

## Finite-temperature properties of quantum antiferromagnets in a uniform magnetic field in one and two dimensions

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(Received 6 January 1994)

Consider a  $d$ -dimensional antiferromagnet with a quantum disordered ground state and a gap to bosonic excitations with nonzero spin. In a finite external magnetic field, this antiferromagnet will undergo a phase transition to a ground state with nonzero magnetization, describable as the condensation of a dilute gas of bosons. The finite-temperature properties of the Bose gas in the vicinity of this transition are argued to obey a hypothesis of *zero scale-factor universality* for  $d < 2$ , with logarithmic violations in  $d = 2$ . Scaling properties of various experimental observables are computed in an expansion in  $\epsilon = 2 - d$ , and exactly in  $d = 1$ .

### I. INTRODUCTION

Numerous experiments<sup>1</sup> have by now examined the properties of  $S = 1$  spin chain antiferromagnets which possess the Haldane gap.<sup>2</sup> More recently, these antiferromagnets have been placed in a strong magnetic field,<sup>3</sup> and found to display evidence of a zero-temperature phase transition to a state with a nonzero ground state magnetization. Theoretical studies<sup>4-7</sup> have also examined this transition at zero temperature, and developed a picture of it as the Bose condensation of magnons with azimuthal spin  $S_z = 1$ . Although this possibility has not been considered before, it is not difficult to see that the condensation of magnons in a finite field should occur in quantum-disordered antiferromagnets in any dimension  $d$  (provided, of course, the magnons continue to behave as bosons). In this paper we shall present a general theory, in dimensions  $d \leq 2$ , of the finite-temperature properties of quantum antiferromagnets in the vicinity of such a field-induced quantum transition.

We begin by elucidating the precise conditions under which our results apply. Consider an antiferromagnet with a quantum-disordered ground state. The Hamiltonian must possess at least an axial symmetry; i.e., at least one component (say  $z$ ) of the total spin must commute with the Hamiltonian. The lowest excitation with nonzero spin must be separated from the ground state by a gap, and behave like a quasiparticle with bosonic statistics. These conditions are clearly satisfied by Haldane gap antiferromagnets in which the in-plane anisotropy can be neglected (the compound NENP does have a rather small in-plane anisotropy,<sup>1</sup> and the restrictions this places on applying our results to experiments in NENP will be discussed later). In  $d = 2$ , the

$S = 1/2$  *kagomé* Heisenberg antiferromagnet<sup>8</sup> is probably the most accessible candidate upon which our results can be tested—it has been argued that the magnons in this system are bosons.<sup>9,10</sup>

Now place this antiferromagnet in a magnetic field pointing along the direction of axial symmetry. The eigenenergy  $\epsilon(k)$  of single magnon (boson) quasiparticles with momentum  $k$  in a field  $H$  then takes the form

$$\epsilon(k) = \Delta + \frac{\hbar^2 k^2}{2m} - g\mu_B S_z H. \quad (1.1)$$

Here  $\Delta$  is the magnon gap and  $m$  is the quasiparticle mass, both determined in the zero field antiferromagnet. The wave vector  $k$  is small, and is measured from ordering wave vector of the corresponding classical antiferromagnet. For the  $S = 1$  Haldane gap chain, we have the azimuthal spin  $S_z = 1$ . In the *kagomé* antiferromagnet, the magnon excitations have been argued to be spinons:<sup>9,10</sup> We therefore expect  $S_z = 1/2$ . Later, we will discuss how the field induced transition in the *kagomé* antiferromagnet may offer a way of experimentally and/or numerically determining the value of  $S_z$ .

The antiferromagnet will undergo a  $T = 0$  field-induced transition at the field  $H = H_c$  which is given exactly by

$$g\mu_B S_z H_c = \Delta. \quad (1.2)$$

In the vicinity of this transition, we may describe low-energy properties of the antiferromagnet by just studying an effective Hamiltonian for the bosonic magnons.<sup>4,5</sup> The remainder of this paper will therefore consider properties of the following coherent state path integral over the Bose field  $\Psi(x, \tau)$ :

$$Z = \int \mathcal{D}\Psi \exp \left( -\frac{1}{\hbar} \int_0^{\hbar/k_B T} d\tau \mathcal{L}(\tau) \right),$$

$$\mathcal{L}(\tau) = \int d^d x \left[ \hbar \Psi^*(x, \tau) \frac{\partial \Psi(x, \tau)}{\partial \tau} - \frac{\hbar^2}{2m} \Psi^*(x, \tau) \nabla^2 \Psi(x, \tau) - \mu |\Psi(x, \tau)|^2 \right] + \frac{1}{2} \int d^d x d^d x' |\Psi(x, \tau)|^2 v(x - x') |\Psi(x', \tau)|^2, \quad (1.3)$$

where  $x$  is the  $d$ -component spatial coordinate,  $\tau$  is the Matsubara time, the chemical potential

$$\mu = g\mu_B S_z H - \Delta, \quad (1.4)$$

and  $v$  is a repulsive interaction of a short-range  $\sim \Lambda^{-1}$ . This field theory has a phase transition exactly at  $T = 0$ ,  $\mu = 0$  which has been studied by Fisher *et al.*<sup>11</sup> They identified the upper-critical dimension as  $d = 2$ , above which the interaction  $v$  is irrelevant. For  $d < 2$ ,  $v$  is relevant, but the exponents were nevertheless found to have trivial values: The dynamic exponent  $z = 2$ , the correlation length exponent  $\nu = 1/2$ , and the field anomalous dimension  $\eta = 0$ . The triviality of the exponents is partially related to the fact that the parameter tuning the system through the transition,  $\mu$ , couples to a conserved quantity—the density of bosons; any such transition<sup>19</sup> must have  $z\nu = 1$ . The structure of the  $d < 2$ , finite- $v$  fixed point is thus very unusual: Despite describing a nontrivial, interacting, critical field theory, the exponents associated with all the relevant directions away from this fixed point are trivial. In this paper, we shall show that the interactions are crucial in determining the finite-temperature properties of the Bose gas near this fixed point. The fixed-point interactions are needed to preserve hyperscaling for  $d < 2$  and lead to highly nontrivial scaling functions for the finite-temperature correlations.

Before we state our zero-scale-factor universality hypothesis for  $Z$  in its most general form, it is helpful to consider one of its simple consequences at  $T = 0$ . Examine the ground state boson density  $n = \langle |\Psi(x, \tau)|^2 \rangle$  as a function of  $\mu$  for small  $\mu$ . This problem was studied many years ago for the  $d = 3$  hard-sphere Bose gas<sup>12</sup> with the result

$$n = \left[ \frac{2m\mu}{\hbar^2} \frac{1}{8\pi a} + \mathcal{O}(\mu^2) \right] \theta(\mu), \quad d = 3, \quad T = 0, \quad (1.5)$$

where  $a$  is the hard-sphere radius, and  $\theta(x)$  is the unit step function. Note that, in addition to its dependence on  $m$  and  $\mu$ , the boson density is sensitive to the nature of the boson-boson interactions (measured by the hard-sphere radius  $a$ ). A different choice for the boson-boson repulsion would lead to different result for  $n$ . The situation in dimensions  $d < 2$  is however strikingly different; one manifestation of the zero-scale-factor universality is that for small  $\mu$

$$n = \left( \frac{2m\mu}{\hbar^2} \right)^{d/2} C \theta(\mu), \quad d < 2, \quad T = 0, \quad (1.6)$$

where  $C$  is a *universal number*, i.e., independent of the details of the interactions between the bosons. We will determine  $C$  in a  $d = 2 - \epsilon$  expansion; its exact value in  $d = 1$  is known:<sup>5</sup>  $C = 1/\pi$ . The universality of  $C$  is a direct consequence of having a finite-coupling fixed point describing the onset at  $\mu = 0$ . For  $d > 2$ , interactions are irrelevant, which leads to a violation of hyperscaling, and a dependence of the density on the nature of the microscopic interactions [as in the  $a$  dependence of (1.5)]. Further,  $n$  will depend linearly on  $\mu$  for all  $d > 2$ . Pre-

cisely in  $d = 2$ , as we shall see below, (1.6) is violated only by a logarithmic dependence on the microscopic interactions.

For  $d < 2$ , the combination of the presence of hyperscaling, and the absence of any anomalous exponents in the leading critical behavior, leads to remarkably universal finite-temperature properties. As in Ref. 13, we may use finite-size scaling to deduce scaling forms away from  $\mu = 0$ ,  $T = 0$ . However, the absence of any anomalous dimensions ( $z = 2$ ,  $\eta = 0$ ,  $\nu = 1/2$ ) means that the usual two-scale-factor universality<sup>14</sup> is now reduced to a *zero-scale-factor universality* described more precisely in the following subsection.

### A. Zero-scale-factor universality

In simple terms, this universality is just the statement that all response functions are universal functions of the *bare* coupling constants  $\mu$  and  $m$ . There are no nonuniversal amplitudes; the usual case has two nonuniversal amplitudes, or scale factors.<sup>14</sup> The universality can be stated more precisely in terms of the the boson Green's function

$$G(x, \tau) = \langle \mathcal{T} \Psi(x, \tau) \Psi^*(0, 0) \rangle, \quad (1.7)$$

where  $\mathcal{T}$  is the ordering symbol in imaginary time  $\tau$ . After Fourier transformation as per

$$G(k, i\omega_n) = \int d^d x \int_0^{\hbar/(k_B T)} d\tau e^{-i(k \cdot x - \omega_n \tau)} G(x, \tau), \quad (1.8)$$

this yields  $G(k, i\omega_n)$  at the Matsubara frequencies  $\omega_n = 2\pi n T / \hbar$ , from which the retarded Green's function  $G^R(k, \omega)$  can be obtained by analytic continuation to real frequencies using

$$G^R(k, \omega) = -G(k, i\omega_n = \omega). \quad (1.9)$$

The Lehman spectral representation of the Green's function implies that  $-\omega \text{Im} G^R(k, \omega) > 0$ , but does not constrain  $G^R$  to be an odd function of  $\omega$ . Our central result is the zero-scale-factor universality of  $G^R$ , which is equivalent to the scaling form

$$G^R(k, \omega) = \frac{\hbar}{k_B T} \mathcal{A} \left( \frac{\hbar \omega}{k_B T}, \frac{\hbar k}{\sqrt{2m k_B T}}, \frac{\mu}{k_B T} \right), \quad (1.10)$$

where  $\mathcal{A}$  is a highly nontrivial, but completely universal complex-valued function; naturally,  $\mathcal{A}$  is independent of the nature of the boson-boson repulsion. An important property of  $\mathcal{A}$  is that it is *analytic at all finite, real, values of all three arguments*. Similar scaling forms hold for other correlators of  $\Psi$ —a particularly instructive observable is the *local* Green's function  $G_l^R$ ,

$$G_l^R(\omega) = \int \frac{d^d k}{(2\pi)^d} G^R(k, \omega). \quad (1.11)$$

If we take the imaginary part of this equation, it is expected that the resulting on-shell contributions on the right-hand side will occur only at small momenta (determined by the frequency  $\omega$ ), and the momentum integral

is ultraviolet convergent. We may therefore deduce from (1.10) the scaling form

$$\text{Im}G_\ell^R(\omega) = -\frac{1}{\omega} \left| \frac{m\omega}{\hbar} \right|^{d/2} F\left(\frac{\hbar\omega}{k_B T}, \frac{\mu}{k_B T}\right), \quad (1.12)$$

where  $F$  is a fully universal, dimensionless, positive function. It is quite instructive to consider the limiting behavior of  $F$  for small and large frequencies. We expect that  $G_\ell^R$  should be analytic at  $\omega = 0$  at any finite  $T$ ; this combined with the positivity condition on the spectral weight noted above implies  $\text{Im}G_\ell^R(\omega) \sim \omega$  for small  $\omega$  at finite  $T$ . Therefore, from (1.12) the scaling function  $F$  must satisfy  $F(\bar{\omega}, t) \sim |\bar{\omega}|^{2-d/2}$  at small  $\bar{\omega}$  [we use here and henceforth the notation  $\bar{\omega} \equiv \hbar\omega/(k_B T)$ , and  $t \equiv \mu/(k_B T)$ ]; the coefficient of the  $|\bar{\omega}|^{2-d/2}$  term is quite difficult to determine, and will be obtained in this paper only in a special limit. For large  $\omega$ , or short times,  $G^R$  should display essentially free particle behavior, as the dilute bosons have not had enough time to interact with each other. Using the free boson spectral weight we can deduce that

$$F(\bar{\omega}, t) = \frac{(2\pi)^{1-d/2}}{\Gamma(d/2)} \theta(\omega) \quad \text{as } |\bar{\omega}| \rightarrow \infty. \quad (1.13)$$

Let us conclude this subsection by noting the precise conditions under which the system is in the critical region and (1.10) and (1.12) are valid. We must have

$$|\mu|, k_B T \ll \frac{\hbar^2 \Lambda^2}{2m}, v(0). \quad (1.14)$$

Further the measurement wave vectors must satisfy

$$k \ll \Lambda. \quad (1.15)$$

In  $d = 2$  the zero-scale-factor universality is violated by a logarithmic dependence on the microscopic interactions. Furthermore, the scaling function  $\mathcal{A}$  will have a singularity associated with the finite-temperature Kosterlitz-Thouless transition. Zero-scale-factor universality does not hold for  $d > 2$ .

## B. Neutron scattering

The dynamic information contained in  $G^R$  is directly observable in neutron scattering experiments. In Appendix A we discuss the relationship between the correlators of  $\Psi$  and antiferromagnetic correlations measured by the neutrons; this discussion is limited to the case where the quantum-disordered phase has confined spinons, i.e.,  $S_z = 1$ , as in Haldane gap antiferromagnets. The relationship for the case of deconfined spinons ( $S_z = 1/2$ ) is quite different<sup>15</sup> and will not be considered in this paper explicitly.

In the following,  $\hbar\omega$  will measure the energy *lost* by the neutrons in their interaction with the antiferromagnet. Consider first a scattering even in which the antiferromagnet undergoes a  $\Delta S_z = +1$  transition. Then, from the discussion in Appendix A and the fluctuation-dissipation theorem we may conclude that the scattering cross section is given by the dynamic structure factor  $S_{+-}(k, \omega)$  where

$$S_{+-}(k, \omega) = \frac{-2Z\text{Im}G^R(k, \omega)}{1 - e^{-\hbar\omega/k_B T}}, \quad (1.16)$$

where  $Z$  is a *nonuniversal* quasiparticle renormalization factor between the magnon operators and the ones that couple to the neutrons. (The wave vector  $k$  on the left-hand side is measured from the antiferromagnetic ordering wave vector.) The scaling result for  $S_{+-}$  thus follows directly from (1.10). Next, consider scattering with  $\Delta S_z = -1$  for the antiferromagnet. The associated dynamic structure factor is then  $S_{-+}(k, \omega)$  which is

$$S_{-+}(k, \omega) = \frac{2Z\text{Im}G^R(k, -\omega)}{1 - e^{-\hbar\omega/k_B T}}. \quad (1.17)$$

By not resolving the energy of the scattered neutron, it is also possible to measure equal-time correlations embodied in the static structure factors. However, the scaling properties of these observables are subtle, and require more careful interpretation. By performing a weighted frequency integral over the scaling limit of  $G^R$ , we are implicitly only sensitive to frequencies much smaller than a high frequency cutoff like  $\Delta/2$ . Thus in the following, our equal-time structure factors actually refer to scattering experiments in which energy transfers greater than  $\Delta/2$  are not integrated over. Not blocking out these events will produce a background structure factor which may (as in the Haldane gap region defined below) overwhelm the universal part of the static structure factor which is considered here. Keeping this caveat in mind, we define the structure factor  $S_{+-}(k)$ ,

$$S_{+-}(k) = \int_{\sim-\Delta/2}^{\sim\Delta/2} \frac{d\omega}{2\pi} S_{+-}(k, \omega), \quad (1.18)$$

where the result is not sensitive to the precise locations of the limits. The correlator  $S_{-+}$  can be defined analogously. From (1.18) and (1.10) we can deduce the scaling result

$$S_{+-}(k) = Z\mathcal{B}_{+-}\left(\frac{\hbar k}{\sqrt{2mk_B T}}, \frac{\mu}{k_B T}\right), \quad (1.19)$$

where  $\mathcal{B}_{+-}$  is a universal scaling function determined completely by  $\mathcal{A}$  [similarly for  $S_{-+}(k)$ ]. Finally, using the fact that  $\Psi$  and  $\Psi^*$  are canonically conjugate fields, it is possible to deduce a frequency sum rule on  $\text{Im}G^R$  which leads to

$$\mathcal{B}_{+-}(r, t) = \mathcal{B}_{-+}(r, t) + 1, \quad (1.20)$$

where we will henceforth use  $r \equiv \hbar k/\sqrt{2mk_B T}$ .

We reiterate that all of the above results in this subsection refer only to the case of antiferromagnets with confined spinons.

## C. Uniform magnetization

A second useful set of observables are those associated with magnetic fluctuations around  $k = 0$ . The uniform magnetization is a conserved quantity and, as a result, the quasiparticle renormalization factor  $Z$  does not appear in their scaling forms. The simplest of these is mean

value of the magnetization density  $M$  itself, which is of course related to the magnon density by

$$M = g\mu_B S_z n. \quad (1.21)$$

This relationship, and the considerations of this subsection, apply to both confined ( $S_z = 1$ ) and deconfined ( $S_z = 1/2$ ) spinons. The mean value of the magnetization is therefore

$$M = g\mu_B S_z G(x = 0, \tau = 0^-) \quad (1.22)$$

and obeys the scaling form

$$M = g\mu_B S_z \left( \frac{2mk_B T}{\hbar^2} \right)^{d/2} \mathcal{M} \left( \frac{\mu}{k_B T} \right), \quad (1.23)$$

where the function  $\mathcal{M}$  is again dependent only on  $\mathcal{A}$ . The result (1.6) follows from (1.23).

#### D. Phase diagram

The above discussion shows that the scaling functions of a large number of experimental observables can be obtained directly from the primary scaling function  $\mathcal{A}$ . The remainder of the paper is therefore devoted primarily to describing  $\mathcal{A}$  and associated scaling functions in different parameter regimes. It is convenient to discuss the properties of  $\mathcal{A}$  separately in three distinct regimes, which are analogous to those found by Chakravarty *et al.*<sup>16</sup> in the  $d = 2$  O(3)  $\sigma$  model. These regimes are illustrated in the phase diagrams in Figs. 1 ( $d = 1$ ) and 2 ( $d = 2$ ). The crossover boundaries between the regimes are delineated by the value of the dimensionless ratio  $\mu/k_B T$  (up to logarithmic terms in  $d = 2$ ). We consider the three cases separately.

##### 1. $\mu \ll -k_B T$

This is the analog of the quantum-disordered regime of Refs. 16 and 13. Only a dilute gas of thermally excited bosons is present, and their mutual interactions are weak. Properties of the quantum antiferromagnet can be described by a low-magnon-density expansion about the quantum-disordered ground state. In  $d = 1$ , we identify this as the Haldane gap regime in Fig. 1.

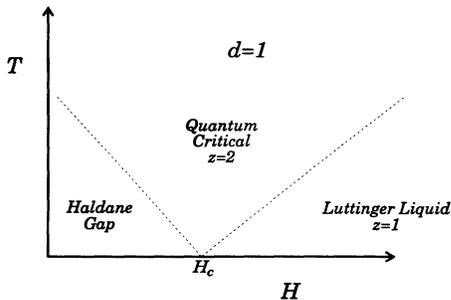


FIG. 1. Phase diagram in  $d = 1$ . There is a line of Luttinger liquid critical fixed points at  $T = 0$  for  $H > H_c$  which have  $z = 1$ . The critical end point at  $T = 0$ ,  $H = H_c$  has  $z = 2$ ; the zero-scale-factor universality is a property of this critical end point. The dashed lines indicate crossovers.

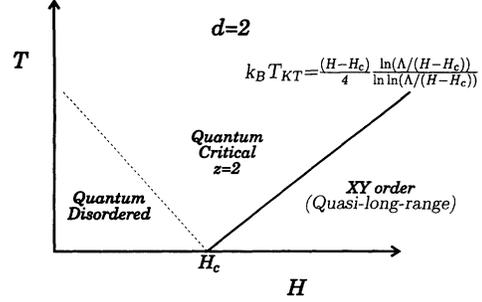


FIG. 2. Phase diagram in  $d = 2$ . The dashed line is a crossover while the solid line is a Kosterlitz-Thouless phase transition. The field  $H$  has absorbed a factor of  $g\mu_B$ . The location of the Kosterlitz-Thouless transition is determined from Refs. 17 and 18.

##### 2. $|\mu| \ll k_B T$

This is the quantum-critical regime, in which  $z = 2$  critical fluctuations are quenched in a universal way by the temperature. The value  $k_B T$  is the dominant energy scale and universally determines everything: the boson density, spectrum, and interactions. The only small parameter which may be used to determine scaling functions is  $\epsilon = d - 2$ . As we will see, this is not particularly effective in  $d = 1$  where, fortunately, exact methods are available.

It is instructive to consider the physics of this region as a function of the measurement frequency  $\omega$  (see Fig. 3). At large frequencies,  $\omega \gg k_B T$ , or short times, the effects of finite temperature have not yet become manifest, and the system displays the physics of the  $\mu = T = 0$  critical field theory; i.e., it is a dilute gas of bosonic quasiparticles

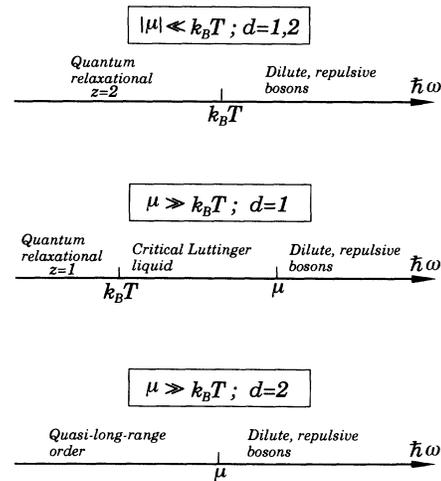


FIG. 3. Properties of the different regimes of Figs. 1 and 2 as a function of the measurement frequency  $\omega$  (the wave vector  $k \approx 0$ ). All crossovers are described by the universal scaling function  $\mathcal{A}$ . The crossover for  $\mu \gg k_B T$  in  $d = 1$  near  $\hbar\omega \sim k_B T$  is also described by the Luttinger liquid scaling function  $\mathcal{A}_L$ .

with repulsive interactions. As one lowers the frequency through  $k_B T/\hbar$  there is a crossover to a novel  $z = 2$ , quantum-relaxational regime (Fig. 3). Now each boson interacts strongly with thermally excited partners, leading to strong dissipation and overdamped quasiparticles.

### 3. $\mu \gg k_B T$

The behavior in  $d = 1$  and  $d = 2$  is quite different and we will therefore consider the two cases separately.

In  $d = 1$ , the ground state is a Luttinger liquid, which is itself a critical phase with  $z = 1$  (Fig. 1). Again, consider the physics as a function of  $\omega$  (Fig. 3). For sufficiently large  $\omega$  ( $\hbar\omega \gg \mu$ ) we have the dilute Bose gas physics of the  $\mu = T = 0$  critical point, similar to that discussed above for the quantum-critical region. At smaller  $\omega$ , there is a crossover (near  $\hbar\omega = \mu$ ) to a Luttinger-liquid-like region where we may as well assume that  $T = 0$ . However, at small enough frequencies,  $\hbar\omega \sim k_B T$ , the effects of a finite temperature finally become apparent. The massless modes of the Luttinger liquid are then quenched into a  $z = 1$  quantum-relaxational regime, rather similar to the  $z = 2$  quantum-relaxational regime discussed above. This last crossover is, strictly speaking, a property of the  $z = 1$  critical point on the Luttinger liquid fixed line determined by the value of  $\mu$ , and not a property of the  $z = 2$  critical end point at  $\mu = T = 0$ . It is thus described by a reduced Luttinger liquid scaling function  $\mathcal{A}_L$ . We will obtain exact results for  $\mathcal{A}_L$  using an argument based on conformal invariance. We will also discuss an important compatibility condition between the scaling functions  $\mathcal{A}$  and  $\mathcal{A}_L$ , and show how the more general  $\mathcal{A}$  collapses into the small  $\mu$  limit of  $\mathcal{A}_L$ .

In  $d = 2$  (Fig. 2) the ground state is a boson superfluid, which survives at finite temperature. There is a Kosterlitz-Thouless phase transition at a finite  $T$ , which has been studied in some detail earlier.<sup>17,18</sup> In the superfluid phase, therefore, one has a large frequency dilute Bose gas behavior crossing over to a small frequency Goldstone phase with quasi-long-range order (Fig. 3).

The outline of the rest of the paper is as follows. We will begin, in Sec. II, with a renormalization group analysis of the partition function  $Z$ . This will allow us to demonstrate the logarithmic corrections to scaling in  $d = 2$  and obtain the leading terms of the in a  $\epsilon = 2 - d$  expansion. These leading terms are consistent with the zero-scale-factor universality. We will then, in Sec. III, turn to a discussion of the exact properties of  $\mathcal{A}$  and  $\mathcal{A}_L$  in  $d = 1$ . A brief summary and a discussion of relevance to experiments appear in Sec. IV. Three appendixes contain discussions of some peripheral points.

## II. RENORMALIZATION GROUP ANALYSIS

The momentum shell renormalization group equations for  $Z$  have already been obtained by Fisher and Hohenberg.<sup>18</sup> However, their analysis of the equations was restricted to  $d = 2$ ,  $\mu > 0$ , and temperatures at or below the Kosterlitz-Thouless transition. In this section

we will extend this analysis to cover the remaining regions in  $d = 2$  (Fig. 2), and to dimensions  $d < 2$  in an  $\epsilon = 2 - d$  expansion (Fig. 1). The analysis<sup>18</sup> proceeds by introducing an upper cutoff  $\Lambda$  in momentum space, and replacing  $v$  by a contact interaction  $u = \Lambda^{-d}v(0)$ . Degrees of freedom in a shell between  $\Lambda$  and  $\Lambda e^{-\ell}$  are integrated out, followed by a rescaling of coordinates and field variables,

$$x' = e^{-\ell}x, \quad \tau' = e^{-2\ell}\tau, \quad (2.1)$$

$$\Psi'(x', \tau') = e^{d\ell/2}\Psi(x, \tau).$$

Note that there is no anomalous dimension in the rescaling factor for  $\Psi$  as  $\eta = 0$ . Further, the scaling dimension of  $|\Psi|^2$  is exactly  $d$ , as it must be for any conserved charge density.<sup>19</sup>

It is convenient to consider the cases  $T = 0$  and  $T > 0$  separately:

### A. $T = 0$

The renormalization group equations are<sup>18,11</sup>

$$\frac{d\mu}{d\ell} = 2\mu, \quad (2.2)$$

$$\frac{du}{d\ell} = (2 - d)u - \frac{mK_d\Lambda^{d-2}}{\hbar^2}u^2, \quad (2.3)$$

where  $K_d = S_d/(2\pi)^d$  and  $S_d$  is the surface area of a unit sphere in  $d$  dimensions. We integrate these equations to a scale  $e^{\ell^*}$  where the system is noncritical, i.e., when  $\mu \sim \alpha\hbar^2\Lambda^2/(2m)$  where  $\alpha$  is significantly smaller than unity, but not so small that the system is still critical. For  $\mu > 0$ , the magnetization  $M$  and the boson density  $n = M/(g\mu_B S_z)$  are then given by

$$n = e^{-d\ell^*} \frac{\mu(\ell^*)}{u(\ell^*)}. \quad (2.4)$$

For  $d < 2$ , and  $\epsilon = 2 - d$  small,  $u$  approaches its fixed-point value

$$u^* = \frac{\hbar^2}{K_d\Lambda^{d-2}m}\epsilon. \quad (2.5)$$

Inserting this fixed-point value and the dependence of  $\ell^*$  on the initial value of  $\mu$  into (2.4) we find to lowest order in  $\epsilon$  that  $n$  indeed has the form (1.6) with the universal number  $\mathcal{C}$  given by

$$\mathcal{C} = \frac{1}{4\pi\epsilon}. \quad (2.6)$$

In  $d = 2$ ,  $u$  approaches 0 logarithmically slowly. For large  $\ell$  we have

$$u(\ell) = \frac{2\pi\hbar^2}{m\ell}. \quad (2.7)$$

Inserting this into (2.4) we find<sup>17</sup>

$$n = \frac{m\mu}{4\pi\hbar^2} \ln \left( \frac{\hbar^2 \Lambda^2}{2m\mu} \right). \quad (2.8)$$

Note the logarithmic violation of the perfect scaling of (1.6).

### B. $T > 0$

We will restrict our analysis to the center of the quantum critical region in  $d = 2$  (Fig. 2) and  $d < 2$  (Fig. 1): The initial value of  $\mu$  will therefore be fixed at  $\mu(\ell = 0) = 0$  and the initial value of the temperature  $T$  will be close to 0. We will only need the finite  $T$  renormalization group equations for  $\mu$  and  $T$  which are<sup>18</sup>

$$\begin{aligned} \frac{dT}{d\ell} &= 2T, \\ \frac{d\mu}{d\ell} &= 2\mu - \frac{2\Lambda^d K_d u}{\exp \left[ \frac{1}{k_B T} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu \right) \right] - 1}. \end{aligned} \quad (2.9)$$

To leading order in  $\epsilon$  (leading logarithms) it is sufficient to assume that for  $d < 2$  ( $d = 2$ )  $u$  is given by Eq. (2.5) [Eq. (2.7)]. We will now integrate the renormalization group equations until a scale  $\ell^*$  where

$$\mu(\ell^*) = -\alpha \frac{\hbar^2 \Lambda^2}{2m}. \quad (2.10)$$

The correlation length  $\xi$  is then given by

$$\xi = \frac{e^{\ell^*}}{\Lambda \sqrt{\alpha}}, \quad (2.11)$$

while the boson density is

$$\begin{aligned} n &= e^{-d\ell^*} \langle |\Psi_{\ell^*}(x=0, \tau=0^-)|^2 \rangle, \\ &= e^{-d\ell^*} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\exp \left[ \frac{1}{k_B T(\ell^*)} \left( \frac{\hbar^2 k^2}{2m} - \mu(\ell^*) \right) \right] - 1} \\ &\approx -\frac{e^{-d\ell^*}}{4\pi} \left( \frac{2mk_B T(\ell^*)}{\hbar^2} \right)^{d/2} \ln(1 - e^{\mu(\ell^*)/T(\ell^*)}). \end{aligned} \quad (2.12)$$

In the last step we have anticipated that to leading order in  $\epsilon$  it is sufficient to evaluate the integral directly in  $d = 2$ .

Let us now examine the results of integrating (2.9) for  $d < 2$ . To leading order in  $\epsilon$  we find

$$\mu(\ell) = -4\epsilon e^{2\ell} \frac{\hbar^2 \Lambda^2}{2m} \int_0^\ell \frac{e^{-2\ell'} d\ell'}{\exp \left( \frac{\hbar^2 \Lambda^2}{2mT(\ell')} \right) - 1}. \quad (2.13)$$

Using  $T(\ell) = T e^{2\ell}$ , it is straightforward to perform the integration and obtain from (2.10) the leading result for  $\ell^*$ :

$$e^{-2\ell^*} = \frac{2mk_B T}{\hbar^2 \Lambda^2} 2\epsilon \ln \left( \frac{1}{2\epsilon} \right). \quad (2.14)$$

From (2.11) we therefore deduce

$$\xi = \frac{1}{[2\epsilon \ln(1/2\epsilon)]^{1/2}} \frac{\hbar}{\sqrt{2mk_B T}} \quad (2.15)$$

and from (2.12) we obtain for the boson density

$$n = \left( \frac{2mk_B T}{\hbar^2} \right)^{d/2} \frac{1}{4\pi} \ln \left( \frac{1}{2\epsilon \ln(1/2\epsilon)} \right). \quad (2.16)$$

These last two results are consistent with the zero-scale-factor universality of (1.19) and (1.23) and yield properties of the scaling functions  $\mathcal{B}_{+-}$  and  $\mathcal{M}$  at  $\mu = 0$ .

Finally consider properties in the quantum-critical region in  $d = 2$ . The analog of Eq. (2.13) is

$$\mu(\ell) = -4e^{2\ell} \frac{\hbar^2 \Lambda^2}{2m} \int_0^\ell \frac{d\ell'}{\ell'} \frac{e^{-2\ell'}}{\exp \left( \frac{\hbar^2 \Lambda^2}{2mT(\ell')} \right) - 1}. \quad (2.17)$$

Integrating this to leading-logarithmic accuracy and using (2.10) we find

$$e^{-2\ell^*} = 4 \frac{2mk_B T}{\hbar^2 \Lambda^2 \alpha} \ln \ln \left( \frac{\hbar^2 \Lambda^2}{2mk_B T} \right). \quad (2.18)$$

We therefore have from (2.11) for the correlation length

$$\xi = \frac{\hbar}{\sqrt{2mk_B T}} \frac{1}{\left\{ 4 \ln \ln [\hbar^2 \Lambda^2 / (2mk_B T)] \right\}^{1/2}}, \quad (2.19)$$

which violates the universality of 1.19 at  $\mu = 0$  by the double logarithms. From (2.12) we get for the boson density

$$n = \frac{2mk_B T}{\hbar^2} \frac{1}{4\pi} \frac{1}{\left\{ \ln [\hbar^2 \Lambda^2 / (2mk_B T)] \right\}^4}, \quad (2.20)$$

again logarithmically violating (1.23) at  $\mu = 0$ .

### III. EXACT RESULTS IN ONE DIMENSION

We have so far determined that for small  $\epsilon = 2 - d$  the  $\mu = T = 0$  critical field theory has a contact interaction of strength  $u^* = \mathcal{O}(\epsilon)$ , and all other two-multiparticle interactions can be neglected. Remarkably, following Haldane,<sup>20</sup> it also possible to determine the exact critical field theory for  $\epsilon = 1$  or  $d = 1$ . The critical field theory then has  $u^*|_{\epsilon=1} = \infty$  (the bosons are thus impenetrable). Moreover, all other boson interactions continue to be irrelevant. In Appendix B we consider a one-dimensional Bose gas in the vicinity of this strong-coupling fixed point and demonstrate this explicitly.

The methods of Appendix B and earlier works<sup>20,21,5</sup> use the well-known equivalence between the  $d = 1$  impenetrable Bose gas and free fermions. The field theory

of the critical end point at  $\mu = T = 0$  is therefore given by the free fermion Hamiltonian

$$H_F = \int dx \Psi_F^\dagger(x) \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \mu \right) \Psi_F(x), \quad (3.1)$$

where  $\Psi_F$  are canonical fermion fields. Correlators of this theory can only depend upon  $m$ ,  $\mu$ , and  $T$ , and the zero-scale-factor universality is therefore manifest. The scaling limit of the correlators of the uniform magnetization can now be obtained almost trivially: The uniform magnetization density just measures the number of particles and its correlators are therefore the same as those of  $g\mu_B S_z \Psi_F^\dagger \Psi_F$ . In particular we have for the scaling function for the uniform magnetization in (1.23)

$$\mathcal{M}(t) = \frac{1}{\pi} \int_0^\infty dy \frac{1}{e^{y^2-t} + 1}, \quad (3.2)$$

where  $t \equiv \mu/(k_B T)$ . This scaling function has the limiting value  $\mathcal{M} = e^t/(2\sqrt{\pi})$  as  $t \rightarrow -\infty$  in the Haldane gap region, and  $\mathcal{M} = \sqrt{t}/\pi$  as  $t \rightarrow \infty$  in the Luttinger liquid region (Fig. 1). This last result combines with (1.23) to yield (1.6) with  $\mathcal{C} = 1/\pi$ .<sup>5</sup>

Observables associated with correlations of the staggered magnetization, like  $G^R$ , are much more difficult to obtain—it is necessary to express the impenetrable Bose fields in terms of the Fermi fields by a continuum Jordan-Wigner transformation<sup>20</sup> and then evaluate the correlator—a naive Wick's theorem expansion of this correlator will yield an infinite number of terms. Recently, Korepin and Slavnov,<sup>22</sup> following earlier work of Lenard,<sup>23</sup> have succeeded in resumming this expansion and showing that all space-time-dependent, finite-temperature correlators of the impenetrable Bose gas can be expressed in terms of the solution and Fredholm determinant of a linear Fredholm integral equation. Thus determination of the universal scaling function  $\mathcal{A}$  in  $d = 1$  has been reduced to the problem of solving completely an integral equation, and taking the Fourier transform of the result. Analytic methods can take us no further, and it is necessary to resort to numerical analysis of the integral equations. We have begun such a numerical program, and have so far obtained essentially exact results for the equal-time correlations—these are described below in Sec. III A. It should be possible to extend our results to obtain local, time-dependent correlations [and hence the scaling function  $F$  in (1.12)] but we have not yet done so. A general picture of the form of  $F$  can be obtained from the asymptotic limits quoted in Sec. I A; should it become experimentally useful to obtain more precise numerical results for  $F$ , we shall be happy to provide them.

We also note that, recently, Korepin and collaborators<sup>24,25</sup> have succeeded in determining exact results for certain asymptotic properties of  $G^R$  by applying the quantum-inverse scattering method to the integral equations noted above. For the equal-time  $G^R$  they obtained results for the leading and next-to-leading terms as  $x \rightarrow \infty$ , while for unequal-time correlators, both  $x$  and  $\tau$  were sent to  $\infty$ . Unfortunately, these asymptotic results

are not very useful in determining experimental observables which require Fourier transformation to functions of momenta and frequency (the large  $x$  behavior of a function implies little about the small  $k$  limit of its Fourier transform). Simply Fourier transforming the asymptotic terms leads to results which compare very poorly with the exact results which we obtained by the alternative means described below. We comment on some features of these exact asymptotic results in Appendix C.

In Sec. III B we will consider the limit  $\mu \gg k_B T$  where it is possible to make much greater analytic progress in determining the scaling functions. As we have already noted, the lower frequency properties in this region are described by Luttinger liquid criticality, and it possible to use conformal invariance arguments to obtain closed-form results.

### A. Equal-time structure factor

The most convenient procedure for determining equal-time correlations begins with Lenard's<sup>23</sup> result for the density matrix of the impenetrable Bose gas, which is tantamount to a formal solution of the integral equation of Ref. 22. His result can be written as<sup>22</sup>

$$G(x, \tau = 0^-) = \langle 0 | \hat{G}_F (1 - 2\hat{G}_F)^{-1} | x \rangle \det(1 - 2\hat{G}_F), \quad (3.3)$$

where the operator  $\hat{G}_F$  acts on the real axis between 0 and  $x$ , and has the matrix elements

$$\langle x | \hat{G}_F | x' \rangle = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik(x-x')}}{e^{[\hbar^2 k^2/(2m) - \mu]/(k_B T)} + 1}, \quad (3.4)$$

i.e., the fermion Green's function. The form (3.3) is amenable to rapid numerical evaluation. We discretize the real line between 0 and  $x$  into  $N$  points, whence the operators in (3.3) become  $N \times N$  Toeplitz matrices. A straightforward extension of Levinson's algorithm<sup>26</sup> was then used to compute the determinant and inverse of  $1 - 2\hat{G}_F$ . The computer time required for this step scales only as  $N^2$ , and we were able to easily use values up to  $N = 10000$ . The results for  $G$  for large  $x$  were compared with the exact asymptotic results of Ref. 24, with excellent agreement. Finally, we performed a spatial Fourier transform, and obtained results for the scaling function  $\mathcal{B}_{-+}$  of the structure factor  $S_{-+}(k)$  defined below (1.19) and by (1.20).

Our results for some representative values of  $t = \mu/(k_B T)$  are shown in Fig. 4. A computer program to obtain numerical values of  $\mathcal{B}_{-+}(r, t)$  ( $r = \hbar k/\sqrt{2mk_B T}$ ) for arbitrary  $r, t$  can be obtained from the authors; the accuracy is limited only by computer time, but it is possible to obtain three significant figure accuracy quite rapidly.

We note that the zero-temperature limit of  $S_{-+}(k)$  was computed by Vaidya and Tracy:<sup>27</sup> They also pointed out that the  $T = 0$ ,  $S_{-+}(k)$  has nonanalyticities<sup>20</sup> (which are, however, unobservably weak) at integer multiples of  $2k_F$ , where  $k_F$  is the Fermi wave vector of  $H_F$ .

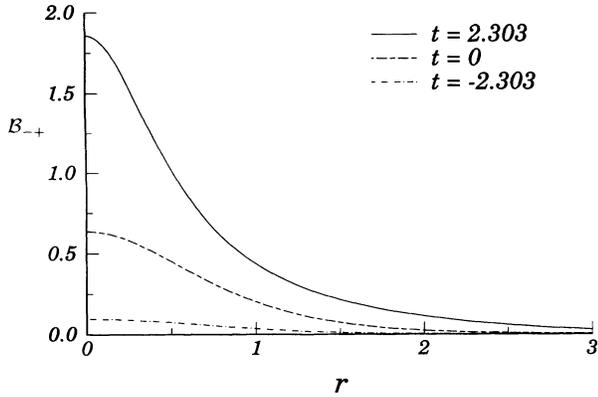


FIG. 4. Exact results for the scaling function  $B_{-+}(r, t)$  for the structure factor [Eq. (1.19)] in  $d = 1$ . We have  $r = \hbar k / \sqrt{2mk_B T}$ , and  $t = \mu / (k_B T)$ . We have chosen some representative values of  $t$ ; a computer program to evaluate  $B_{-+}$  for arbitrary  $t$  is available from the authors.

### B. Luttinger liquid

This is the regime where  $\mu \gg k_B T$ . In the fermionic description of the problem, this means the Fermi sea is much deeper than the temperature. In this case the ground state and its pertinent excitations can be described by a theory in which the spectrum is linearized near the Fermi points  $k = \pm k_F$  determined by

$$\mu = \frac{\hbar^2 k_F^2}{2m}. \quad (3.5)$$

By the same token, as long as the probe frequency  $\omega$  and momentum  $k$  do not cause excitations that probe the sea deeply (i.e., stay in the linearized region) the Bose gas will exhibit characteristics of the line of finite- $\mu$ ,  $z = 1$  Luttinger liquid critical points at  $T = 0$ —the Luttinger liquid scaling function for the Green's function will be denoted by  $\mathcal{A}_L$ . In terms of Fig. 3, it means that the  $\mathcal{A}_L$  will describe the lower frequency crossover around  $\hbar\omega \sim k_B T$ . The ratio  $\hbar\omega/k_B T$  can take arbitrary values as long as both  $\hbar\omega$  and  $k_B T$  remain significantly smaller than  $\mu$ . The depth of the Fermi sea will not enter any of the calculations of  $\mathcal{A}_L$  and  $\mu$  will enter only via the Fermi velocity  $c$ . For example, in the low density limit,  $c$  is given by

$$c = \frac{k_F}{m} = \left( \frac{2\mu}{m} \right)^{1/2}. \quad (3.6)$$

We shall see all this happen as we analyze the exact results momentarily. The crossover around  $\hbar\omega \sim \mu$  in Fig. 3 is *not* part of the Luttinger liquid criticality, and is instead associated with the  $z = 2$  critical end point, and the scaling function  $\mathcal{A}$ . It should be apparent from this discussion that the limits  $\hbar\omega/k_B T \rightarrow \infty$  and  $\mu/k_B T \rightarrow \infty$  of  $\mathcal{A}$  do *not* commute.

Note that Luttinger liquid criticality and associated scaling forms hold even when the condition that  $\mu$  be small in (1.14) is violated. Suppose, however, that

$k_B T, \hbar\omega$  and  $\hbar^2 k^2 / (2m)$  are much smaller than  $\mu$ , and that  $\mu$  itself is small so that Eq. (1.14) is satisfied. Then zero-scale-factor universality of the  $z = 2$  critical point at  $\mu = 0, T = 0$  must be simultaneously satisfied. This will lead to a compatibility condition between the scaling functions  $\mathcal{A}$  and  $\mathcal{A}_L$ , which we shall shortly examine.

Let us begin by writing down the scaling forms of the Luttinger liquid at  $T = 0$ . We know that at equal times

$$G(x, \tau = 0^-) = \frac{D}{x^\eta}, \quad x \rightarrow \infty, T = 0, \mu > 0. \quad (3.7)$$

The constant  $D$  and the exponent  $\eta$  will, in general, have a nonuniversal dependence upon the microscopic couplings. However, knowledge of  $D, \eta$ , and a zero-sound velocity  $c$  will universally determine all remaining hydrodynamic properties in the Luttinger liquid regime. For example the unequal-time correlation function will have exactly the same form as above with  $x$  replaced by the Euclidean distance  $\sqrt{x^2 + c^2 \tau^2}$ . As for  $\eta$ , it has a value that depends on the Luttinger coupling. At the point  $\mu = 0$ , the bosons are impenetrable and equivalent to free fermions. For  $\mu > 0$ , the deviation from the impenetrability condition can be translated into a residual interaction between fermions by integrating out doubly occupied states (see Appendix B). This is the marginal coupling of the Luttinger liquid. Let us note for future reference that  $\eta = 1/2$  for zero Luttinger coupling. Some readers may have trouble reconciling this with the fact that fermion-fermion correlation functions fall as  $1/x$  in free field theory. However, we have already noted that there is a rather complicated relationship between the Fermi and Bose fields and we remind the reader of the chain of transformations relating the two. First the hard-core bosons are described by the Pauli matrices  $\sigma_\pm$ . The latter are then converted to a single component fermions  $\Psi_F$  by a Jordan-Wigner transformation. These fermions are filled up to some Fermi momentum determined by the chemical potential. When linearized near the Fermi points, the spinless Fermi field turns into a pair of relativistic fields  $\Psi_{L,R}$ . The hard-core boson correlation function at equal times is

$$\langle \Psi(0) \Psi^\dagger(x) \rangle = \langle \Psi_F(0) e^{i\pi \int_0^x \Psi_F^\dagger(x') \Psi_F(x') dx'} \Psi_F^\dagger(x) \rangle. \quad (3.8)$$

If we now write

$$\Psi_F(x) = \Psi_L e^{-ik_F x} + \Psi_R e^{ik_F x} \quad (3.9)$$

and drop terms that oscillate at  $k_f$ , we obtain

$$\langle \Psi(0) \Psi^\dagger(x) \rangle \simeq \langle \Psi_L(0) e^{i\pi \int_0^x [\Psi_L^\dagger(x') \Psi_L(x') + \Psi_R^\dagger(x') \Psi_R(x')] dx'} \times \Psi_L^\dagger(x) + L \leftrightarrow R \rangle. \quad (3.10)$$

Clearly this is a complicated object in the Fermi theory. To evaluate it one uses bosonization. Using the standard dictionary it is possible to show that it is proportional to the two-point function

$$\langle e^{-i\sqrt{\pi}\phi(0)} e^{i\sqrt{\pi}\phi(x)} \rangle \propto \frac{1}{x^{1/2}}, \quad (3.11)$$

where  $\tilde{\phi}$  is the field dual to the usual boson field. (See for example Ref. 28.)

We now consider the correlations at finite temperatures. In general the passage from zero to nonzero temperatures is nontrivial since in the latter case not only the ground state but excited state correlations enter. However, in two Euclidean dimensions, in a relativistically invariant theory such as this one, we have the remarkable result from conformal field theory that was first pointed out by Cardy.<sup>29</sup> Using the conformal mapping between the infinite plane and the strip of finite length  $L_\tau = \hbar c/k_B T$  along the imaginary-time direction, one can obtain from Eq. (3.7)

$$G_L(x, \tau) = \frac{D}{2^{\eta/2}} \left( \frac{2\pi k_B T}{\hbar c} \right)^\eta \left[ \cosh \left( \frac{2\pi k_B T x}{\hbar c} \right) - \cos \left( \frac{2\pi k_B T \tau}{\hbar} \right) \right]^{-\eta/2}. \quad (3.12)$$

(The operator  $e^{-i\sqrt{\pi}\tilde{\phi}(0)}$  is a primary field, which allows us use of conformal invariance as above.) The subscript  $L$  has been placed to emphasize that this formula is valid only near the lower frequency crossover in Fig. 3.

The ease with we obtain this result should not detract us from noting its importance—rarely does one have the thermally averaged correlation functions of an interacting system. We shall therefore spend some time analyzing this result.

As a first step let us extract from this the correlation function at equal time, for long distances. It is readily seen that

$$G(x, 0^-) = D \left( \frac{2\pi k_B T}{\hbar c} \right)^\eta \exp \left( \frac{-\eta\pi k_B T x}{\hbar c} \right), \quad T > 0, \mu > 0, x \rightarrow \infty. \quad (3.13)$$

Now let us consider the regime where (1.14) is also satisfied and so zero-scale-factor universality holds. Thus  $D$ ,  $\eta$ , and  $c$  can no longer be nonuniversal, but must be universal functions of  $\mu$  and  $m$ . Let us now invoke the asymptotic (long distance) hard-core boson scaling functions at the  $\mu = 0$  critical point which are known by exact solution:<sup>27,24</sup>

$$G(x, 0^-) = \frac{\sqrt{2mk_B T}}{\hbar} \frac{\rho_\infty}{\sqrt{\pi}} \exp \left( -\frac{\sqrt{2mk_B T}}{\hbar} \frac{\pi x}{4\sqrt{t}} \right), \quad (3.14)$$

where  $t = \mu/(k_B T)$ . The constant  $\rho_\infty$  is a known universal number; more details on this formula are relegated to Appendix C. We see that this agrees with the Luttinger liquid result Eq. (3.13) if

$$\eta = \frac{1}{2}, \quad c = \left( \frac{2\mu}{m} \right)^{1/2}, \quad D = \frac{\rho_\infty}{\pi} \left( \frac{2m\mu}{\hbar^2} \right)^{1/4}. \quad (3.15)$$

Notice also how the chemical potential entered only via

the Fermi velocity as anticipated (excluding the  $\mu$  dependence of the prefactor  $D$ ).

We now consider the Fourier transform of (3.12) to obtain the corresponding  $G_L(k, i\omega_n)$  at the Matsubara frequencies along the imaginary frequency axis,

$$G_L(k, i\omega_n) = \int_{-\infty}^{\infty} dx \int_0^{\hbar/(k_B T)} d\tau e^{-i(kx - \omega_n \tau)} G_L(x, \tau). \quad (3.16)$$

Given the scaling form of  $G_L(x, \tau)$  it follows that the two integrals involved in the transform lead us to the scaling form

$$G_L^R(k, \omega) = \frac{D}{c} \left( \frac{\hbar c}{k_B T} \right)^{2-\eta} \mathcal{A}_L \left( \frac{\hbar\omega}{k_B T}, \frac{\hbar ck}{k_B T} \right), \quad (3.17)$$

where  $\mathcal{A}_L$  is a completely universal scaling function, dependent only upon the value of  $\eta$ . We will determine  $\mathcal{A}_L$  in closed form below.

Now if we take the limit  $\mu/k_B T \rightarrow \infty$  at fixed  $\omega/k_B T$  and  $\hbar ck/k_B T$  [while satisfying (1.14) of course], the system is described simultaneously by the Luttinger liquid result (3.17) and the zero-scale-factor universality of (1.10). Comparing these two results and using (3.15) we obtain immediately the compatibility condition between the reduced scaling function  $\mathcal{A}_L$  at  $\eta = 1/2$  and the scaling function  $\mathcal{A}$ ,

$$\mathcal{A}_L(\bar{\omega}, \bar{k})|_{\eta=1/2} = \frac{\pi}{\sqrt{2}\rho_\infty} \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \mathcal{A} \left( \bar{\omega}, \frac{\bar{k}}{2\sqrt{t}}, t \right). \quad (3.18)$$

We are using here, as before, the notation  $\bar{\omega} \equiv \hbar\omega/(k_B T)$ ,  $\bar{k} \equiv \hbar ck/(k_B T)$ , and  $t \equiv \mu/(k_B T)$ . We have demanded here that  $\mu$  enters  $\mathcal{A}_L$  only through  $c$ . Notice that the  $t \rightarrow \infty$  limit is taken at fixed  $\bar{\omega}$ , and as we have noted before and shall see explicitly below, the  $\bar{\omega} \rightarrow \infty$  limit of  $\mathcal{A}_L$  does not agree with the  $\bar{\omega} \rightarrow \infty$  limit of  $\mathcal{A}$  which was implicit in (1.13).

The remainder of this subsection is devoted to obtaining explicit results for  $\mathcal{A}_L$  and associated scaling functions in Minkowski space. There are two ways to proceed.

The first approach is to perform the transform for complex (Matsubara) frequencies and then make the substitution  $i\omega_n = \omega$  to obtain  $-G^R(\omega)$ . The transforms are tedious to perform but the interpretation of the results is instructive and we shall do so soon. The general principles we learn about analytic continuation into the complex plane are usually illustrated with trivial examples (i.e., noninteracting propagators) and here we have one of the few nontrivial and hence instructive cases.

The second approach is peculiar to this problem and relevant because conformal invariance methods always give the correlations in coordinate and not momentum space. Thus one can continue the results from imaginary to real time first and then take the transform. That calculation may be found in Sec. 3.3 of Ref. 30 and has its own pedagogical value.

Returning to the first approach, we used the identity

$$X^{-\eta/2} = \frac{1}{\Gamma(\eta/2)} \int_0^\infty d\lambda \lambda^{\eta/2-1} e^{-\lambda X} \quad (3.19)$$

to put the cosh and sin terms in (3.12) up in the ex-

$$\mathcal{A}_L(i\bar{\omega}, \bar{k}) = \frac{\pi^{\eta-1} \Gamma\left(1 - \frac{\eta}{2}\right)}{2^{2-\eta} \Gamma\left(\frac{\eta}{2}\right)} \frac{\Gamma\left(\frac{\eta}{4} + \frac{|\bar{\omega}| + i\bar{k}}{4\pi}\right) \Gamma\left(\frac{\eta}{4} + \frac{|\bar{\omega}| - i\bar{k}}{4\pi}\right)}{\Gamma\left(1 - \frac{\eta}{4} + \frac{|\bar{\omega}| + i\bar{k}}{4\pi}\right) \Gamma\left(1 - \frac{\eta}{4} + \frac{|\bar{\omega}| - i\bar{k}}{4\pi}\right)}. \quad (3.20)$$

This result was obtained earlier by Schulz<sup>4</sup> and Shankar<sup>30</sup> in a different context but not analyzed in any detail.

First, the above function  $\mathcal{A}_L(|\bar{\omega}|)$  specifies our knowledge at positive and negative Matsubara points. Our goal is to construct the physical real frequency correlation function and its singularity structure from it. As it stands,  $\mathcal{A}_L$  can only be used for numerical purposes and not for studying analytic structure since  $|\bar{\omega}|$  is neither analytic nor antianalytic. In other words we can use  $\mathcal{A}_L$  to calculate values of the putative function at the Matsubara points. The “mod” symbol tells us that the function we are seeking has the same value at any positive Matsubara point and its negated image.

Now, on the real frequency axis we have a retarded correlation function, which we assume is well defined. Since the factor  $e^{i\omega t}$  converges in the upper half-plane (UHP), the function on the real axis has an analytic extension to the UHP which is free of singularities. Its values at the Matsubara points  $\bar{\omega} = 2\pi n$ , with  $n$  an integer are given by  $\mathcal{A}_L$ . Such a function is readily found: Simply drop the mod symbol on  $\bar{\omega}$  in the formula for  $\mathcal{A}_L$ . Let us call the function (with the mod symbol dropped)  $\mathcal{A}_L^{\text{UHP}}(\bar{\omega})$ . This  $\mathcal{A}_L^{\text{UHP}}(\bar{\omega})$  is the unique analytic function (with good behaviour in the UHP) determined by our knowledge at positive Matsubara frequencies. Being an analytic function it has a continuation to the lower half-plane (LHP) which is however not guaranteed to be free of singularities or to have anything to do with the original problem. In particular the poles that the  $\Gamma$  functions have in the LHP are not germane to the physical response function. In fact this continuation to the LHP of  $\mathcal{A}_L^{\text{UHP}}(\bar{\omega})$  does not even agree with the data we have in Eq. (3.20) for negative Matsubara points: Since  $\mathcal{A}_L^{\text{UHP}}(\bar{\omega}) \neq \mathcal{A}_L^{\text{UHP}}(-\bar{\omega})$ , it is not invariant under the change of sign of frequency as the given data are. However, there is an analytic function which will duplicate the given data in the LHP: It is obtained by replacing  $|\bar{\omega}|$  by  $-\bar{\omega}$  in Eq. (3.20). Such a function  $\mathcal{A}_L^{\text{LHP}}(\bar{\omega})$  satisfies

$$\mathcal{A}_L^{\text{LHP}}(\bar{\omega}) = \mathcal{A}_L^{\text{UHP}}(-\bar{\omega}). \quad (3.21)$$

This function will agree with  $\mathcal{A}_L$  of Eq. (3.20) at points with negative Matsubara frequencies and be free of singularities in the LHP. However, its poles in the UHP have no physical significance. Thus the function at real frequencies is the limit of two different functions as we approach the real axis from above or below. *The true*

ponent. The  $x$  and  $\tau$  integrals were then analytically performed, followed finally by the  $\lambda$  integration. After using (3.17), the final result gave us values of the scaling function  $\mathcal{A}_L$  at the Matsubara frequencies along the imaginary frequency axis. We found

*singularities of the physical response function are due to the mismatch of these two functions and not due to the poles they have in regions where they no longer represent the physical function or a continuation of it.* So we must consider the difference between these two functions  $\mathcal{A}_L^{\text{UHP}}(\bar{\omega})$  and  $\mathcal{A}_L^{\text{LHP}}(\bar{\omega})$  on the real frequency axis. It is readily shown that the two functions are conjugates of each other there so that the discontinuity is just twice the imaginary part. The physical response function’s real singularity is therefore a cut and not poles. This is reasonable since a theory with gapless excitations on top of the ground state (and hence a cut at zero temperature) cannot lose its spectral weight in these regions by the inclusion of higher states in the thermal average at finite temperatures. (The converse is possible: A cut-free region on the real axis at  $T = 0$  can close up at  $T \neq 0$ .)

In Fig. 5 we plot  $-\text{Im}\mathcal{A}_L(\bar{\omega}, \bar{k})/\bar{\omega}$  as a function of  $\bar{\omega}$  and at a representative set of values of  $\bar{k}$ . Notice that for large  $\bar{k}$  there is a well-defined peak at  $\bar{\omega} \sim \bar{k}$ : This is a signature of the propagating modes in the Luttinger liquid ground state and represents the behavior of the intermediate region  $k_B T \ll \omega \ll \mu$  in Fig. 3. At smaller values of  $\bar{k}$  notice that the peak in Fig. 5 remains at

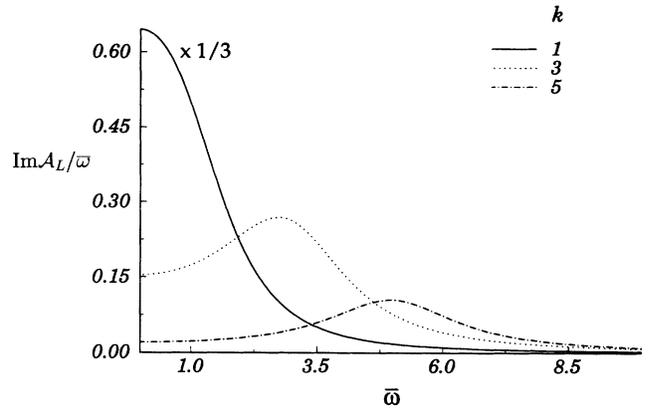


FIG. 5. Exact values of the scaling function  $-\text{Im}\mathcal{A}_L(\bar{\omega}, \bar{k})/\bar{\omega}$  [given in Eq. (3.20)] for the Green’s function in  $d = 1$  in the Luttinger liquid regime at  $\eta = 1/2$  [Eq. (3.17)]. We have  $\bar{\omega} = \hbar\omega/(k_B T)$ ,  $\bar{k} = \hbar ck/(k_B T)$ . The values for  $\bar{k} = 1$  are 3 times larger than those on the graph. Notice how the spectrum evolves from an overdamped, relaxational peak at small  $\bar{k}$  (as for  $\bar{k} = 1$ ) to a damped, propagating mode at large  $\bar{k}$  (as for  $\bar{k} = 3, 5$ ).

$\bar{\omega} = 0$ . This is the  $z = 1$  quantum-relaxational behavior (Fig. 3) where the strong interaction between the thermally excited Luttinger modes has left only overdamped excitations.

We now consider a couple of other experimental observables, related to local and equal-time correlations, respectively.

### 1. Local Green's function

The local Green's function  $G_{\ell}^R$  was defined in (1.11). In the Luttinger liquid regime, we can deduce that, provided  $\omega \ll \mu$ , this observable satisfies the scaling form

$$\text{Im}G_{\ell L}^R(\omega) = -\text{sgn}(\omega) \frac{D}{c^\eta} |\omega|^{\eta-1} F_L\left(\frac{\hbar\omega}{k_B T}\right), \quad (3.22)$$

where  $F_L$  is a universal function, specified completely by the function  $\mathcal{A}_L$  in (1.11). Moreover, as the Luttinger liquid criticality has a particle-hole symmetry,  $F_L$  must be an even, positive function of  $\bar{\omega}$ . As already noted, we expect on general grounds that  $\text{Im}G_{\ell}^R(\omega) \sim \omega$  for small  $\omega$  at finite  $T$ : Therefore  $F_L(\bar{\omega}) \sim |\bar{\omega}|^{2-\eta}$  at small  $\bar{\omega}$ . We also note that for  $\eta > 1$  the *real* part of the local Green's function will not satisfy an analogous scaling form because the integral in (1.11) is then dominated by large momentum contributions.

There is again a compatibility condition between the Luttinger liquid scaling function  $F_L$  and the scaling function  $F$  in (1.12) quite analogous to that for  $\mathcal{A}_L$ ,  $\mathcal{A}$  in (3.18); we have

$$F_L(\bar{\omega})|_{\eta=1/2} = \frac{\pi}{\rho_\infty} \lim_{t \rightarrow \infty} F(\bar{\omega}, t). \quad (3.23)$$

As before, the limits  $\bar{\omega} \rightarrow \infty$  and  $t \rightarrow \infty$  do not commute, and the  $\bar{\omega} \rightarrow \infty$  limit of the exact  $F_L$  computed below will not agree with that of  $F$  in (1.13).

Let us finally present the exact computation of  $F_L$ . We use the result (3.12) at  $x = 0$ , Fourier transform to Matsubara frequencies, analytically continue, and take the imaginary part to obtain the following result for  $F_L$ :

$$F_L(\bar{\omega}) = |\bar{\omega}|^{1-\eta} \pi^{\eta-1/2} \sinh\left(\frac{|\bar{\omega}|}{2}\right) \frac{\left|\Gamma\left(\frac{\eta}{2} - \frac{i|\bar{\omega}|}{2\pi}\right)\right|^2}{\Gamma\left(\frac{1+\eta}{2}\right) \Gamma\left(\frac{\eta}{2}\right)}. \quad (3.24)$$

A plot of this function is shown in Fig. 6.

For small  $\bar{\omega}$  we have

$$F_L(\bar{\omega}) = \frac{\pi^{\eta-1/2} \Gamma\left(\frac{\eta}{2}\right)}{2\Gamma\left(\frac{1+\eta}{2}\right)} |\bar{\omega}|^{2-\eta}, \quad |\bar{\omega}| \rightarrow 0. \quad (3.25)$$

This is the behavior characteristic of the  $z = 1$  quantum-relaxational regime of Fig. 3. It is expected that the limits  $\bar{\omega} \rightarrow 0$  and  $t \rightarrow \infty$  *do commute*, and so combined with (3.23), the above result gives us the small  $\bar{\omega}$  behavior

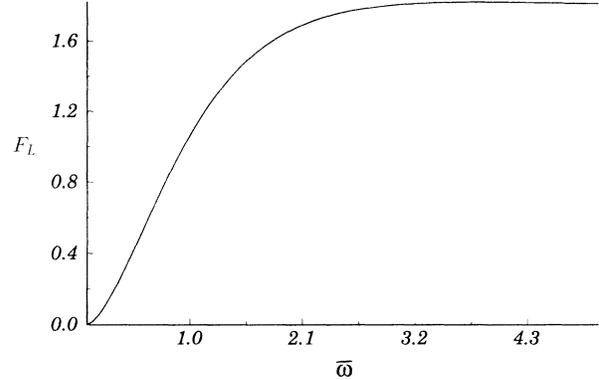


FIG. 6. Exact scaling function  $F_L(\bar{\omega})$  [given in Eq. (3.24)] for the imaginary part of the local susceptibility in  $d = 1$  in the Luttinger liquid regime at  $\eta = 1/2$  [Eq. (3.22)]. We have  $\bar{\omega} = \hbar\omega/(k_B T)$ .

of  $F(\bar{\omega}, t)$  at large values of  $t$ .

In the opposite limit of large  $\bar{\omega}$  we crossover to the critical correlations of the Luttinger liquid ground state in which case

$$F_L(\bar{\omega}) = \frac{2^{1-\eta} \pi^{3/2}}{\Gamma\left(\frac{1+\eta}{2}\right) \Gamma\left(\frac{\eta}{2}\right)}, \quad |\bar{\omega}| \rightarrow \infty. \quad (3.26)$$

This last result can also be obtained by a Fourier transform of the relativistic zero-temperature correlator.

### 2. Structure factor

The two structure factors  $S_{+-}(k)$  and  $S_{-+}$  of the antiferromagnet were defined in Eq. (1.18). The Luttinger liquid behavior has particle-hole symmetry, and so in this regime, the two structure factors are essentially equal and will be denoted by the common value  $S_L(k)$ . The scaling form for  $S_L(k)$  follows from (1.16), (1.18), and the scaling of  $G_{\ell}^R$  in (3.17):

$$S_L(k) = ZD \left(\frac{\hbar c}{k_B T}\right)^{1-\eta} \mathcal{B}_L\left(\frac{\hbar c k}{k_B T}\right), \quad (3.27)$$

where the constant  $Z$  was introduced in (1.16) and  $\mathcal{B}_L$  is a universal function obtained below. There is a compatibility condition between the Luttinger liquid scaling function  $\mathcal{B}_L$  and the scaling function  $\mathcal{B}_{+-}$  in (1.19) which is quite analogous to that for  $\mathcal{A}_L$ ,  $\mathcal{A}$  in (3.18):

$$\mathcal{B}_L(\bar{k})|_{\eta=1/2} = \frac{\pi}{\sqrt{2}\rho_\infty} \lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \mathcal{B}_{+-}\left(\frac{\bar{k}}{2\sqrt{t}}, t\right). \quad (3.28)$$

Using the other scaling function  $\mathcal{B}_{-+}$  on the right-hand side would yield, from (1.20), an identical result. Again the limits  $t \rightarrow \infty$  and  $\bar{k} \rightarrow \infty$  are not expected to commute.

Finally, the exact determination of  $\mathcal{B}_L$ : We simply perform a spatial Fourier transform of (3.12) at  $\tau = 0$ ; we

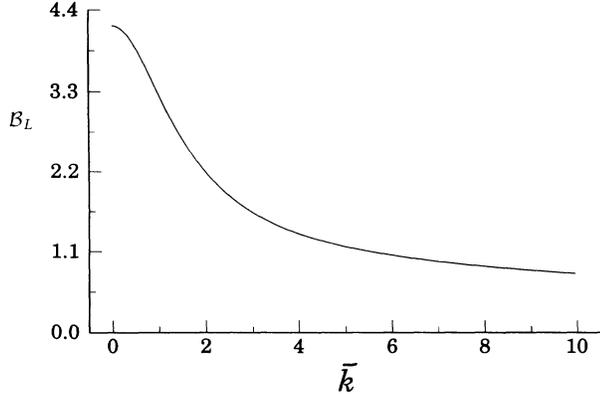


FIG. 7. Exact scaling function  $\mathcal{B}_L(\bar{k})$  [given in Eq. (3.29)] for the structure factor in  $d = 1$  in the Luttinger liquid regime at  $\eta = 1/2$  [Eq. (3.27)]. We have  $\bar{k} = \hbar ck / (k_B T)$ .

obtain in this manner

$$\mathcal{B}_L(\bar{k}) = \pi^{\eta-1/2} \frac{\Gamma\left(\frac{1-\eta}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)} \left| \frac{\Gamma\left(\frac{\eta}{2} + \frac{i\bar{k}}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + \frac{i\bar{k}}{2\pi}\right)} \right|^2. \quad (3.29)$$

This result is illustrated in Fig. 7. For small  $\bar{k}$ ,  $\mathcal{B}_L$  reaches a constant whose value is easily obtainable from (3.29). For large  $\bar{k}$  we find from (3.29)

$$\mathcal{B}_L(\bar{k}) = \frac{2^{1-\eta} \pi^{1/2} \Gamma\left(\frac{1-\eta}{2}\right)}{\Gamma\left(\frac{\eta}{2}\right)} \frac{1}{\bar{k}^{1-\eta}}, \quad |\bar{k}| \rightarrow \infty. \quad (3.30)$$

Again this last result could have also been obtained by direct computation at  $T = 0$ .

#### IV. CONCLUSIONS

This paper has studied the universal, finite-temperature properties of a dilute Bose gas with repulsive interactions in dimensions less than or equal to 2. In the vicinity of the  $T = 0$  onset at zero chemical potential  $\mu$ , it was argued that the leading scaling properties obey, for  $d < 2$ , a hypothesis of zero-scale-factor universality. This means that the entire two-point correlator is a universal function of just  $\mu$ ,  $T$ , and the bare boson mass  $m$ .

The main motivation behind this study is the mapping onto it of the properties of quantum-disordered antiferromagnets in a finite field. In particular, in  $d = 1$ , Haldane gap antiferromagnets undergo a magnetization onset at a critical field which is expected to be in the universality class of the Bose gas transition. Applicability of our theory requires that there be no spin anisotropy in the plane perpendicular to the applied field. Most materials do have some anisotropy—in this case we would require that the temperature  $T$  be larger than the anisotropy gap, before applying our results.

We have described the rather complicated properties of numerous scaling functions, which may be rather difficult to disentangle experimentally. A useful starting point for neutron scattering experiments appears to be the following. Perform the experiment somewhere in the Luttinger liquid region where the absolute value of the scattering cross section is also the largest. Measure the local susceptibility  $G_L^R(\omega)$  and see if it collapses onto the scaling form (1.12). For large  $\mu/k_B T$ , we have a rather complete picture of the scaling function  $F$ : For  $\omega$  smaller than or around  $k_B T$ , we can deduce  $F$  from (3.23) and (3.24), while for extremely large  $\omega$  we can use (1.13).

Another possible application of our results may be to quantum-disordered antiferromagnets in  $d = 2$ . By measuring the ground state magnetization in a field, and comparing the result to (1.4), (1.21), and (2.8) it may be possible to determine the spin  $S_z$  of the elementary excitations above the ground state. Of course, we would also need an independent determination of the quasiparticle mass  $m$ .

#### ACKNOWLEDGMENTS

We thank I. Affleck, I. Gruzberg, B.I. Halperin, P. Hohenberg, and V. Korepin for helpful discussions and correspondence. The research was supported by NSF Grants No. DMR-9120525 and DMR-9224290.

#### APPENDIX A: MAGNON OPERATORS

It is clear that the predominant coupling of neutrons will be to the antiferromagnetic order parameter  $\phi_\alpha$ , with  $\alpha = x, y, z$ . In a zero-field spin-fluid phase with confined spinons, this order parameter corresponds to a real, massive, bosonic triplet. In this appendix, we want to explore in some more detail the relationship between  $\phi_\alpha$  and the complex bosonic field,  $\Psi$  in Eq. (1.3). The subtle point we wish to elaborate on is the following. In the presence of a magnetic field, only one of the three real components of  $\phi_\alpha$  will move down to lower energies, and one might naively conclude that the low-energy theory should therefore include only this real field: So then why does our low-energy theory (1.3) have a complex scalar  $\Psi$ ?

Before addressing this issue, we note in passing that the field theory (1.3) will also describe the magnetization onset transition in antiferromagnets with deconfined spinons; however, in this case, the relationship between the neutron scattering cross section and the field  $\Psi$  will be quite different,<sup>15</sup> and will not be considered in this paper explicitly.

For simplicity, we consider  $d = 1$ , although the analysis is quite general. First we expand the real triplet  $\phi_\alpha$  in terms of magnon creation and destruction operators as usual:

$$\phi_\alpha(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{2\omega_k}} [a_\alpha(k) e^{ikx} + a_\alpha^\dagger e^{-ikx}], \quad (A1)$$

where

$$\omega_k = \sqrt{\Delta^2 + k^2}, \quad (\text{A2})$$

$\Delta$  is the Haldane gap, and  $a_\alpha, a_\alpha^\dagger$  the magnon destruction and creation operators for each of three polarizations. Let us focus on the two combinations

$$\begin{aligned} \phi_\pm(x) &= \frac{\phi_x \pm \phi_y}{\sqrt{2}} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{4\omega_k}} \{ [a_x(k) \pm ia_y(k)] e^{ikx} \\ &+ [a_x^\dagger(k) \pm ia_y^\dagger(k)] e^{-ikx} \}. \end{aligned} \quad (\text{A3})$$

Observe that  $\phi_\pm$  are adjoints of each other, but commute with each other. Let us now consider an effective theory for energies far below the Haldane gap. In this case we can make the replacement

$$\omega_k \simeq \Delta \quad (\text{A4})$$

and obtain

$$\begin{aligned} \phi_\pm(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{4\Delta}} \{ [a_x(k) \pm ia_y(k)] e^{ikx} \\ &+ [a_x^\dagger(k) \mp ia_y^\dagger(k)] e^{-ikx} \} \end{aligned} \quad (\text{A5})$$

$$\equiv \frac{1}{\sqrt{2\Delta}} [\Psi_\pm(x) + \Psi_\pm^\dagger(x)], \quad (\text{A6})$$

where  $\Psi_\pm(x)$  destroys a spin  $\pm 1$  magnon at  $x$  while  $\Psi_\pm^\dagger(x)$  creates a spin  $\mp 1$  magnon at  $x$ . Suppose we next argue that, when the applied uniform field is near its critical value, only the spin up magnon (very light) will either be easily created or destroyed, so that we may drop the spin down creation and destruction operator in the above expressions. Then we obtain

$$\phi_+ = \frac{1}{\sqrt{2\Delta}} \Psi_+(x), \quad (\text{A7})$$

$$\phi_- = \frac{1}{\sqrt{2\Delta}} \Psi_+^\dagger(x). \quad (\text{A8})$$

Observe that now (up to a scale factor)  $\phi_\pm$ , which were previously commuting, *are now canonically conjugate fields*. This is just like in the Hall effect wherein  $x$  and  $y$ , which are commuting coordinates in the full Hilbert space become conjugates in the lowest Landau level. It is also clear from the discussion that the field  $\Psi$  in the coherent space integral in Eq. (1.3) is precisely this complex conjugate pair.

In other words, the answer to the question raised in the first paragraph of this appendix is the following: One also has to include a high-energy component of  $\phi$  because it is canonically conjugate to the low-energy mode. The two together make up the complex scalar  $\Psi$ .

## APPENDIX B: BOSON HUBBARD MODEL IN ONE DIMENSION

Consider bosons  $b_i$  moving on the sites,  $i$  of a chain described by the Hamiltonian

$$\begin{aligned} H &= -w \sum_i \left( b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i - 2b_i^\dagger b_i \right) \\ &+ \sum_i \left( \frac{V}{2} n_i (n_i - 1) - \mu n_i \right), \end{aligned} \quad (\text{B1})$$

where  $n_i = b_i^\dagger b_i$  is the number operator,  $w$  is the hopping matrix element, and  $V$  is the on-site repulsion between the bosons. In the limit of large  $V$ , states with more than one boson on a site will only occur rarely, and it should pay to restrict the Hilbert space by projecting out such states. However, the elimination will generate a residual interaction of order  $w^2/V$  between the states on the restricted space. This interaction can be determined by the usual second-order perturbation theory and leads to the effective Hamiltonian

$$\begin{aligned} H_e &= -w \sum_i \left( b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i - 2b_i^\dagger b_i \right) - \mu \sum_i n_i \\ &- \frac{2w^2}{V} \sum_i \left( 2b_i^\dagger b_{i+1}^\dagger b_{i+1} b_i + b_i^\dagger b_{i-1}^\dagger b_{i-1} b_i \right. \\ &\left. + b_i^\dagger b_{i+1}^\dagger b_{i-1} b_i \right). \end{aligned} \quad (\text{B2})$$

We reiterate that  $H_e$  is nonzero only on states with at most one boson per site. Notice now that this reduced Hilbert space is identical to that of spinless fermions. The transformation between the  $b_i$  and the spinless fermion operators  $f_i$  is of course the Jordan-Wigner mapping

$$b_i = \prod_{j<i} (1 - 2f_j^\dagger f_j) f_i. \quad (\text{B3})$$

We now insert (B3) in (B2) and take the continuum limit with  $f_i = \sqrt{a} \Psi_F(x = ia)$ ,  $w = \hbar^2/(2ma^2)$  ( $a$  is the lattice spacing), and obtain

$$\begin{aligned} H_F &= \int dx \left[ \Psi_F^\dagger \left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \mu \right) \Psi_F \right. \\ &\left. - \frac{8w^2 a^3}{V} \frac{d\Psi_F^\dagger}{dx} \Psi_F^\dagger \Psi_F \frac{d\Psi_F}{dx} \right]. \end{aligned} \quad (\text{B4})$$

It is now clear by power counting that the four-fermion coupling term is clearly an irrelevant perturbation to the  $\mu = T = 0$  fixed point. It is in fact also not difficult to see that all interactions between the fermions are irrelevant, and that this result is not special to the boson Hubbard model considered here. The key point is of course that a term like  $\Psi_F^\dagger \Psi_F^\dagger \Psi_F \Psi_F$ , which is the only interaction term which is relevant by power counting about the free fermion fixed point at  $\mu = 0$ , vanishes identically because of the fermion anticommutation relations.

The significance of the four-fermion coupling changes when we consider the scaling dimensions of operators about the Luttinger liquid fixed points. In this case we decompose the fermion field into left- ( $\Psi_L$ ) and right- ( $\Psi_R$ ) moving excitations with a linear dispersion, and obtain the long-wavelength Hamiltonian

$$H_L = \int dx \left[ \hbar c \left( \Psi_R^\dagger \frac{d\Psi_R}{dx} - \Psi_L^\dagger \frac{d\Psi_L}{dx} \right) - \frac{32w^2 a^3 k_F^2}{V} \Psi_R^\dagger \Psi_L^\dagger \Psi_L \Psi_R \right], \quad (\text{B5})$$

where  $c = \hbar k_F/m$  and the Fermi wave vector  $k_F$  is given by  $\hbar^2 k_F^2/(2m) = \mu$ . Performing power counting on the  $z = 1$  free field part of  $H_L$  we now find that the four-fermion coupling is now marginal. Note however that the coefficient of this four-fermion coupling is suppressed by a factor of  $k_F^2$ , which vanishes as one approaches the  $z = 2$  critical end point. By the usual logarithmic perturbation theory at  $T = 0$  we can determine that the four-fermion interaction modifies the exponent  $\eta$  of Sec. III B by

$$\eta = \frac{1}{2} - \gamma \frac{\sqrt{w\mu}}{V}, \quad (\text{B6})$$

where  $\gamma$  is a numerical constant of order unity. (The perturbation theory yields exponents for the fermionic correlators: These can be related to the exponents of the bosonic correlators by the exponent identities of Ref. 20.) Notice, as expected, that the correction to  $\eta$  vanishes as  $\mu \rightarrow 0$ . It is also apparent that the impenetrable limit  $V \rightarrow \infty$  is equivalent to the vanishing density ( $\mu \rightarrow 0$ ) limit.

At finite  $T$ , there will be corrections to the correlators with terms like  $(\sqrt{t}\mu/V) \ln(\mu/k_B T)$ . These cannot be neglected when

$$k_B T < \mu \exp\left(-\gamma' \frac{V}{\sqrt{w\mu}}\right), \quad (\text{B7})$$

where  $\gamma'$  is of order unity. Zero-scale-factor universality is thus violated for arbitrarily small  $\mu$ , when  $T$  is smaller still and satisfies (B7). Notice however that the boundary specified by (B7) lies well below  $k_B T \sim \mu$  crossover to the Luttinger liquid regime (Fig. 1).

### APPENDIX C: IMPENETRABLE BOSE GAS IN ONE DIMENSION

Its *et al.*<sup>24</sup> have recently obtained some exact asymptotic results for the equal-time boson Green's function of the  $d = 1$  impenetrable Bose gas. Recall that this model is precisely the scaling limit describing the  $z = 2$  quantum phase transition with zero-scale-factor universality. In this appendix, we show how the requirement that the scaling functions be analytic in  $\mu/k_B T$  can lead to a considerable simplification of their results.

The asymptotic results of Its *et al.* can be written in the form

$$G(x, \tau = 0^{-1}) = \frac{\sqrt{2mk_B T}}{\hbar} A\left(\frac{\mu}{k_B T}\right) \times \exp\left[-\frac{\sqrt{2mk_B T}}{\hbar} f\left(\frac{\mu}{k_B T}\right) x\right] \quad \text{as } x \rightarrow \infty, \quad (\text{C1})$$

where  $A(t)$  and  $f(t)$  are functions to be determined (as before  $t \equiv \mu/k_B T$ ). From the arguments in Sec. III, it is clear that  $A(t)$  and  $f(t)$  are also universal crossover functions of the  $\mu = 0, T = 0$ , quantum phase transition the repulsive,  $d = 1$  Bose gas with *arbitrary*, short-range interactions. Its *et al.* obtained two separate, closed-form, integral expressions for  $f_{\pm}(t)$  and  $A_{\pm}(t)$  valid respectively for  $t > 0$  and  $t < 0$ . The two expressions were quite distinct and there appeared to be no straightforward relationship between them.

Here, we point out that the absence of any singularity in the finite- $T$  Bose gas in fact requires that  $f(t)$  and  $A(t)$  be analytic for all finite, real values of  $t$ . In other words, the functions  $f_+(t)$  and  $f_-(t)$  must be analytic continuations of each other [similarly for  $A_+(t)$  and  $A_-(t)$ ]. We have in fact succeeded in proving that the expression of Its *et al.* for  $f_+(t)$  is the analytic continuation of their result for  $f_-(t)$ . We have been unable to establish a similar result for  $A_{\pm}(t)$ , but have performed numerical tests on their expressions, which leave essentially no doubt that  $A$  is also analytic.

With the help of the above considerations, it is possible to deduce from Ref. 24 a simple closed-form result for  $f(t)$  and  $A(t)$  which is valid for *all*  $t$ :

$$f(t) = 1 + \frac{1}{\pi} \int_0^\infty dy \ln \left( \frac{(e^{y^2-t} + 1)(y^2 - t)}{(e^{y^2-t} - 1)(y^2 + 1)} \right),$$

$$A(t) = \frac{\rho_\infty}{\sqrt{\pi}} \exp \left[ -2 \int_t^\infty dy \left( \frac{df(y)}{dy} \right)^2 \right]. \quad (\text{C2})$$

The value of the constant  $\rho_\infty$  was obtained by matching to the  $T = 0$  result of Vaidya and Tracy,<sup>27</sup>

$$\rho_\infty = \pi e^{1/2} 2^{-1/3} A_G^{-6} = 0.924 182 203 782 \dots, \quad (\text{C3})$$

$A_G$  being Glaisher's constant.<sup>31</sup> Note that the analyticity of  $f$  and  $A$  for all real  $t$  is manifest. We have plotted the functions  $f(t)$  and  $A(t)$  in Fig. 8. They obey the asymptotic limits

$$f(t) = \begin{cases} \pi/(4\sqrt{t}), & t \rightarrow \infty, \\ \sqrt{-t}, & t \rightarrow -\infty, \end{cases}$$

$$A(t) = \begin{cases} \rho_\infty/\sqrt{\pi}, & t \rightarrow \infty, \\ 1/(2\sqrt{-t}), & t \rightarrow -\infty. \end{cases} \quad (\text{C4})$$

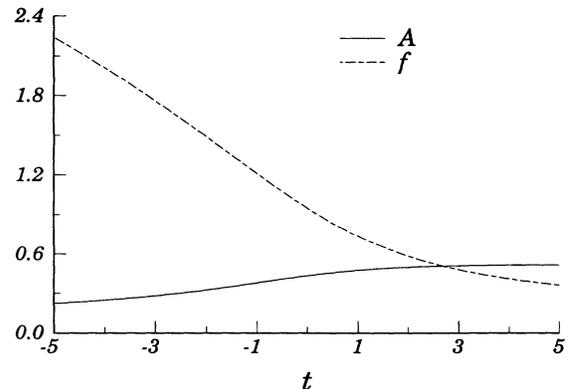


FIG. 8. The scaling functions  $f(t)$  and  $A(t)$  [ $t = \mu/(k_B T)$ ] of the  $d = 1$  Bose gas defined by Eqs. (C1) and (C2).

Both asymptotic limits of  $f(t)$ , and the  $t \rightarrow +\infty$  limit of  $A(t)$ , can be obtained directly from (C2). The  $t \rightarrow -\infty$  limit of  $A(t)$  is more difficult to obtain from (C2), and we used instead the second expression for  $A(t)$  in Ref. 24. Demanding that these two methods of obtaining the limit be identical in fact provides one with an independent derivation of the value of the constant  $\rho_\infty$ .

We also recall<sup>24</sup> that

$$f(0) = \frac{\zeta(3/2)}{\sqrt{\pi}} \left( 1 - \frac{1}{2\sqrt{2}} \right) = 0.952\,781\,470\,610\,75\dots \quad (C5)$$

We have already noted that taking a Fourier transform of the asymptotic results (C1) to obtain the structure factor yields results which compare very poorly with numerically exact results of Fig. 4.

Finally, we note that the requirement of analyticity as a function of  $t$  should apply to essentially all of the equal-time and unequal-time results of Its *et al.*<sup>24,25</sup> In every case, they have obtained separate expressions for  $t < 0$  and  $t > 0$ : Proving that these are analytic continuations of each other will lead to highly nontrivial checks on the results, and should also produce some fascinating mathematical identities.

- <sup>1</sup> W.J.L. Buyers *et al.*, Phys. Rev. Lett. **56**, 371 (1986); M. Steiner *et al.*, J. Appl. Phys. **61**, 3953 (1987); J.P. Renard *et al.*, Europhys. Lett. **3**, 945 (1986); J.P. Renard *et al.*, J. Appl. Phys. **63**, 3538 (1988); K. Katsumata *et al.*, Phys. Rev. Lett. **63**, 1424 (1989); M. Hagiwara *et al.*, *ibid.* **65**, 3181 (1990).
- <sup>2</sup> F.D.M. Haldane, Phys. Rev. Lett. **50**, 1153 (1983).
- <sup>3</sup> W. Lu *et al.*, Phys. Rev. Lett. **67**, 3716 (1991); A. Meyer *et al.*, Inorg. Chem. **21**, 1729 (1982); L.P. Legnault *et al.*, Physica B **180**, 188 (1992); M. Chiba *et al.*, Phys. Rev. B **44**, 2838 (1991); M. Date and K. Kindo, Phys. Rev. Lett. **65**, 1659 (1990).
- <sup>4</sup> H.J. Schulz, Phys. Rev. B **34**, 6372 (1986).
- <sup>5</sup> I. Affleck, Phys. Rev. B **41**, 6697 (1990); **43**, 3215 (1991); E. Sorenson and I. Affleck, Phys. Rev. Lett. **71**, 1633 (1993).
- <sup>6</sup> A.M. Tselvik, Phys. Rev. B **42**, 10499 (1990).
- <sup>7</sup> P. Mitra and B.I. Halperin, Phys. Rev. Lett. **72**, 912 (1994).
- <sup>8</sup> C. Zeng and V. Elser, Phys. Rev. B **42**, 8436 (1990); J.B. Marston and C. Zeng, J. Appl. Phys. **69**, 5962 (B) (1991); A.B. Harris, C. Kallin, and A.J. Berlinsky, Phys. Rev. B **45**, 2889 (1992); J.T. Chalker, P.S. Holdsworth, and E.F. Shender, Phys. Rev. Lett. **68**, 855 (1992); P. Chandra, P. Coleman, and I. Ritchey, Phys. Rev. B **47**, 15342 (1993); A.V. Chubukov, Phys. Rev. Lett. **69**, 832 (1992); J. von Delft and C.L. Henley, *ibid.* **69**, 3236 (1992); J.N. Reimers, A.J. Berlinsky, and A.-C. Shi, Phys. Rev. B **43**, 865 (1991); R.R.P. Singh and D. Huse, Phys. Rev. Lett. **68**, 1706 (1992); D.L. Huber and W.Y. Cheong, Phys. Rev. B **47**, 3220 (1993).
- <sup>9</sup> N. Read and S. Sachdev, Phys. Rev. Lett. **66**, 1773 (1991); S. Sachdev and N. Read, Int. J. Mod. Phys. B **5**, 219 (1991).
- <sup>10</sup> S. Sachdev, Phys. Rev. B **45**, 12377 (1992).
- <sup>11</sup> M.P.A. Fisher, P.B. Weichmann, G. Grinstein, and D.S. Fisher, Phys. Rev. B **40**, 546 (1989).
- <sup>12</sup> K. Huang and C.N. Yang, Phys. Rev. **105**, 767 (1957); T.D. Lee, K. Huang, and C.N. Yang, *ibid.* **106**, 1135 (1957).
- <sup>13</sup> A.V. Chubukov and S. Sachdev, Phys. Rev. Lett. **71**, 169 (1993); **71**, 2680(E) (1993); S. Sachdev and Jinwu Ye, *ibid.* **69**, 2411 (1992); A.V. Chubukov, S. Sachdev, and J. Ye, Phys. Rev. B **49**, 11 919 (1994).
- <sup>14</sup> D. Stauffer, M. Ferer, and M. Wortis, Phys. Rev. Lett. **29**, 345 (1972); P.C. Hohenberg, A. Aharony, B.I. Halperin, and E.D. Siggia, Phys. Rev. B **13**, 2986 (1976); C. Bervillier, *ibid.* **14**, 4964 (1976); M.-C. Cha *et al.*, *ibid.* **44**, 6883 (1991).
- <sup>15</sup> A.V. Chubukov, S. Sachdev, and T. Senthil, Phys. Rev. Lett. **72**, 2089 (1994).
- <sup>16</sup> S. Chakravarty, B.I. Halperin, and D.R. Nelson, Phys. Rev. B **39**, 2344 (1989).
- <sup>17</sup> V.N. Popov, *Functional Integrals in Quantum Field Theory and Statistical Physics* (D. Reidel, Boston, 1983).
- <sup>18</sup> D.S. Fisher and P.C. Hohenberg, Phys. Rev. B **37**, 4936 (1988).
- <sup>19</sup> S. Sachdev, Z. Phys. B (to be published).
- <sup>20</sup> F.D.M. Haldane, Phys. Rev. Lett. **47**, 1840 (1981).
- <sup>21</sup> E.H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1965); E.H. Lieb, *ibid.* **130**, 1616 (1965).
- <sup>22</sup> V.E. Korepin and N.A. Slavnov, Commun. Math. Phys. **129**, 103 (1990).
- <sup>23</sup> A. Lenard, J. Math. Phys. **7**, 1268 (1966).
- <sup>24</sup> A.R. Its, A.G. Izergin, and V.E. Korepin, Physica D **53**, 187 (1991).
- <sup>25</sup> A.R. Its, A.G. Izergin, V.E. Korepin, and G.G. Varzugin, Physica D **54**, 351 (1992).
- <sup>26</sup> E.A. Robinson and S. Treitel, *Geophysical Signal Analysis* (Prentice-Hall, Englewood Cliffs, NJ, 1980).
- <sup>27</sup> H.G. Vaidya and C.A. Tracy, J. Math. Phys. **20**, 2291 (1979); M. Jimbo, T. Miwa, Y. Mori, and M. Sato, Physica D **1**, 80 (1980).
- <sup>28</sup> *Bosonization: How to make it work for you in Condensed Matter Physics*, BCSPIN School at Kathmandu (1991) in Current Topics in Condensed Matter and Particle Physics, edited by J. Pati, Q. Shafi, and Yu Lu (World Scientific, Singapore, 1993).
- <sup>29</sup> J.L. Cardy, J. Phys. A **17**, L385 (1984).
- <sup>30</sup> R. Shankar, Int. J. Mod. Phys. B **4**, 2371 (1990).
- <sup>31</sup> B.M. McCoy and T.T. Wu, *The Two Dimensional Ising Model* (Oxford University Press, Oxford, 1973).