\tilde{N}_\downarrow-1), \quad (15)$$

where $\tilde{N}_\uparrow$ and $\tilde{N}_\downarrow$ are the numbers of the up-spin electrons and the down-spin electrons respectively. For fixed number of electrons on the chain, i.e., for fixed N_e, the minimum of the energy given in Eq. (15) is obtained when $S_z=0$ for even N_e, or when $S_z=\pm 1/2$ for odd N_e. Therefore, the ground state is a spin singlet (respectively spin 1/2) state for even (respectively odd) number of electrons on the chain. In particular, for an even number of electrons on the chain, the ground state energy is

$$E_G=(-1/4)L(L-1)+N_e^2/4-N_e/2, \quad (16)$$

while for an odd number of electrons it is

$$E_G=(-1/4)L(L-1)+\left(\frac{N_e-1}{2}\right)^2. \quad (17)$$

The charge susceptibility of the ground state χ_c is given by $\chi_c^{-1}=\partial^2E_G/\partial N_e^2=1/2$, independent of the electron concentration. Very unexpectedly, the charge susceptibility is also finite at half-filling $N_e=L$, in spite of the existence of a metal-insulator phase transition at half-filling for this system. This is in contrast to the case of the periodic $1/r^2$ supersymmetric t-J model, where the charge susceptibility is divergent at half-filling, at which the metal-insulator phase transition occurs.

To study the spectrum of the system away from half-filling, we follow the idea introduced in Ref. 14. Let us define the operators

$$\pi_j=i\sum_{k(\neq j)}(q_j-q_k)^{-1}M_{jk}=\pi_j^\dagger,$$
$$a_j^\dagger=\pi_j+iq_j,$$
$$a_j=\pi_j-iq_j, \quad (18)$$

which satisfy the following commutation relations:

$$[\pi_j,\pi_k]=0,$$
$$[q_j,H]=i\pi_j,$$
$$[\pi_j,H]=-2i\sum_{k(\neq j)}(q_j-q_k)^{-3}. \quad (19)$$

Then using the property of the roots of the Hermite polynomial we have

$$[\pi_j,H]=-iq_j \quad (20)$$

and

$$[a_j^\dagger,H]=-a_j^\dagger,$$
$$[a_j,H]=a_j. \quad (21)$$

Therefore the operators $A_i^\dagger(\nu)=a_i^\dagger S_i^{(\nu)}, i=1,2,\ldots,N_e$, where $\nu=0,\pm,z$ for the SU(2) case with $S_i^{(0)}=1$, will act as raising operators, while their Hermitian conjugate $A_i(\nu)$ will act as lowering operators. It thus follows that the wave function

$$\phi_{\{n\},\{\nu\}}=\sum_P\prod_{i=1}^{N_e}[A_i^\dagger(\nu_i)]^{n_{P_i}}\phi, \quad (22)$$

with $\{n\}=(n_1,n_2,\ldots,n_{N_e}), n_i\geq 0, \{\nu\}=(\nu_1,\nu_2,\ldots,\nu_{N_e})$, is either an eigenstate with energy

$$E_{\{n\}} = E + \sum_{i=1}^{N_e} n_i \tag{23}$$

or zero.

Moreover, it is shown in Appendix B that the operators $\sum_{i=1}^{N_e} A_i(\nu_i)$ with $\nu_i = 0$ or z and a_α annihilate the wave function ϕ, and also $\sum_{i=1}^{N_e} A_i(\pm)\phi_G = 0$, confirming the conjecture that ϕ is the lowest-energy state in the subspace. We then arrive at the conjecture that the excitation spectrum of the system is of the form

$$E(s) = E + s, \quad s \in (0, 1, 2, \ldots, s_{\max}); \tag{24}$$

i.e., the spectrum of this t-J model consists of equal-spaced energy levels. Since the model is on a finite chain, $s_{\max}$ is finite. In the special case of SU(2), the small lattice diagonalization up to eight sites suggests that the highest-energy level is given by

$$E_{\max}(Q) = L(L-1)/4 - Q(Q-1)/2; \tag{25}$$

i.e., for an even number of electrons,

$$E_{\max} = E_G + (1/4)N_e(4L - 3N_e), \tag{26}$$

where Q is the number of holes on the chain and $N_e = L - Q$ is the number of electrons. For $N_e = 1$ or $N_e = L$ (half-filling), this formula gives right results; moreover, at half-filling, this corresponds to all spins polarized in one direction.

The feature of the t-J model spectrum consisting of equal-distant energy levels may also be seen by taking the strong interaction limit of the Sutherland-Calogero-Morse quantum system for a mixture of fermions and bosons;

$$H = (-1/2)\sum_{i=1}^{L} \partial^2/\partial q_i^2 + \sum_{i=1}^{L} l^2 q_i^2/2 + \sum_{i<j} l(l - M_{ij})/(q_i - q_j)^2, \tag{27}$$

where there are N_e fermions with spins and Q spinless bosons; M_{ij} permutes the positions of the particles i and j only. The mixture gas has equal-distant energy levels described in terms of effective harmonic oscillators. In the strong interaction limit, the elastic modes decouple from the internal degrees of freedom. Since elastic modes also consists of equal-distant energy levels, we thus are led to the conclusion that the spectrum of the internal dynamics, which is that of our t-J model, also consists of equal-distant energy levels. Further work is necessary for a fully complete proof that the t-J model full spectrum takes the form Eq. (24).

Finally, we would like to point out that the states of the t-J model in the whole Hilbert space are grouped into a structure of "spin supermultiplets," as indicated by the small lattice diagonalization, similar to that of the periodic $1/r^2$ supersymmetric t-J model. Such pattern of the Hilbert space is related to the symmetries associated with the Hamiltonian. It is highly worth while to identify them more explicitly, and we would like to study these aspects in further work.

In summary, a set of Jastrow eigenfunctions have been found for the t-J model, with the eigenenergies explicitly calculated. The expected ground state of the t-J model is included in this set of wave functions. The full spectrum of the t-J model is found to have equal-distant energy levels which are highly degenerate. It would be very interesting to understand the underlying symmetry principles that give rise to such simple Hilbert space structure. It remains to study various correlation functions, as well as the thermodynamics, for this strongly correlated electron system. It would also be very interesting to study the effective field theory for the low-lying excitations for this t-J model.

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APPENDIX A

In this appendix, we provide a brief proof for the sum rule Eq. (9) for reader's convenience. The same argument can also be found in previous works Refs. 8, 19. Let $X = (x_1, x_2, \ldots, x_n), X_i = X/x_i$; we wish to show

$$\sum_{i=1}^{n} x_i^t \prod_{x_j \in X_i} \frac{1}{x_i - x_j} = 0, \forall t = 0, 1, 2, \ldots, (n-2). \tag{A1}$$

Indeed, the Vandermonde determinant $V(X)$ has the property

$$\prod_{j=1}^{n}(x - x_j) = \frac{V(Xx)}{V(X)}, \tag{A2}$$

where $X = (x_1, x_2, \ldots, x_n), Xx = (x_1, x_2, \ldots, x_n, x)$. Therefore we obtain

$$\sum_{i=1}^{n} x_i^t \prod_{x_j \in X_i} \frac{1}{x_i - x_j} = \sum_{i=1}^{n} x_i^t \frac{V(X_i)}{V(X_i x_i)} = \sum_{i=1}^{n} (-1)^{i-n} x_i^t \frac{V(X_i)}{V(X)} = \frac{(-1)^{n-1}}{V(X)} \det \begin{pmatrix} x_1^t & x_2^t & \ldots & x_n^t \\ 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ \vdots & \vdots & \ldots & \vdots \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \end{pmatrix} = 0.$$

This thus proves the sum rule Eq. (9).

APPENDIX B

In this appendix, we shall show that the lowering operators $A_i(z)$ and a_α give zero when acting on the Jastrow wave function ϕ given by Eq. (4). This will yield a partial confirmation that the wave function ϕ is the lowest-energy state in the subspace where the number of particles of each spin component is fixed. For the b boson degrees of freedom, we have the property

$$a_i\phi = 0, i \in (N_e + 1, N_e + 2, \ldots, L), \tag{B1}$$

which is shown to be true using the sum rule Eq. (9) and the property of the Hermite polynomial roots $\sum_{j(\neq i)}(r_i - r_j)^{-1} = r_i$. The procedure to deal with the permutation operator M_{ij} in a_i is very similar to that of proving ϕ to be the eigenenergy state of the Hamiltonian, but we do not write the full details here. Combining the Eq. (B1) with the fact $\sum_{i=1}^{L} a_i = 0$, we thus arrive at the following results:

$$\sum_{\alpha=N_e+1}^{L} a_\alpha\phi = 0,$$
$$\sum_{i=1}^{N_e} a_i\phi = 0. \tag{B2}$$

Furthermore, we realize that

$$\left[\sum_{i=1}^{N_e} A_i(z)\right]\phi = 0. \tag{B3}$$

We have been able to show this to be true, following the similar approach to handle the effect of the permutation operator M_{ij} acting on the Jastrow wave function ϕ. In the particular case where $\tilde{N}_\uparrow = \tilde{N}_\downarrow$, the wave function ϕ is a spin singlet and we may globally rotate Eq. (B3) in the spin space, giving us

$$\left[\sum_{i=1}^{N_e}(A_i(\pm))\right]\phi = 0. \tag{B4}$$

In summary, we have proved that it is impossible to construct nonvanishing eigenstates with the lowering operators and the wave functions ϕ in the subspace where the number of electrons of each flavor is fixed.

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