Impurity scattering in mesoscopic quantum wires and the Laguerre ensemble

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A maximum-entropy model describing the conductance statistics of disordered mesoscopic quantum wires is presented. The model, closely related to the Laguerre ensemble of random matrices, predicts a weak localization effect and universal conductance fluctuations, in qualitative agreement with microscopic theory. A simple interpretation of the maximum-entropy model is given and the relation with other maximum-entropy models is discussed.

I. INTRODUCTION

This paper is concerned with a statistical description of the electrical conductance of disordered mesoscopic quantum wires. A statistical approach is appropriate because the conductance G of a mesoscopic sample depends not only on the mean impurity concentration but also on the exact positions of the impurities within the sample. As a result, over an ensemble of macroscopically similar samples, G fluctuates from sample to sample. The magnitude of the conductance fluctuations is independent of the mean conductance provided that this is large, a phenomenon referred to as universal conductance fluctuations (UCF).[1] If UCF's are to be observable, the electron motion within the sample must be phase coherent. For this reason experiments are usually performed on very small (μm) metal or semiconductor samples at very low temperatures (mk) in which the dominant scattering mode is elastic. Under such conditions the conductance can be related to the elastic-scattering properties of the system by the Landauer formula.[2,3] Imry[4] was the first to suggest that random matrix theory might provide a very natural and general explanation of UCF's. This suggestion prompted the formulation of two different random matrix theories of scattering in disordered mesoscopic systems: the so-called local[5-7] and global[8-10] maximum-entropy models. In these models the elastic scattering of electrons by the sample is described by a transfer matrix T (defined in Sec. II). Underlying both theories is the observation that, as a consequence of current conservation, the transfer matrices T form a Lie group under matrix multiplication.

The local maximum-entropy model predicts a weak-localization effect and conductance fluctuations in exact quantitative agreement with those of microscopic calculations for "long" mesoscopic quantum wires. (By "long" we mean that the length of the wires is much greater than its diameter. For short wires there are qualitative differences with microscopic calculations.) In the local theory the central quantity $P(T)$, the probability distribution of the transfer matrix, appears as the solution of a Fokker-Planck equation describing the evolution of $P(T)$ with the length of wire. Unfortunately the exact solution of this equation is known only for a wire in an applied magnetic field.[11] In contrast, the global maximum-entropy theory yields immediately a very simple expression for $P(T)$ as a maximum-entropy distribution on the group formed by the transfer matrices under multiplication. The disadvantage is that the global model makes no prediction concerning the weak localization effect and is only in qualitative agreement with microscopic calculations concerning the conductance fluctuations.

In this paper we describe in detail a maximum-entropy model[12] which is simple in that it is also a maximum-entropy theory on a Lie group and yet predicts not only conductance fluctuations but also a weak-localization effect in qualitative agreement with microscopic theory. The starting point for the model we present is the existence of a second group structure in the transfer matrix description of the elastic scattering. We propose a maximum-entropy distribution not on the group formed by the transfer matrices T under multiplication but that formed by the matrices Ω, defined by $\Omega=\ln TT^{\dagger}$, under matrix addition. The two group structures, which we denote by $(T,\times)$ and $(\Omega,+)$, respectively, are not isomorphic and the corresponding maximum-entropy models are not equivalent. In addition to the qualitative agreement with microscopic theory we find that the model describes, to a very good approximation, the conductance statistics of numerically simulated quantum wires.

The model we describe is related to the Laguerre ensemble of random matrices (see Appendix A). Even though the matrix Ω is Hermitian, the relevant ensemble turns out to be the Laguerre rather than the Gaussian ensemble because current conservation imposes an additional symmetry constraint on Ω. The Laguerre ensemble can be defined equivalently both as a maximum-entropy distribution on a Lie group or by the requirements that the distribution be invariant under certain types of transformation and that matrix elements not otherwise related by symmetry be statistically independent (see Appendix A). This equivalence makes possible a clear statement of the assumptions underlying the maximum-entropy distribution on $(\Omega,+)$ and allows us to

understand in a simple way the origin of the difference with the maximum-entropy distribution on $(T, \times)$.

II. SCATTERING THEORY

In this section we outline the scattering theory for a quantum wire. This is done first in order to define the transfer matrix T and second to derive two relations which T must satisfy as a consequence of current conservation and time-reversal symmetry. Though these relations have been derived previously in Ref. 5, we rederive them here for model (1), taking explicit account of nonpropagating (evanescent) states and the more complicated structure of the wave functions in magnetic field, details which were not considered in the original derivation. Later we shall compare the maximum-entropy model with a numerical simulation of model (1).

We consider the scattering of electrons by a disordered mesoscopic system in the two-probe geometry depicted in Fig. 1 in which impurity free contacts supply an electric current to a disordered sample. We shall suppose that the spin and spatial degrees of freedom of the electrons are uncoupled. We shall ignore inelastic effects within the sample and consider only the elastic scattering of electrons at the Fermi energy E_F. As a physical model of the system we take a tight-binding Hamiltonian

$$H=\sum_{xyz} \varepsilon_{x,y,x}|xyz\rangle\langle xyz| + |xyz\rangle\langle x\pm 1yz| + |xyz\rangle\langle xy\pm 1z| + |xyz\rangle\langle xyz+1|e^{i2\pi\alpha x} + |xyz\rangle\langle xyz-1|e^{-i2\pi\alpha x} . \quad (1)$$

The coordinates are chosen so that electrons are confined in the transverse directions within $-(L_x-1)/2 \le x \le (L_x-1)/2$ and $1 \le y \le L_y$. The potential ε is random in the sample $1 \le z \le L_z$ and zero outside. Lengths are measured in units of the lattice spacing and energies are measured in units of the nearest-neighbor hopping energy, both taken as unity. As shown in Fig. 1, there is a magnetic flux in the y direction. The gauge is chosen so that the vector potential $A=(0,0,-Bx)$. The strength of the field is given in terms of the number α of magnetic flux quanta $\phi_0=e/h$ penetrating each lattice cell. Note that the magnetic field is applied both in the contracts and in the sample so that there is no discontinuity in the field at the sample boundaries.

In the contact to the left we decompose the wave function in terms of the Bloch states at the Fermi energy E_F

$$\Psi(x,y,z)=\sum_{n=1}^{L_x\times L_y} [c_{+n}\psi_n(x,y)e^{ik_n z} + c_{-n}\psi_n(-x,y)e^{-ik_n z}] .$$

Here we have made use of the fact that for our symmetrical choice of x coordinates the Bloch states occur in pairs of opposite wave number k, more precisely if $\Psi(x,y,z)=\psi(x,y)e^{ikz}$ is a solution of $H\Psi=E\Psi$, so is $\Psi(x,y,z)=\psi(-x,y)e^{-ikz}$. As a boundary condition in the left contact we require that the wave function remains finite as $z\to-\infty$, which is achieved by setting

$$c_{\pm n}=0 \quad \text{if } \mathrm{Im}[\pm k_n]>0 . \quad (2)$$

From the time derivative of the probability to find an electron in some given volume we derive the following expression for the total electric current in the left contact:

$$J(z)=\frac{e}{\hbar}\sum_{x,y}\frac{-i}{\hbar}[\Psi^*(x,y,z)\Psi(x,y,z+1)e^{i\alpha x} - \Psi(x,y,z)\Psi^*(x,y,z+1)e^{-i\alpha x}] .$$

After a lengthy but elementary calculation this can be reduced to[13]

$$J=\sum_{n,\mathrm{Im}[k_n]=0}|c_{+}n|^2-|c_{-n}|^2 .$$

The boundary condition (2) ensures that the only contributions to the total current in the contact come from states with $\mathrm{Im}[k]=0$. For these states we can show that the state $+k_n$ has an exactly opposite current to the state $-k_n$. In writing the expression for the total current we therefore assume a labeling and normalization such that $+k_n$ has a unit current in the $+z$ direction and $-k_n$ a unit current in the $-z$ direction. The number of states for which $\mathrm{Im}[k]=0$ we denote as N. In the right contact we expand the wave function in the same way as in the left contact, but with primed coefficients c'_{+n} and c'_{-n}. We impose the boundary condition that Ψ remain finite as $z\to+\infty$, which is satisfied if

$$c'_{\pm n}=0 \quad \text{if } \mathrm{Im}[\pm k_n]<0 . \quad (3)$$

Proceeding as before we find an exactly analogous expression for the total current in the right contact.

A knowledge of the N incoming and N outgoing flux amplitudes $(c_{+n}, c_{-n}; \mathrm{Im}[k_n]=0)$ in the left contact and the boundary conditions (2) and (3) uniquely determines the corresponding flux amplitudes in the right contact through a linear relation of the form

$$c'_{+}=tc_{+}+r'c'_{-} , \quad c_{-}=rc_{+}+t'c'_{-} . \quad (4)$$

Here c_+ is the N component vector with components $\{c_{+n}; \mathrm{Im}k_n=0\}$ and c_- is the N component vector with components $\{c_{-n}; \mathrm{Im}k_n=0\}$. This defines two $N\times N$ transmission matrices t and t' and two $N\times N$ reflections matrices r and r'. The zero temperature dc conductance in the two-probe geometry can be obtained from the transmission matrix by using the Landauer formula[3]

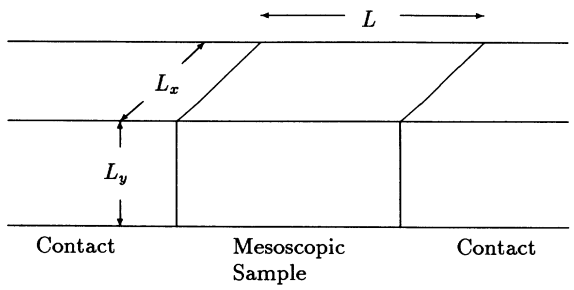

FIG. 1. An idealization of a two-probe measuring geometry for transport measurements on a mesoscopic sample.

$$g=\mathrm{tr}(tt^{\dagger}) . \tag{5}$$

Here g is a dimensionless conductance related to the actual conductance G by $G=2(e^2/h)g$, where a factor of 2 is included to take account of spin degeneracy. Only propagating states of the contacts with $\mathrm{Im}[k]=0$ contribute to the conductance, as is made clear in the derivation of the Landauer formula from the Kubo formula.[2] We can now define a $2N\times 2N$ transfer matrix T by rewriting (4) as

$$T\begin{bmatrix} c_+ \\ c_- \end{bmatrix}=\begin{bmatrix} c'_+ \\ c'_- \end{bmatrix} . \tag{6}$$

In terms of transmission and reflection matrices T has the form

$$T=\begin{bmatrix} [t^{\dagger}]^{-1} & r't'^{-1} \\ -t'^{-1}r & t'^{-1} \end{bmatrix} . \tag{7}$$

We now examine the general restrictions on the form of T due to the requirements of current conservation and time-reversal symmetry. First we discuss current conservation. We rewrite the current in the left contact in the form

$$J=(c_+^{\dagger}c_-^{\dagger})\begin{bmatrix} 1_N & 0 \\ 0 & -1_N \end{bmatrix}\begin{bmatrix} c_+ \\ c_- \end{bmatrix} ,$$

where 1_N is the $N\times N$ unit matrix. Thus if the currents in the right contact and left contact are always to be equal, T must satisfy

$$T^{\dagger}\Sigma_c T=\Sigma_c, \quad T\Sigma_c T^{\dagger}=\Sigma_c, \quad \Sigma_c=\begin{bmatrix} 1_N & 0 \\ 0 & -1_N \end{bmatrix} . \tag{8}$$

Second we discuss time-reversal symmetry. In the absence of any applied magnetic flux ($\alpha=0$) the Hamiltonian (1) is real and commutes $[H,\mathcal{T}]=0$ with the operation $\mathcal{T}$ of complex conjugation $\mathcal{T}\Psi=\Psi^*$. This is referred to as time-reversal symmetry. It follows that if Ψ is a solution to the scattering problem, then $\mathcal{T}\Psi$ must also be a solution. Looking at the left contact we have

$$\mathcal{T}\Psi=\sum_{m,\mathrm{Im}[k]=0}\left[\psi_m(-x,y)e^{-ik_m z}\sum_n d_{m,n}c^*_{+n}+\psi_m(x,y)e^{ik_m z}\sum_n d_{m,n}c^*_{-n}\right]+\sum_{n,\mathrm{Re}[k]=0} c^*_{+n}\psi^*_n(x,y)e^{+|k_n|z} ,$$

where d is an $N\times N$ symmetric unitary matrix with elements

$$d_{m,n}=\delta_{k_n,k_m}\sum_{x,y}\psi^*_m(x,y)\psi^*_n(-x,y) .$$

Note that when $B=0$, the wave numbers k are either purely real or purely imaginary. There is an analogous expression for the right contact. Thus if H and $\mathcal{T}$ commute and we have a set of flux amplitudes satisfying (6), then the flux amplitudes corresponding to the time-reversed state must also satisfy (6). This can only be true in general if T satisfies

$$T\begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix}=\begin{bmatrix} 0 & d \\ d & 0 \end{bmatrix}T^* .$$

Since d is a symmetric unitary matrix we can decompose it as $d=ee^T$, where e is unitary, and then make the transformation

$$T\to E^{\dagger}TE, \quad E=\begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}$$

after which T satisfies

$$\Sigma_+ T\Sigma_+=T^*, \quad \Sigma_+=\begin{bmatrix} 0 & 1_N \\ 1_N & 0 \end{bmatrix} . \tag{9}$$

The important point is that the form of the current conservation condition is not changed by this transformation so that we may choose a representation of the states in which T satisfies both (8) and (9) *at the same time.*

Following the general symmetry classification introduced by Dyson,[14] we identify two symmetry classes applicable for zero and nonzero applied magnetic fields: orthogonal if $[H,\mathcal{T}]=0$ and $\mathcal{T}^2=+1$ and unitary if $[H,\mathcal{T}]\neq 0$. These are the only two possibilities which need to be considered if there is no interaction between the spin degrees of freedom of the electron and its spatial motion. We shall not consider here materials in which spin-orbit scattering is important and the electron spin and spatial motion are coupled. We then need to consider the possibility that $\mathcal{T}^2=-1$, referred to as symplectic symmetry, and the time-reversal condition on the transfer matrix takes a different form.[9]

III. MAXIMUM ENTROPY ON $(\Omega, +)$

The set of $2N\times 2N$ complex matrices satisfying (8) forms the group $U(N,N)$ of pseudounitary matrices under matrix multiplication.[15] Matrices satisfying both (8) and (9) form a subgroup of $U(N,N)$. This group structure, for which we shall use the notation $(T,\times)$, underlies both the local and global maximum-entropy models. As stated in the Introduction, this is not the basis of the theory we present here. Rather we identify a second group structure in the scattering theory and develop a maximum-entropy model on this.

We shall first identify the elements of this group. These are not the transfer matrices T themselves, but the set of related matrices Ω defined by

$$\Omega=\ln TT^{\dagger} .$$

As a consequence of the definition Ω is Hermitian. Conservation of current imposes an additional symmetry conditional on Ω, which we can derive from the corresponding condition (8) on T as follows. From (8) we have $TT^{\dagger}\Sigma_c TT^{\dagger}=\Sigma_c$. We may rewrite this in the form $\Sigma_c\exp[\Omega]\Sigma_c=\exp[-\Omega]$. Since Σ_c is unitary, we may take the logarithm immediately and obtain

$$\Sigma_c\Omega\Sigma_c=-\Omega . \tag{10}$$

Hermitian matrices satisfying this condition form a group under matrix addition. We shall use the notation $(\Omega,+)$ for this group structure. It is not isomorphic to $(T,\times)$ since $(\Omega,+)$ is Abelian and $(T,\times)$ is not. The general form of the group elements is

$$\Omega=\begin{bmatrix}0 & \omega\\ \omega^{\dagger} & 0\end{bmatrix}, \tag{11}$$

where ω is an arbitrary $N\times N$ complex matrix. The decomposition (11) defines an isomorphism of $(\Omega,+)$ with the group of arbitrary $N\times N$ complex matrices under addition. The group elements may be parametrized by $2N^2$ real parameters $\{\omega^R_{n,m},\omega^I_{n,m};n,m=1,\ldots,N\}$, where R and I refer to real and imaginary parts. The group composition law can be written in the form $\Omega[\omega]+\Omega[\omega']=\Omega[\chi(\omega,\omega')]$ with $\chi(\omega,\omega')=\omega+\omega'$ clearly an analytic function. This is sufficient to show that $(\Omega,+)$ is a Lie group. From (10) it is possible to show that the $2N$ real eigenvalues of Ω occur in pairs of opposite sign. We shall refer to them by the notation $\{\pm\nu_i;\ i=1,\ldots,N\}$.

In a similar way to that presented above, the time-reversal condition on T can be shown to imply the following condition on Ω:

$$\Sigma_+\Omega\Sigma_+=\Omega^* . \tag{12}$$

Matrices satisfying both (10) and (12) also form a Lie group. The general form of the group elements is as before, but with ω now symmetric $\omega=\omega^T$. The group elements are now parametrized by $N(N+1)$ real parameters $\{\omega^R_{n,m},\omega^I_{n,m};\ n\geq m\}$. Regardless of whether we are referring to matrices satisfying (10) alone or both (10) and (12) we shall refer to the group structure as $(\Omega,+)$, making an explicit distinction only when necessary.

Given a probability distribution $P(\Omega)=p(\Omega)d\mu(\Omega)$ on the group, we define an entropy $S[p(\Omega)]$ by

$$S=-\int_{(\Omega,+)} d\mu(\Omega)p(\Omega)\ln p(\Omega) .$$

Here $d\mu(\Omega)$ is the invariant measure on the group which is (see Appendix B)

$$d\mu(\Omega)=\prod_{n,m} d\omega^R_{n,m}\prod_{n,m} d\omega^I_{n,m} ,$$

where R and I refer to real and imaginary parts. For the subgroup of matrices satisfying both (10) and (12) the invariant measure is

$$d\mu(\Omega)=\prod_{n\geq m} d\omega^R_{n,m}\prod_{n\geq m} d\omega^I_{n,m}$$

(since ω is then symmetric and not all elements may be considered independent, see Appendix B). For our purposes it is more useful to rewrite the measures in terms of the eigenvalues of Ω. This is done by noting that the N positive eigenvalues $\{\nu_1,\ldots,\nu_N\}$ of Ω are also the singular values which occur in the singular value decomposition[16] of ω as

$$\omega=u_1\nu u_2^{\dagger} . \tag{13}$$

Note that any complex matrix can be decomposed in this way as product of two unitary matrices u_1 and u_2 and a real positive diagonal matrix ν with elements $\{\nu_1,\ldots,\nu_N\}$. For an arbitrary complex matrix the singular value decomposition is not unique since any transformation of the form $u_1\rightarrow u_1e^{i\theta}$, $u_2\rightarrow u_2e^{i\theta}$ with θ real diagonal leaves ω unchanged. In terms of these parameters the invariant measure may be rewritten (see Appendix B)

$$d\mu(\Omega)=\prod_n \nu_n d\nu_n \prod_{n>m}|\nu_n^2-\nu_m^2|^2 d\mu(u_1)d\mu(u_2) ,$$

where any one of the measures on u_1 or u_2 may be taken to be the invariant measure on the unitary group $U(N)$ and the measure on the remaining matrix must be modified to take account of the degeneracy in the singular value decomposition (see Appendix B). For matrices satisfying both (10) and (12) ω is symmetric and the singular value decomposition is unique

$$\omega=u_1\nu u_1^T . \tag{14}$$

The invariant measure becomes

$$d\mu(\Omega)=\prod_n \nu_n d\nu_n \prod_{n>m}|\nu_n^2-\nu_m^2| d\mu(u_1) ,$$

where the measure on u_1 is the invariant measure on $U(N)$.

In principle the probability density $p(\Omega)$ could be determined directly from the probability distribution of the Hamiltonian. However, in practice this is at least very difficult if not impossible. A simple alternative, borrowed from statistical mechanics, is to make an unbiased guess for the probability density using a maximum-entropy procedure. This requires us to define an entropy S and then to choose one or more constraints subject to which we then maximize S. In the present context the resulting probability density depends not only on the choice of the constraints but also on which group $(T,\times)$ or $(\Omega,+)$ we decide to define the density and the entropy. Here we shall assume that density $p(\Omega)$ on $(\Omega,+)$ is such as to maximize the entropy $S[p(\Omega)]$ defined on $(\Omega,+)$. The group is not compact, so at least one constraint, in addition to the normalization of the probability distribution, must be imposed. Perhaps the simplest choice would be a constraint on the trace $\langle \mathrm{tr}\Omega\rangle$; however, since this is exactly zero by virtue of (10), this is already satisfied. We therefore maximize S subject to

$$\langle \mathrm{tr}\Omega^2\rangle=\frac{[\beta N(N-1)+2N]}{\beta a} . \tag{15}$$

The result is

$$P(\Omega)=C_{N,\beta}\exp\left[-\frac{\beta a}{2}\mathrm{tr}\Omega^2\right]d\mu(\Omega) , \tag{16}$$

where $C_{N,\beta}$ is a normalization constant. Maximizing the entropy does not introduce any correlations between the unitary matrices u_1 and u_2 in the decomposition and the parameters $\{\nu_1,\ldots,\nu_N\}$ since no constraint is imposed on either u_1 or u_2. This being the case the integration over u_1 and u_2 is trivial and we obtain the joint probability distribution of $\{\nu_1,\ldots,\nu_N\}$

$$P(\nu_1,\ldots,\nu_N)=C_{N,\beta}\prod_{n>m}|\nu_n^2-\nu_m^2|^\beta\prod_n \nu_n e^{-\beta a\nu_n^2}d\nu_n \ . \tag{17}$$

Making the transformation $x_n=2a\nu_n^2$, the distribution can be identified as that of the Laguerre ensemble (see Appendix A)

$$P(x_1,\ldots,x_N)=C_{N,\beta}\prod_{n>m}|x_n-x_m|^\beta\prod_n e^{-\beta(x_n/2)}dx_n \ . \tag{18}$$

The two symmetry classes $\beta=1$ and 2 are referred to as the Laguerre orthogonal ensemble (LOE) and the Laguerre unitary ensemble (LUE), respectively. In common with the Gaussian and circular ensembles there is a characteristic dependence of the eigenvalue repulsion on the symmetry of the ensemble: linear ($\beta=1$) for the LOE and quadratic ($\beta=2$) for the LUE.

To make the significance of the maximum-entropy assumption clearer it is helpful to observe that (16) is the only distribution, for example, for unitary symmetry ($\beta=2$), satisfying the following two conditions.

(i) All elements of Ω not otherwise related by symmetry are statistically independent.

(ii) The probability distribution $P(\Omega)$ is invariant under transformations of the form

$$P(\Omega')=P(\Omega),\quad \Omega'=U\Omega U^\dagger\quad \forall U=\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix},$$

where u and u' are arbitrary $N\times N$ unitary matrices.

This follows from a corresponding statement for the LUE. Analogous conditions for orthogonal symmetry ($\beta=1$) can be deduced by reference to those for the LOE. Unfortunately condition (i) has no obvious physical interpretation. Condition (ii), as we discuss in more detail later, is equivalent to the assumption that scattering among the various channels is statistically equivalent. Whether these are reasonable assumptions must be determined by comparison with microscopic models. It is interesting to note that the local maximum-entropy model, which is in exact quantitative agreement with the microscopic theory for long quantum wires, incorporates an assumption equivalent to (ii). As we shall show in Sec. IV, with the model we present here we achieve only qualitative agreement with microscopic theory so that it would seem that (i) is only approximately satisfied in practice.

Having derived a distribution for the eigenvalues of Ω it remains to relate them to the dimensionless conductance g. From the current conservation condition on T (8) and the decomposition of T in terms of transmission and reflection matrices (7) we can derive the relation

$$2[\cosh\Omega+I_{2N}]^{-1}=\begin{bmatrix} tt^\dagger & 0 \\ 0 & t'^\dagger t' \end{bmatrix} .$$

This tells us the relationship between the conductance and the eigenvalues of Ω

$$g=\mathrm{tr}tt^\dagger=\sum_{n=1}^{N}\frac{2}{1+\cosh\nu_n} \ .$$

IV. COMPARISON OF THE MAXIMUM-ENTROPY MODEL WITH MICROSCOPIC MODELS

In this section we discuss the extent to which the predictions of the maximum-entropy model are in agreement with those of more traditional microscopic models. First we compare with the results of calculations of weak localization and conductance fluctuations for electrons in a random Gaussian white noise potential.[1] Second we compare the eigenvalue density and correlations of the maximum-entropy model with those obtained in a numerical simulation of the Hamiltonian (1) using a numerical method.[17]

A. Weak localization

First we examine the prediction of the theory concerning the weak-localization correction to the mean conductance. The mean conductance $\langle g\rangle$ is

$$\langle g\rangle=\int_0^\infty d\nu\, g(\nu)\sigma(\nu) \ ,$$

where σ is the eigenvalue density defined as

$$\sigma(\nu)=\int_0^\infty p(\nu,\nu_2,\ldots,\nu_n)\prod_{n=2}^{N}d\nu_n$$

and

$$g(\nu)=\frac{2}{1+\cosh\nu} \ .$$

The simplest way to extract the weak-localization correction predicted by the theory is to calculate the change in the conductance, to leading order in the large N limit, when time-reversal symmetry is broken by the application of a weak magnetic field. The weak-localization correction δg should thus be equal to

$$\delta g=\int_0^\infty d\nu\, g(\nu)[\sigma_{\mathrm{LOE}}(N,a,\nu)-\sigma_{\mathrm{LUE}}(N,a,\nu)]$$

with $N\gg 1$. We assume here that a is independent of magnetic field at least for weak fields. The eigenvalue density for the LOE (Ref. 18) is

$$\sigma_{\mathrm{LOE}}(N,a,\nu)=2a\nu S(a\nu^2,a\nu^2) \ ,$$

where $S(x,y)$ is the function

$$\begin{aligned} S(x,y)=&2\exp(-x)\exp(-y)\sum_{n=1}^{N=1}L_n(2x)L_n(2y) \\ &+\exp(-x)\exp(-y)L_{N-1}(2x)L_N'(2y) \\ &-\exp(-y)\int_0^x\exp(-z)L_{N-1}(2z)dzL_N'(2y) \ . \end{aligned}$$

For the LUE the density has a simpler analytic form[19]

$$\sigma_{\mathrm{LUE}}(N,a,\nu)=4a\nu K(2a\nu^2,2a\nu^2) \ ,$$

where

$$K(x,y)=\exp(-x/2)\exp(-y/2)\sum_{n=0}^{N-1}L_n(x)L_n(y) \ .$$

Making use of asymptotic forms for the Laguerre polynomials the density in the bulk is found to be quarter circle independent of symmetry in the limit as $N\to\infty$

$$\sigma(\nu)=\frac{4}{\pi}\sqrt{(2aN-a^2\nu^2)}\ . \tag{19}$$

There is thus no contribution to δg from the bulk of the spectrum. There is, however, a nonzero contribution from the region $\nu\sim O(\sqrt{1/N})$. In this region the asymptotic forms of the densities are[18]

$$\rho(\nu)=2Na\nu\left[J_1^2(\sqrt{Na}\,\nu)-J_0(2\sqrt{Na}\,\nu)J_2(2\sqrt{Na}\,\nu)+\frac{J_0(2\sqrt{Na}\,\nu)J_1(2\sqrt{Na}\,\nu)}{2\sqrt{Na}\,\nu}\right]$$

for the LOE and[19]

$$\rho(\nu)=2Na\nu\left[J_1^2(2\sqrt{Na}\,\nu)-J_0(2\sqrt{Na}\,\nu)J_2(2\sqrt{Na}\,\nu)+2\frac{J_0(2\sqrt{Na}\,\nu)J_1(2\sqrt{Na}\,\nu)}{2\sqrt{Na}\,\nu}\right]$$

for the LUE. Taking the difference and making a suitable change of variable we find

$$\delta g=-\tfrac{1}{2}\int_0^\infty dz\,J_0(z)J_1(z)=-\tfrac{1}{4}\ .$$

This is in qualitative agreement (the correction has the correct sign and is independent of parameters such as system size and mean free path, etc.) with microscopic calculations. There is, however, a quantitative discrepancy, with the latter yielding $\delta g=-\frac{1}{3}$.

B. Conductance fluctuations

In common with the global maximum-entropy theory the Laguerre ensemble provides a natural explanation for the phenomena of universal conductance fluctuations. This follows from the Dyson-Mehta theorem[20] concerning the asymptotic values of linear statistics of random matrix ensembles and its generalization to positive definite ensembles.[21] A linear statistic of the spectrum is any function F of the eigenvalues which has the form

$$F=\sum_{n=1}^{N}f(\nu_n)\ .$$

The variance of a linear statistic may be calculated as

$$\mathrm{Var}[F]=\int d\nu\,d\nu' f(\nu)f(\nu')\times[\sigma(\nu)\delta(\nu-\nu')-T_2(\nu,\nu')]\ .$$

This requires a knowledge of both the density and the two-point correlation function

$$R_2(\nu,\nu')=N(N-1)\int p(\nu,\nu',\nu_3,\ldots,\nu_N)\prod_{n=3}^{N}d\nu_n$$

to which the two-level cluster function T_2 is related by $R_2(\nu,\nu')=\sigma(\nu)\sigma(\nu')-T_2(\nu,\nu')$. We are interested in particular in the value of the invariance in the limit that $N\to\infty$ with a fixed. We calculate the variance in this limit using the functional derivative method of Ref. 21; however, in order to put the final result in a very simple form, the derivation we give here differs somewhat in detail from that of Ref. 21. Suppose that the distribution of the eigenvalues has the general form

$$P(\nu_1,\ldots,\nu_N)=C_{N,\beta}\prod_{n>m}|\nu_n^2-\nu_m^2|^\beta\times\prod_{n=1}^{N}\exp[-\beta V(\nu_n)]d\nu_n\ . \tag{20}$$

The variance is evaluated by noticing the following relationship between T_2, σ, and the functional derivative of σ with respect to the potential $V(\lambda)$:

$$\frac{1}{\beta}\frac{\delta\sigma(\nu)}{\delta V(\nu')}=T_2(\nu,\nu')-\sigma(\nu)\delta(\nu-\nu')\ .$$

The functional derivative is determined with the aid of an approximate relation between $V(\lambda)$ and σ

$$V(\nu)+V_0=\int_0^\infty d\nu'\ln|\nu^2-\nu'^2|\sigma(\nu')\ .$$

Here V_0 is a constant chosen so the density is normalized to N. The use of this relationship restricts the analysis to smooth functions f and the asymptotic limit $N\to\infty$. We deduce that the change $\delta\sigma(\nu)$ in the function $\sigma(\nu)$ due to a change $\delta V(\lambda)$ in $V(\lambda)$ satisfies

$$\delta V(\nu)+V_0'=\int_0^\infty d\nu'\ln|\nu^2-\nu'^2|\delta\sigma(\nu')\ .$$

The constant V_0' is chosen so that

$$\int_0^\infty d\nu\,\delta\sigma(\nu)=0\ .$$

The solution of this equation is of the form

$$\delta\sigma(\nu)=\int_0^\infty\Theta(\nu,\nu')\delta V(\nu')\ ,$$

where the kernel Θ is clearly the functional derivative we require

$$\Theta(\nu,\nu')=\frac{\delta\sigma(\nu)}{\delta V(\nu')}\ .$$

The only difficulty in solving the integral equation is the range of integration which prevents a straightforward solution by Fourier transform. We surmount this difficulty by observing that Θ can be related to the solution of the integral equation

$$\delta V(\nu)+V_0'=\int_{-\infty}^{+\infty}d\nu'\ln|\nu-\nu'|\delta\sigma(\nu')\ ,$$

where V_0' is the same constant as before [and ensures that $\delta\sigma(\nu)$ integrates to zero over the full real axis]. Denoting the solving kernel of this equation as $\theta(\nu,\nu')$, it is possible to show by direct substitution that

$$\Theta(\nu,\nu')=\theta(\nu,\nu')+\theta(\nu,-\nu')\ .$$

This gives a relation between the correlation function for the positive random matrix ensemble with the distribution (20) and a random matrix ensemble with eigenvalues

on the full line distributed according to

$$P(\nu_1,\ldots,\nu_N) = C_{N,\beta}\prod_{n>m}|\nu_n-\nu_m|^{\beta}\prod_{n=1}^{N}\exp[-\beta V(\nu_n)]d\nu_n \ .$$

We now solve this related problem by Fourier transform. Assuming that $V_0'=0$, which we justify later, we find $\delta\sigma(k)=-(|k|/\pi)\delta V(k)$. We deduce that $\theta(\nu,\nu')\equiv\theta(|\nu-\nu'|)$ and that $\theta(k)=-|k|/\pi$. The normalization condition is equivalent to

$$\int_{-\infty}^{+\infty}d\nu\,\delta\sigma(\nu)=\lim_{k\to 0}-\frac{1}{2\pi^2}|k|\delta V(k)$$

and so the assumption that $V_0'=0$ is correct provided $\delta V(k)$ is not singular at the origin. Returning to the original problem we find in the asymptotic limit a simple formula for the variance

$$\mathrm{Var}[f]=\frac{1}{4\pi^2\beta}\int_{-\infty}^{\infty}dk\,f(k)f(-k)|k| \ , \tag{21}$$

where $f(k)$ is the Fourier transform of $f(\nu)$,

$$f(k)=\int_{-\infty}^{\infty}dk\,\exp(ikx)f(x) \ .$$

The asymptotic value is seen to be independent of the choice of V and has a characteristic dependence on the symmetry class. For the conductance fluctuations we set $f(\nu)=g(\nu)$ and find

$$\lim_{N\to\infty}\mathrm{Var}[g]=0.148/\beta \ .$$

The Laguerre ensemble thus predicts the correct qualitative behavior: the sample to sample fluctuations in the conductance are independent of the sample average conductance provided this is much greater than unity. There is, however, a quantitative discrepancy with the microscopic theory for a quantum wire[1]

$$\lim_{N\to\infty}\mathrm{Var}[g]=0.133/\beta \ .$$

We shall discuss this discrepancy later.

C. Eigenvalue density

In this and the next section we compare the transmission statistics predicted by the Laguerre ensemble with those obtained in a numerical simulation of the Hamiltonian (1). The transmission matrix is calculated using a standard Green's-function iteration technique described in Ref. 17. The eigenvalue densities for the LOE and LUE are evaluated using the formulas already given and the correlation functions with the aid of formulas given below.

In Fig. 2 we show the density $\sigma(\nu)$ obtained in a simulation of a long thin wire. The length of the wire is much longer than the mean free path l, but much shorter than the localization length so that the motion of the electron is diffusive along the direction of the wire.[22] Good quantitative agreement between the LOE and the microscopic model is obtained and the tendency to converge to a quarter circle [see Eq. (19)] in the asymptotic limit is

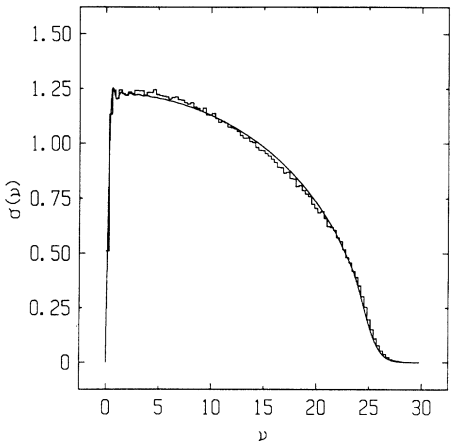

FIG. 2. The eigenvalue density of the matrix Ω obtained in a numerical simulation of model (1) of a quantum wire with parameters $L_x=6$, $L_y=6$, and $L_z=100$. The Fermi energy is at the band center $E=0$ and zero magnetic field. For these parameters $N=24$. The site energies ϵ_{xyz} are random with a rectangular distribution of $W=2$ and mean zero. The mean free path and localization length are estimated to be $l\sim 11{\cdot}5$ and $\xi\sim 280$, respectively. A comparison is made with the density of the Laguerre orthogonal ensemble for which the free parameters has been determined from the value of $\langle \mathrm{tr}\Omega^2\rangle$ obtained numerically.

clearly demonstrated. We find that this agreement persists provided that the length of the wire is not shorter than or comparable to the mean free path or longer than the localization length.[23] In addition differences between the LOE and the microscopic density also became apparent for sufficiently strong disorder.[24]

In Fig. 3 we show the density when a magnetic field is applied to the wire in the direction perpendicular to that of current flow. For a field strength large enough that several flux quanta penetrate the wire we expect that time-reversal symmetry is broken and that the LUE is the appropriate ensemble. The oscillations in the density ob-

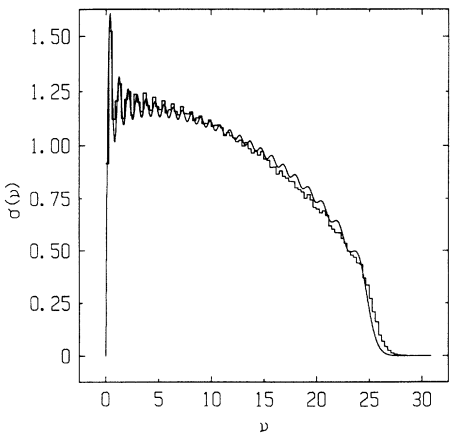

FIG. 3. The eigenvalue density for the quantum wire studied in Fig. 2, but in an applied magnetic field of $\alpha=\frac{1}{100}$. A comparison is made with the eigenvalue density of the LUE.

servable here in contrast to Fig. 2 are a consequence of the stronger eigenvalue repulsion expected for unitary symmetry (and the infinite value of N).

D. Eigenvalue correlations

We evaluate the correlation functions for the Laguerre ensemble using the following standard expressions: for the LOE

$$T_2(\nu,\nu')=4a^2\nu\nu' S(a\nu^2,a\nu'^2)S(a\nu'^2,a\nu^2) - D(a\nu^2,a\nu'^2)I(a\nu^2,a\nu'^2) ,$$

where $S(x,y)$ has already been defined and

$$D(x,y)=\frac{\partial}{\partial x}S(x,y) ,$$

$$I(x,y)=-\int_x^y S(x,z)dz-\epsilon(x-y) ,$$

$$\epsilon(x)=\begin{cases}\frac{1}{2}, & x>0\\ 0, & x=0\\ -\frac{1}{2}, & x<0 ,\end{cases}$$

and for the LUE

$$T_2(\nu,\nu')=16a^2\nu\nu' K(2a\nu^2,2a\nu'^2) .$$

In Fig. 4 we show some typical results for the diffusive quantum wire whose eigenvalue density was shown in Fig. 2. The expected linear short-range eigenvalue repulsion typical of systems with orthogonal symmetry is apparent and the LOE describes quite accurately the short-range eigenvalue correlations. In Fig. 5, after breaking time-reversal symmetry by applying a magnetic field, we find an equally satisfactory agreement with the LUE for the short-range eigenvalue correlations. The linear repulsion of the orthogonal symmetry is transformed into the quadratic repulsion typical of unitary symmetry.

Taken together the results for the correlation function

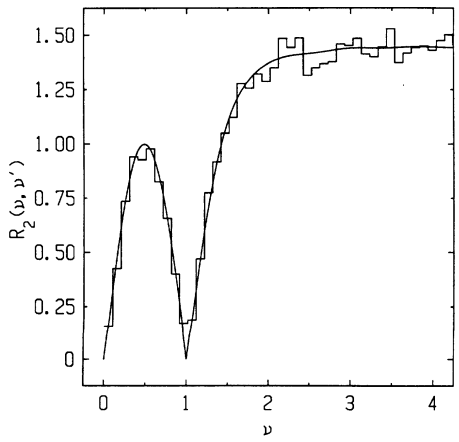

FIG. 4. The eigenvalue correlation function $R_2(\nu,\nu')$ for the quantum wire whose density is shown in Fig. 2. A comparison is made with the correlation function of the LOE. Here $\nu'=1$. Similar results are obtained for other values of ν'.

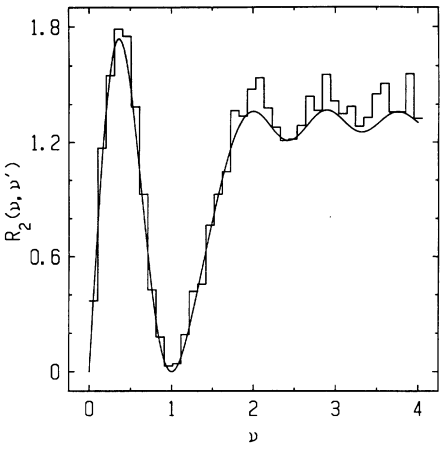

FIG. 5. The eigenvalue correlation function for the wire whose density is shown in Fig. 3. The comparison is with the correlation function of the LUE, as opposed to the LOE in Fig. 4.

and the conductance fluctuations seem to be somewhat contradictory. Looking at Figs. 4 and 5 it seems reasonable to expect not just qualitative but also quantitative agreement for the magnitude of the conductance fluctuations with microscopic theory. Why is this not the case? One possible explanation is that the behavior of the model (1) is different from that of the random white-noise potential studied in Ref. 1. To rule this out we present in Fig. 6 the conductance fluctuations obtained in numerical simulations of wires of various lengths. The most noticeable feature is a sharp increase (roughly a factor of 2 compared with a longer wire) in the magnitude of the conductance fluctuations for a short wire whose length is comparable to its diameter. This is exactly the behavior predicted in Ref. 1 for the random white-noise potential. Also shown are the magnitudes of the fluctuations predicted by the LOE. For long wires the LOE slightly

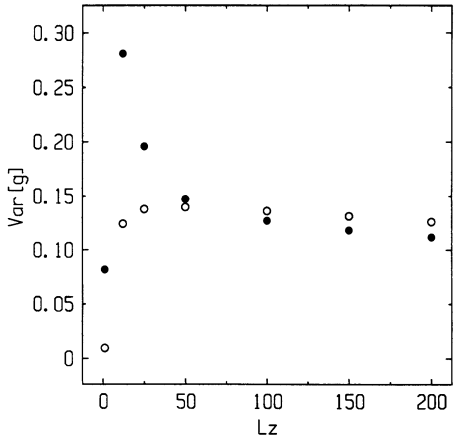

FIG. 6. The conductance fluctuations (solid circles) obtained in numerical simulations of quantum wires of various lengths and all other physical parameters as in Fig. 2. Also shown (empty circles) are the conductance fluctuations calculated for the LOE with $N=24$ and the parameter a determined using numerical values of $\langle \mathrm{tr}\Omega^2 \rangle$.

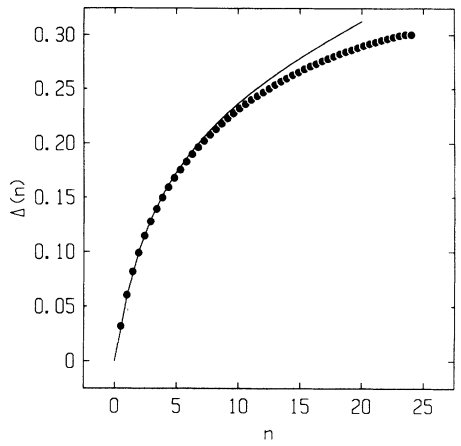

FIG. 7. The Δ statistics, defined in the text, for the wire studied in Fig. 2 (circles) and for the LOE (line).

overestimates the conductance fluctuations, in agreement with the asymptotic results in Sec. IV B. We believe the origin of the discrepancy to be the failure of the Laguerre ensemble to describe accurately enough the long-range eigenvalue correlations of the microscopic model. (Such a discrepancy would not be evident in Figs. 4 or 5 since it is only possible to probe the short-range eigenvalue correlations with the type of histogram plotted there.) To try to make this at least plausible we present in Fig. 7 some results for the so-called Δ statistic of random matrix theory

$$\Delta(n)=\left\langle \min_{A,B} \int_0^{\infty} dy\,[N(y)-Ay-B]^2 \right\rangle ,$$

where

$$y(v)=\int_0^{v} dv'\sigma(v')$$

and

$$N(y)=\sum_{m=1}^{N} \theta(y-y_m) .$$

The Δ statistic describes the tendency of the eigenvalues to deviate from a regular array or, more roughly speaking, the long-range rigidity of the eigenvalue spectrum. The value of the statistic at n is sensitive to the form of the eigenvalue corrections up to a range of n levels. The deviations visible at large n indicate that the spectrum of the wire is more rigid than predicted by the LOE. This is in agreement with the idea that the LOE fails to describe the long-range correlations exactly.

V. COMPARISON WITH OTHER MAXIMUM-ENTROPY THEORIES

There are two alternative maximum-entropy models to the Laguerre ensemble presented here. We shall discuss first the global maximum-entropy model. This is a maximum-entropy distribution on the group $(T,\times)$ formed by the transfer matrices T under multiplication. In the absence of time-reversal symmetry $(T,\times)$ is the Lie group $U(N,N)$ of $2N\times 2N$ pseudounitary matrices and if time reversal applies it is a subgroup of $U(N,N)$. Given a probability distribution on the group $P(T)=p(T)d\mu(T)$ the entropy $S[p(T)]$ is defined in the usual way.

$$S[p(T)]=-\int_{(T,\times)} d\mu(T)p(T)\ln p(T) ,$$

where $d\mu(T)$ is the invariant measure on $(T,\times)$. This may be derived as follows: taking the differential of (8) we have $dT^{\dagger}\Sigma_c T+T^{\dagger}\Sigma dT=0$. We define an anti-Hermitian infinitesimal matrix dA by $dA=dT^{\dagger}\Sigma_c T$. For unitary symmetry there are no further restrictions on dA and we may define a measure on $(T,\times)$ as a product over the $4N^2$ independent elements of dA by

$$d\mu(T)=\prod_{n>m} dA^{R}_{n,m}dA^{I}_{n,m}\prod_{n} dA_{n,n} .$$

Here R and I refer to real and imaginary parts. If we take any fixed $T_0\in U(N,N)$, we have

$$dA'=d(T_0T)^{\dagger}\Sigma_c(T_0T)dA=dT^{\dagger}T_0^{\dagger}\Sigma_c T_0T=dT^{\dagger}\Sigma_c T =dA ,$$

so that $d\mu(T)=d\mu(T_0T)\ \forall T\in U(N,N)$, which proves that this measure is the invariant measure of the group. It is convenient to rewrite the invariant measure in terms of the parameters occurring in the polar decomposition of T as

$$T=\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix}\begin{pmatrix} \sqrt{1+\lambda} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{1+\lambda} \end{pmatrix}\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix} .$$

Here u_1 and u_2 are the unitary matrices which appeared in (13), v_1 and v_2 are $N\times N$ unity matrices, and λ is an $N\times N$ real positive diagonal matrix with elements λ_n related to the ν_n by $\lambda_n=(\cosh\nu_n-1)/2$. After calculation of the required Jacobian we find

$$d\mu(T)=\prod_{n>m}|\lambda_n-\lambda_m|^2\prod_{n=1}^{N} d\lambda_n d\mu(u_1)d\mu(u_2) \times d\mu(v_1)d\mu(v_2) ,$$

where measures on u_1 and u_2 are the same as those in the corresponding expression for $d\mu(\Omega)$ and on v_1 and v_2 are the invariant measures on $U(N)$. A similar calculation is possible for orthogonal symmetry.

We now follow[8] and maximize the entropy on $(T,\times)$ subject to a fixed mean density $\sigma(\lambda)$ for the λ and obtain

$$P(\lambda)=C_{N,\beta}\prod_{n>m}|\lambda_n-\lambda_m|^{\beta}\prod_{n=1}^{N}\exp[-\beta V(\lambda_n)]d\lambda_n , \quad (22)$$

where $V(\lambda)$ is a Lagrange multiplier chosen to reproduce the density and with $\beta=2$ for unitary symmetry and $\beta=1$ for orthogonal symmetry. Since the eigenvalue density enters as the constraint, the theory makes no prediction concerning either the density or the weak-localization correction. Using the same argument as that for the Laguerre ensemble the asymptotic value of the conductance fluctuations is independent of $V(\lambda)$. Evaluating (21) we find

$$\lim_{N\to\infty}\mathrm{Var}[g]=0.125/\beta ,$$

which is different from both the microscopic theory and

the maximum-entropy theory on $(\Omega,+)$. The calculation of the correlation function requires a prior knowledge of the density $\sigma(\lambda)$ and is in practice only possible for unitary symmetry.[25] In previous work a comparison was made between the correlations predicted by the theory and those obtained in numerical simulations of quantum wires in magnetic fields, the numerically simulated density function $\sigma(\lambda)$ being used as the input to the theory. In general the quality of the agreement obtained is similar to that obtained with the maximum entropy on $(\Omega,+)$.

The maximum-entropy distributions on $(T,\times)$ and $(\Omega,+)$ make a common assumption about the statistical equivalence of different channels. This can be understood by noticing that the maximum-entropy distribution on $(T,\times)$ satisfies the following condition.

(iii) The probability distribution $P(T)$ is invariant under transformations of the form

$$P(T')=P(T),\quad T'=\begin{bmatrix} u & 0 \\ 0 & u' \end{bmatrix} T \begin{bmatrix} v & 0 \\ 0 & v' \end{bmatrix},$$

where u, u', v, and v' are arbitrary $N\times N$ unitary matrices.

Using the decomposition of T we can derive from this very similar conditions on the distributions of the transmission t and reflection r matrices: $P(t)=P(t')$, where $t'=utv$, and $P(r)=P(r')$, where $r'=urv$. These must hold for any arbitrary $N\times N$ unitary matrices u and v. Since condition (iii) on $P(T)$ implies (ii) on $P(\Omega)$ this is also a property of the maximum-entropy model on $(\Omega,+)$. This assumption is intuitively reasonable for long wires and is supported by numerical work.[10] However, the two models involve different assumptions about statistical correlations among the elements of Ω. The maximum-entropy model on $(\Omega,+)$ assumes that these are, apart from symmetry, statistically independent. The correlations implied by maximizing the entropy on $(T,\times)$ depend on the choice of $V(\lambda)$; however, there is no choice of $V(\lambda)$ which implies statistical independence of the elements of Ω.

It is interesting to ask if there is some simple assumption about the statistical independence of the elements of T which could reasonably be made in analogy with (i) for $P(\Omega)$. This would of course imply some particular form for $V(\lambda)$. This seems, however, to be difficult. The symmetry relations among the elements of Ω are particularly simple and there is an obvious subset, the elements of ω, which can be assumed to be statistically independent. Moreover, the product of the differentials of these elements is also the invariant measure on $(\Omega,+)$ so that the assumptions of statistical independence and invariance under transformations are equivalent to the maximum-entropy assumption. The symmetry relations among the elements of T are much more complicated and there is no obvious choice for statistically independent subset. It is for this reason that we defined an infinitesimal matrix dA in deriving the invariant measure on $(T,\times)$ rather than first parametrizing T and then using the standard formula in Appendix B.

The distribution (17) is sometimes presented[26] as an approximation to (22), the argument being as follows. We assume that all the $\nu_n \ll 1$ and make an expansion $\lambda_n=\nu_n^2/4+O(\nu_n^4)$. Making this approximation, changing variables in (22) and assuming that $V(\lambda)=a\lambda$, we obtain (17). Within the approximation that all the $\nu_n \ll 1$, this choice of V is equivalent to maximizing the entropy on $(T,\times)$ subject to a constraint on the ensemble average of the total reflection

$$\langle rr^\dagger\rangle=\left\langle \sum_{i=1}^{N}\lambda_n+O(\lambda_n^2)\right\rangle .$$

(This condition is used below to define a mean free path in the local maximum-entropy theory.) However, we do not see in this way that (17) has a more fundamental origin as a maximum-entropy distribution on the Lie group $(\Omega,+)$. In addition the necessary condition for the approximation $\nu_n \ll 1$ is not met unless the length of the conductor is much shorter than the mean free path.

A simple multiplicative composition law is often assumed for the transfer matrix so that the transfer matrix $T_{[A+B]}$ of a sample formed from two samples $[A]$ and $[B]$ connected in series is simply

$$T_{[A+B]}=T_{[A]}T_{[B]},$$

where $T_{[A]}$ and $T_{[B]}$ are the transfer matrices of samples $[A]$ and $[B]$. The group operation on $(T,\times)$ then has an obvious physical interpretation as the composition rule when two samples are placed in series whereas the group operation on $(\Omega,+)$ does not. It might therefore be argued that a maximum-entropy model on $(T,\times)$ is more physically meaningful. In defense we point out that the multiplicative combination law for transfer matrices holds only under special circumstances and not at all in general. Referring to Sec. II we see that the boundary conditions (2) and (3) are crucial in constructing a transfer matrix which refers only to the propagating states of the contacts, which are the only states which contribute to the conductance, and which at the same time conserves current. In general these conditions will not be satisfied at the junction of two samples connected in series and the multiplicative composition law will not hold. The only exception occurs when all the states of the contacts are propagating, which for model (1), for example, happens only rarely at very special values of the parameters. In general then, the group operation on $(T,\times)$ has no more physical significance than that on $(\Omega,+)$.

The local maximum-entropy theory is based on two assumptions. First the mesoscopic sample is imagined to be divided into sections of length ΔL. Then the transfer-matrix distribution for each section is assumed to maximize the entropy on $(T,\times)$ subject to the constraint that $\langle\sum_{i=1}^{N}\lambda_i\rangle=N\Delta L/l$, where the length of each section ΔL is assumed to be much less than the mean free path l. Under this condition the definition is equivalent to $\langle \mathrm{tr}rr^\dagger\rangle=N\Delta L/l$, so that $1/l$ is the probability of reflection per unit length per channel. Second a multiplicative composition law is assumed for the transfer matrices so that the transfer matrix of the whole sample is

then the product of those of the sections. From these two assumptions it is possible to develop an evolution equation, as a function of the length L of the wire, for the probability distribution of the transfer matrix. Though a maximum-entropy distribution for each slice is assumed, this property is not conserved when two transfer matrices are multiplied together so that the solution to the evolution equation is not the maximum-entropy distribution on $(T,\times)$. The solution must, however, satisfy condition (iii) since this condition is satisfied by the distributions for each slice and this property is not destroyed by multiplication. What then is the specific form of the distribution implied by these assumptions and does it resemble either of the maximum-entropy distributions on the two Lie groups? For orthogonal symmetry we do not know since no solution has so far been possible. For unitary symmetry a solution has recently been obtained which reduces to the following, in the diffusive regime $l \ll L \ll Nl$:

$$P(\nu_1,\ldots,\nu_N)=\prod_{n>m}|\lambda_n-\lambda_m||\nu_n^2-\nu_m^2|\prod_{n=1}^{N}\exp\left[-\frac{\nu_n^2 Nl}{4L}\right]\sqrt{\nu_n\sinh\nu_n}\,d\nu_n \ .$$

The eigenvalue interaction turns out to be the geometric mean of that arising from the invariant measures on $(T,\times)$ and $(\Omega,+)$. It seems that the action of multiplying transfer matrices together introduces correlations among the matrix elements of T which are somewhat intermediate between those of the maximum-entropy distributions on $(T,\times)$ and $(\Omega,+)$.

Before concluding we mention that the weak-localization effect and conductance fluctuations are different for short quantum wires. None of the maximum-entropy models we have discussed so far are able to describe the behavior in this regime and it is speculated that it is assumption (iii) [or equivalently (ii) for T] which is then questionable. This has prompted some efforts to develop a maximum-entropy model which does not incorporate this assumption.[27]

VI. CONCLUSION

A maximum-entropy model describing the conductance statistics of disordered quantum wires was presented in detail. A simple expression for the distribution of the matrix Ω, where $\Omega=\ln TT^{\dagger}$ and T is the transfer matrix of the conductor, was derived. This distribution was shown to be closely related to that of the Laguerre ensemble of random matrices. It was shown that a weak-localization effect and universal conductance fluctuations are predicted, which are in qualitative agreement with microscopic theory. Further we find that the model describes to a good approximation both the global (eigenvalue density) and local (eigenvalue correlations) spectral properties of Ω found in numerical studies of a tight-binding model of a quantum wire. It was shown that the maximum-entropy model is equivalent to the assumption that $P(\Omega)$ is invariant under certain transformations and that elements of Ω not related by symmetry are statistically independent.

Note added in proof. The asymptotic eigenvalue density and average conductance for the Laguerre ensemble has also been considered by Chen *et al.*, J. Phys. Condens. Matter **5**, 177 (1993).

APPENDIX A: THE LAGUERRE ENSEMBLE

The Laguerre ensemble is less familiar to physicists than the more well-known Gaussian and circular ensembles.[28] These later ensembles are appropriate for matrices with Hermitian symmetry such as Hamiltonians and matrices with unitary symmetry such as scattering matrices. Though Ω is Hermitian there is an additional symmetry constraint which arises as a consequence of current conservation so that the Gaussian ensembles are not appropriate. As shown in the main text there is, in the absence of time-reversal symmetry, an isomorphism between the group $(\Omega,+)$ of $2N\times 2N$ matrices and the group of $N\times N$ complex matrices under addition. The appropriate ensemble for matrices ω with this symmetry is the Laguerre unitary ensemble. If time-reversal symmetry applies, then the isomorphism is with the group of $N\times N$ complex symmetric matrices under addition and the relevant ensemble is the Laguerre orthogonal ensemble. In this appendix we give a precise definition of these ensembles.

The LUE of $N\times N$ complex matrices can be defined by the following two requirements.

(i) All the elements $\omega_{i,j}$ of ω are statistically independent (including real $\omega_{i,j}^{R}$ and imaginary $\omega_{i,j}^{I}$ parts of the same element).

(ii) The probability $P(\omega)=p(\omega)d\mu(\omega)$ that a member of the ensemble is in the volume $d\mu(\omega)$

$$d\mu(\omega)=\prod_{i,j}d\omega_{i,j}^{R}\prod_{i,j}d\omega_{i,j}^{I}$$

near ω is invariant under transformations of the form $P(\omega)=P(\omega')$, where $\omega'=U\omega V$ and U and V are arbitrary $N\times N$ unitary matrices.

Using arguments similar to that in Ref. 28 for the Gaussian unitary ensemble it is possible to show these two postulates restrict $p(\omega)$ to the generic form

$$p(\omega)=C\exp(-a\ \mathrm{tr}\omega\omega^{\dagger})$$

with a a real constant. In applications it is more useful to rewrite the probability density in terms of the parameters appearing in the singular value decomposition $\omega=u_1\sqrt{x}\,u_2^{\dagger}$. Here u_1 and u_2 are unitary matrices and x is real positive diagonal matrix whose elements are the eigenvalues $\{x_1,\ldots,x_N\}$ of the matrix $\omega\omega^{\dagger}$. We emit the full calculation (since it is almost identical to the calculation in Appendix B) and simply give the result

$$P(\omega)=C_{N,\beta}\prod_{n>m}|x_n-x_m|^{\beta}\times\prod_{n}e^{-\beta(x_n/2)}dx_n\,d\mu(u_1)\,d\mu(u_2) \ .$$

The measures on the unitary matrices are given in Appendix B. Integrating out both of them we obtain (18) with $\beta=2$.

Just as for the GUE the LUE can be derived from a maximum-entropy argument. The matrices ω form a group $(\omega,+)$ under matrix addition. We may define an entropy on the group

$$S[p(\omega)]=-\int_{(\omega,+)} d\mu(\omega)p(\omega)\ln p(\omega) .$$

Here the measure $d\mu(\omega)$ is the invariant measure of the group which (see Appendix B) has the form already given in (ii) above. If the entropy is now maximized subject to the constraint

$$\langle \mathrm{tr}\omega\omega^{\dagger}\rangle=\frac{N^2}{2a} ,$$

we arrive at the LUE distribution.

The LOE is an ensemble of $N\times N$ complex symmetric matrices satisfying the following conditions.

(i) All the elements $\omega_{i,j}$ of ω not related by symmetry are statistically independent (including real $\omega^R_{i,j}$ and imaginary $\omega^I_{i,j}$ parts of the same element).

(ii) The probability $P(\omega)=p(\omega)d\mu(\omega)$ that a member of the ensemble is in the volume $d\mu(\omega)$

$$d\mu(\omega)=\prod_{i\geq j} d\omega^R_{i,j}\prod_{i\geq j} d\omega^I_{i,j}$$

near ω is invariant under transformations of the form $P(\omega)=P(\omega')$, where $\omega'=U\omega U^T$ and U is an arbitrary $N\times N$ unitary matrix.

In an analogous manner to the discussion above for the LUE it is possible to derive from these two conditions the joint probability distribution (18) for the eigenvalues with $\beta=1$. The LOE may also be defined in an exactly analogous way as maximum-entropy ensemble on the group of $N\times N$ complex symmetric matrices under addition.

APPENDIX B: LIE GROUP PROPERTY AND THE INVARIANT MEASURE ON $(\Omega, +)$

We recall the definition of a Lie group. A Lie group G is a set of elements $\{g\}$ on which there is a well-defined binary composition law which satisfies, in addition to the usual conditions for a group, the following two conditions. First the elements of $G=\{g[a]\}$ must be completely parametrized by n real parameters $\{a_1,\ldots,a_n\}$ varying in a continuous range. Second the group composition law must be of the form $g[a]g[b]=g[c(a,b)]$ with c an analytic function of both a and b. These conditions are easily verified for $(\Omega, +)$.

The concept of the invariant measure arises in defining integrals on a Lie group. If $f(g)$ is a function and $d\mu$ some measure on the group, then the integral of f over the group is

$$I=\int_G d\mu(g)f(g) .$$

The measure $d\mu$ is said to be left invariant if $d\mu_L(g)=d\mu_L(g_0g)$ for all $g\in G$ and any fixed $g_0\in G$. This ensures that the value of integrals is independent of the choice of parametrization of the group elements. There is also a right invariant measure satisfying $d\mu_R(g)=d\mu_R(gg_0)$ for all g in the group. The left invariant measure is unique up to a multiplicative constant and can be shown to be equal to

$$d\mu_L(g)=\left[\det\left[\frac{\partial c_\alpha(a,b)}{\partial b_\beta}\right]\Bigg|_{b=0}\right]^{-1}\prod_n da_n .$$

Similarly the right invariant measure is also unique and a there is a corresponding expression. Since $(\Omega,+)$ is Abelian, the left and right measures are by definition identical and applying this formula we find immediately the expressions given in the main text.

As stated in the main text the invariant measures may be rewritten in terms of the eigenvalues of Ω. We take the case with ω symmetric first. The differential of the singular value decomposition (14) is

$$d\omega=du_1\nu u_1^T+u_1\nu du_1^T+u_1 d\nu u_1^T .$$

We define a matrix da by $da=u_1^{\dagger}du_1$. Since u is unitary, da is anti-Hermitian $da=-da^{\dagger}$. We use this to obtain the invariant measure $d\mu(u_1)$ on $U(N)$ the group of $N\times N$ unitary matrices under multiplication

$$d\mu(u_1)=\prod_i da^I_{i,i}\prod_{i>j} da^R_{i,j}\prod_{i>j} da^I_{i,j} .$$

The invariance within the unitary group is easily established from the definition of da. We now rewrite the differential in terms of da

$$d\omega=u_1(d\nu+da\nu-\nu da^*)u_1^T .$$

We will calculate the Jacobian of the outer transformation first. The transformation is of the form $d\omega'=ud\omega u^T$ with u unitary. We rewrite the transformation in terms of a direct product $d\omega'=[u\times u]d\omega$. The Jacobian of this transformation is clearly just the determinant of $u\times u$. Making use of the matrix identity $[a\times c][b\times d]=ab\times cd$ we can show that $[u\times u][u\times u]^{\dagger}=[u\times u][u^{\dagger}\times u^{\dagger}]=I$ and hence that the determinant of the transformation is unity $|\det[u\times u]|=1$. This means that we can ignore the outer transformation and consider only

$$d\omega=d\nu+da\nu-\nu da^* .$$

Equating real and imaginary parts we have

$$d\omega^R_{i,i}=d\nu_i , \quad d\omega^I_{i,i}=2\nu_i da^I_{i,i} ,$$

$$d\omega^R_{i,j}=(\nu_j-\nu_i)da^R_{i,j} , \quad d\omega^I_{i,j}=(\nu_j+\nu_i)da^I_{i,j} .$$

The Jacobian of this transformation is the determinant of a diagonal matrix and so we easily find

$$d\mu(\Omega)=C\prod_i \nu_i d\nu_i\prod_{i>j}|\nu_i^2-\nu_j^2|\mu(du_1) ,$$

where C is a constant.

For complex matrices the singular values decomposition (13) is not unique since it may be rewritten in many ways as $\omega=u_1e^{i\theta}\nu(u_2e^{i\theta})^{\dagger}$ where θ is any $N\times N$ real diagonal matrix. There is thus an N-fold phase degeneracy in the singular value decomposition. To obtain the invariant measure we first take the differential

$$d\omega = d(u_1 e^{i\theta}) v (u_2 e^{i\theta})^\dagger + (u_1 e^{i\theta}) dv (u_2 e^{i\theta})^\dagger + (u_1 e^{i\theta}) v d(u_2 e^{i\theta})^\dagger .$$

We define $da = (u_1 e^{i\theta})^\dagger d(u_1 e^{i\theta})$ and $db = (u_2 e^{i\theta})^\dagger d(u_2 e^{i\theta})$. As before da and db are anti-Hermitian. The differential can be rewritten in terms of these matrices as

$$d\omega = u_1 e^{i\theta} (da\, v + dv + v db^\dagger)(u_2 e^{i\theta})^\dagger .$$

Since ω is independent of θ we may make any convenient choice. We use this freedom to make the singular value decomposition unique by choosing the variation of the N real elements of θ in such a way that

$$db_{i,i} = e^{-i\theta_i} ([u_2^\dagger du_2]_{i,i} + i d\theta_i) e^{i\theta_i} = 0 ,$$

leaving only $N^2 - N$ free parameters in $(u_2 e^{i\theta})$. As before the outer transformation may be ignored and we need to consider only

$$d\omega = da\, v + dv - v db .$$

Equating real and imaginary parts ($j > i$)

$$d\omega^R_{i,i} = dv_i , \quad d\omega^I_{i,i} = v_i da^I_{i,i} ,$$
$$d\omega^R_{i,j} = da^R_{i,j} v_j - v_i db^R_{i,j} , \quad d\omega^I_{i,j} = da^I_{i,j} v_j - v_i db^I_{i,j} ,$$
$$d\omega^R_{j,i} = -da^R_{i,j} v_i + v_j db^R_{i,j} , \quad d\omega^I_{j,i} = da^I_{i,j} v_i - v_j db^I_{i,j} .$$

The Jacobian is a product of 2×2 determinants, which is easily calculated giving the result

$$d\mu(\Omega) = \prod_i v_i dv_i \prod_{i>j} |v_i^2 - v_j^2|^2 \mu(du_1)\mu(du_2) .$$

Here $\mu(du_1)$ is the invariant measure on the unitary group and

$$d\mu(u_2) = \prod_{i>j} db^R_{i,j} \prod_{i>j} db^I_{i,j} .$$

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