Response function of the fractional quantized Hall state on a sphere. I. Fermion Chern-Simons theory

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Using a well-known singular gauge transformation, certain fractional quantized Hall states can be modeled as integer quantized Hall states of transformed fermions interacting with a Chern-Simons field. In previous work we have calculated the electromagnetic response function of these states at arbitrary frequency and wave vector by using the random-phase approximation in combination with a Landau-Fermi-liquid approach. We now adopt these calculations to a spherical geometry in order to facilitate comparison with exact diagonalizations performed on finite-size systems.

I. INTRODUCTION

Calculational methods based on transforming electrons to fermions interacting with a Chern-Simons field have shown themselves to be very powerful for understanding fractional quantized Hall states and related unquantized states.[1–3] In a previous paper[1] we have developed a set of approximations for calculating the current and density response functions within this fermion Chern-Simons model at filling fractions corresponding to the principal fractional quantized Hall states. In particular, we use the random-phase approximation (RPA) to account for fluctuations in the Chern-Simons field, as well as to account for the direct Coulomb interaction. The large effective mass renormalization that we expect to occur in the Chern-Simons theory is then accounted for by using a Landau-Fermi-liquid theory approach. This set of approximations, which we call the "modified RPA," has a number of desirable features. To begin with, the f-sum rule, a result of Galilean invariance, is automatically satisfied. In addition, Kohn's theorem — which says that a mode at the bare cyclotron frequency must have all the weight of the f-sum rule in the long wavelength limit — is also satisfied within this approach. Finally, this approach predicts a discrete series of quasiexciton lines, with the lowest branch in qualitative agreement with exact diagonalization of finite-size systems.[1] More quantitative comparison of this theory with exact numerical work could not be made since the theory considers an infinite planar system, whereas exact diagonalizations necessarily consider only small systems. The purpose of the present work is to formulate an analogous fermion Chern-Simons theory on a sphere, so as to facilitate comparison with exact diagonalization results. A quantitative comparison between our results and the results of exact diagonalization of small spherical systems will be made in a following paper.[4]

The outline of this paper is as follows. In Sec. II A we begin by discussing the Dirac string and the Dirac quantization condition for a spherical system with a magnetic monopole in its center. We then write down the Hamiltonian that describes interacting electrons on the surface of this sphere in Sec. II B. In Sec. II C a singular gauge transformation is made that maps our system into a system of "composite fermions" — fermions bound to an even number of Chern-Simons flux quanta. We treat the Chern-Simons flux within mean-field theory in Sec. II D and review how certain fractional quantized Hall states can be viewed as integer quantized Hall states of these composite fermions as originally proposed by Jain.[5]

In Secs. III A and III B we define the current and density response function in terms of a convenient spherical basis. We make use here of some restrictions on the form of the response function which are derived in Appendixes A and B using rotational symmetry, gauge invariance, and current conservation. In Sec. III C we use standard linear response theory to calculate the response function within the mean-field approximation. This mean-field response also serves as a starting point for the RPA calculation. (The most tedious part of the mean-field calculation, the evaluation of a correlation function, is relegated to Appendixes C and D.) In Sec. IV A we use a self-consistent approximation to derive the RPA equation for the response function. (Certain angular momentum components of the Coulomb and Chern-Simons interactions are evaluated in Appendix E, and are used to establish the form of the induced effective electric and magnetic fields in Appendix F.) In Sec. IV B we discuss the issue of quasiparticle effective mass renormalization. In general, we expect that the RPA results will either not properly account for this mass renormalization, or will violate the f-sum rule. To repair this problem, we follow the results of Ref. 1 and give a modified RPA prescription that incorporates the effects of mass renormalization and satisfies the f-sum rule. Finally, in Sec. V we give a very brief summary of our work.

II. THE BASIC PROBLEM

A. The magnetic monopole

We begin by considering some of the features of a spherical system with a magnetic monopole in its center.

Since we want the magnetic field to be given by the curl of a vector potential, and the divergence of a curl is zero, a monopole must correspond to a vector potential that has a singularity. Specifically, a one-dimensional "Dirac string," which is essentially an infinitely thin flux tube carrying magnetic flux corresponding to the monopole's magnetic charge *into* the monopole, stretches from the monopole to spatial infinity.[6] The singular string can be moved, but not eliminated, by a gauge transformation. Formally, one should consider the vector potential of the monopole in two different gauges simultaneously such that the vector potential is well defined at each point in space in at least one of the gauges. In this way the vector potential can be considered a well-defined connection on a $U(1)$ bundle rather than a singular function.[7] Although this formal mathematical construction eliminates the singular Dirac string, it also complicates calculations. We will thus work in a singular gauge being careful to remember — when necessary — that we are really working with a more complicated mathematical object.

Another well-known property of the monopole system is the Dirac quantization condition[6] that tells us that the strength of the magnetic monopole is quantized in units of the magnetic flux quantum $\phi_0 = 2\pi/e$ where $-e$ is the charge on an electron; and, here and elsewhere in this paper, we have set $\hbar = c = 1$.

We will work in a gauge where the vector potential has no radial component. In spherical polar coordinates, one simple gauge choice is the spherical analog of Landau gauge where we choose the vector potential to point in the polar direction (i.e., in the $\hat{\phi}$ direction, directed around the $\hat{z}$ axis):

$$\mathbf{A} = \hat{\phi} A_\phi \ , \tag{1}$$

where

$$A_\phi(\theta,\phi) = \frac{S}{e\sin\theta}(1-\cos\theta). \tag{2}$$

This gauge places the singular Dirac string at the south pole of the sphere ($\theta = \pi$). It is easy to check that the magnetic field is given by $\mathbf{B} = \nabla\times\mathbf{A} = \frac{S}{e}\mathbf{\Omega}$ at all points $\mathbf{\Omega}$ on the unit sphere except at the south pole where the Dirac string cuts through the sphere. The Dirac quantization condition then dictates that $2S$ is an integer.

More generally, by rotating the above solution, we can construct a gauge where the Dirac string is at an arbitrary point $\mathbf{\Omega}'$ on the sphere. The vector potential $\mathbf{A}_{\mathbf{\Omega}'}$ for such a gauge is given by

$$\mathbf{A}_{\mathbf{\Omega}'}(\mathbf{\Omega}) = \frac{2S}{e}\frac{\mathbf{\Omega}\times\mathbf{\Omega}'}{|\mathbf{\Omega}-\mathbf{\Omega}'|^2}. \tag{3}$$

B. Defining the system

We consider a monopole of magnetic charge $2S$ flux quanta at the center of a unit sphere with N electrons of charge $-e$ restricted to the surface of the sphere. The points on the unit sphere are represented by unit vectors $\mathbf{\Omega}$ from the origin. The length scale of the system is determined by the magnetic length $l_0 = (eB)^{-(1/2)}$ which can be altered by changing the charge on the monopole. The Hamiltonian for this system can be written as

$$H = T + V \ , \tag{4}$$

where the kinetic energy is given by

$$T = \frac{1}{2m_b}\sum_{j=1}^{N}[\mathbf{p}_j + e\mathbf{A}(\mathbf{\Omega}_j)]^2 \ , \tag{5}$$

where $\mathbf{p}_j$ and $\mathbf{\Omega}_j$ are the momentum and position of the jth particle, $\mathbf{A}$ is the vector potential, and m_b is the band mass of the electron. The potential energy V is given by

$$V = \frac{1}{2}\sum_{i\neq j} v(\mathbf{\Omega}_i,\mathbf{\Omega}_j) \ , \tag{6}$$

where we will take the interaction to be a Coulombic $1/r$ potential with r the chord distance, i.e.,

$$v(\mathbf{\Omega},\mathbf{\Omega}') = \frac{e^2}{\epsilon}\frac{1}{|\mathbf{\Omega}-\mathbf{\Omega}'|} \ , \tag{7}$$

where ϵ is the dielectric constant of the medium (which may be allowed to change as a function of the magnetic length). The restriction of the system to a sphere is best represented by rewriting

$$T = \frac{1}{2m_b}\sum_{j=1}^{N}|\boldsymbol{\mathcal{P}}_j + e\boldsymbol{\mathcal{A}}(\mathbf{\Omega}_j)|^2 \ , \tag{8}$$

where

$$\boldsymbol{\mathcal{P}}_j = \mathbf{\Omega}\times\mathbf{p}_j \ , \tag{9}$$

$$\boldsymbol{\mathcal{A}} = \mathbf{\Omega}\times\mathbf{A} \ . \tag{10}$$

In terms of this "angular" vector potential, the radial component of the magnetic field $\mathbf{B}$ is given by

$$\mathbf{\Omega}\cdot\mathbf{B} = \mathbf{\Omega}\cdot(\nabla\times\mathbf{A}) = -\nabla\cdot\boldsymbol{\mathcal{A}}. \tag{11}$$

The tangential component of the magnetic field does not couple to our system of electrons confined to the sphere and can therefore be ignored.

It is convenient to rewrite the kinetic and potential energies in terms of the operator $\psi_e^\dagger(\mathbf{\Omega})$ that creates an electron at the point $\mathbf{\Omega}$. In this second quantized notation, we have

$$T = \frac{1}{2m_b}\int d\mathbf{\Omega}\,\psi_e^\dagger(\mathbf{\Omega})\ |\hat{\boldsymbol{\mathcal{P}}} + e\boldsymbol{\mathcal{A}}(\mathbf{\Omega})|^2\ \psi_e(\mathbf{\Omega}) \ , \tag{12}$$

$$V = \frac{1}{2}\int d\mathbf{\Omega}\int d\mathbf{\Omega}'\ v(\mathbf{\Omega},\mathbf{\Omega}')\ :n_e(\mathbf{\Omega})n_e(\mathbf{\Omega}'): \ , \tag{13}$$

where the colons represent normal ordering of the operators, and $\hat{\boldsymbol{\mathcal{P}}}$ is the canonical angular momentum operator

$$\hat{\boldsymbol{\mathcal{P}}} = \mathbf{\Omega}\times\hat{\mathbf{p}} = \mathbf{\Omega}\times -i\nabla, \tag{14}$$

and n_e is the density given by

$$n_e(\mathbf{\Omega}) = \psi_e^\dagger(\mathbf{\Omega})\psi_e(\mathbf{\Omega}). \tag{15}$$

C. Chern-Simons transformation

As described originally by Lopez and Fradkin,[3] a singular gauge transformation can be used to convert the electrons into composite fermions—electrons bound to an even number $\tilde{\phi} = 2m$ of Chern-Simons flux quanta, where m is an integer. Such a transformation provides a natural way of understanding many of the features of the fractional quantized Hall effect.[1–3,5] To make such a transformation, we define a vector valued Green's function on the sphere [see Eq. (3)] for two unit vectors $\mathbf{\Omega}$ and $\mathbf{\Omega}'$ which correspond to points on the sphere

$$\mathbf{g}(\mathbf{\Omega},\mathbf{\Omega}') = -\frac{\mathbf{\Omega}\times\mathbf{\Omega}'}{|\mathbf{\Omega}-\mathbf{\Omega}'|^2} + \frac{\mathbf{\Omega}\times\mathbf{S}}{|\mathbf{\Omega}-\mathbf{S}|^2} , \tag{16}$$

where $\mathbf{S}$ is the unit vector pointing toward the south pole, and the point $\mathbf{\Omega} = \mathbf{\Omega}'$ should be excluded from the Green's function. The function $\mathbf{g}$ is the vector potential associated with an infinitely thin magnetic flux tube carrying a single magnetic flux entering the sphere at the south pole, and leaving the sphere at the point $\mathbf{\Omega}'$. Note that just as in the case of the magnetic monopole, we cannot write a vector potential that represents a flux quantum entering the sphere but not leaving the sphere somewhere else. As in the case of the monopole, we have chosen the Dirac string to leave the sphere through the south pole. As we will see below, the Dirac string through the south pole is an artifact of the gauge choice and can essentially be ignored, whereas the flux quanta elsewhere through the sphere will have important physical effects.

Defining an auxiliary (singular and multivalued) function $f(\mathbf{\Omega},\mathbf{\Omega}')$ such that

$$\mathbf{g}(\mathbf{\Omega},\mathbf{\Omega}') = \nabla_{\mathbf{\Omega}} f(\mathbf{\Omega},\mathbf{\Omega}'), \tag{17}$$

we use the function f (which we will never actually need to evaluate) to define the quasiparticle creation operator

$$\psi^\dagger(\mathbf{\Omega}) = \psi_e^\dagger(\mathbf{\Omega}) \exp\left[-i\tilde{\phi}\int d\mathbf{\Omega}' f(\mathbf{\Omega},\mathbf{\Omega}') n_e(\mathbf{\Omega}')\right] , \tag{18}$$

which creates an electron bound to $\tilde{\phi}$ inward directed Chern-Simons flux quanta at the point $\mathbf{\Omega}$, as well as creating a Dirac string carrying the same flux out the south pole. Although the function f is multivalued, the exponential in Eq. (18) is well defined and single valued since $n_e(\mathbf{\Omega}')$ is a sum of δ functions, and $e^{-i\tilde{\phi}f}$ is single valued. As in the case of the simple magnetic monopole, the string carrying flux quanta out of the south pole is an artifact of our gauge choice and has no physical effects as long as the net flux through the string satisfies the Dirac quantization condition. One should note, however, that although the flux entering the sphere at the position of the quasiparticle looks very similar to the Dirac string, its position is a dynamical variable. We find that, unlike the Dirac string, these flux quanta do in fact have physical effects.

Since the density of electrons is equal to the density of quasiparticles, the density operator is given by

$$n_e(\mathbf{\Omega}) = \psi_e^\dagger(\mathbf{\Omega})\psi_e(\mathbf{\Omega}) = \psi^\dagger(\mathbf{\Omega})\psi(\mathbf{\Omega}), \tag{19}$$

and the equation for the potential energy (13) remains unchanged. On the other hand, in terms of the quasiparticle operators, the kinetic energy is now written as

$$T = \frac{1}{2m_b}\int d\mathbf{\Omega}\, \psi^\dagger(\mathbf{\Omega})\, |\hat{\boldsymbol{\mathcal{P}}} + e\boldsymbol{\mathcal{A}}(\mathbf{\Omega}) - \boldsymbol{\mathfrak{a}}(\mathbf{\Omega})|^2\, \psi(\mathbf{\Omega}) , \tag{20}$$

where

$$\boldsymbol{\mathfrak{a}}(\mathbf{\Omega}) = \mathbf{\Omega}\times\mathbf{a}(\mathbf{\Omega}) , \tag{21}$$

$$\mathbf{a}(\mathbf{\Omega}) = \tilde{\phi}\int d\mathbf{\Omega}'\mathbf{g}(\mathbf{\Omega},\mathbf{\Omega}') n_e(\mathbf{\Omega}') , \tag{22}$$

and $\mathbf{a}$ is the Chern-Simons vector potential.

D. Mean-field theory

We begin our analysis by considering a simple mean-field approach. The total magnetic flux through the sphere affecting a quasiparticle is an integer number $2S$ of quanta from the monopole charge minus $\tilde{\phi}(N-1) = 2m(N-1)$ Chern-Simons flux quanta bound to the *other* quasiparticles. (The quasiparticle is not affected by the flux bound to itself.) As discussed above, the flux through the Dirac string is static, and therefore should have no physical effects. Thus the flux through the string is not "seen" by the quasiparticle. We conclude that a given quasiparticle is affected by a total field corresponding to a monopole charge of

$$2\Delta S = 2S - 2m(N-1) \tag{23}$$

flux quanta. Within mean-field theory, the associated magnetic field is considered to be constant and uniform. The effective Hamiltonian for this mean-field system is then given by

$$H_0 = \frac{1}{2m_b}\int d\mathbf{\Omega}\, \psi^\dagger(\mathbf{\Omega})\, |\hat{\boldsymbol{\mathcal{P}}} + e\Delta\boldsymbol{\mathcal{A}}(\mathbf{\Omega})|^2\, \psi(\mathbf{\Omega}) , \tag{24}$$

where $\Delta\boldsymbol{\mathcal{A}}$ is the angular vector potential associated with the magnetic monopole charge of $2\Delta S$ quanta [see Eqs. (3), (10), and (23)], and we have clearly neglected the Coulomb interaction. This is simply the Hamiltonian for a system of noninteracting quasiparticles on the unit sphere in the field of a monopole of charge $2\Delta S$ flux quanta. Such a problem has been solved previously[7,8] and the eigenstates for such a system are given by the monopole spherical harmonics $Y_{lm}^{(\Delta S)}$ of Wu and Yang,[7] where l takes on the values ΔS, $\Delta S+1$, $\Delta S+2$, ..., and m takes on the values $-l, -l+1, \ldots, l-1, l$. The energies of these eigenstates are given by

$$\omega_l \equiv E(l) = \frac{1}{2m_b}[l(l+1) - (\Delta S)^2] \tag{25}$$

and thus the degeneracies are given by

$$\text{Degeneracy}(l) = 2l+1. \tag{26}$$

These l states should be thought of as analogous to the usual Landau levels on a plane. We expect that when the number of electrons (and thus the number of quasiparticles) is such that an integer number of these quasiparticle Landau levels are filled, we will have an integer quantized Hall state for the quasiparticles, and hence a stable fractional quantized Hall state for the original electron system.[3,5] This condition occurs for $N = 2\Delta S + 1, 4\Delta S + 4, \ldots$. The general formula is given by

$$N = 2p\Delta S + p^2 , \tag{27}$$

where p is a positive integer. Substituting in the definition (23) of ΔS, we find full quasiparticle Landau levels and hence stable quantized Hall states when

$$N = 2p[S - m(N-1)] + p^2. \tag{28}$$

For an infinitely large system (the $S \to \infty$ limit) this equation predicts quantized Hall states at the Jain filling fractions[5]

$$\nu(m,p) = \frac{N}{2S} = \frac{p}{2mp+1} , \tag{29}$$

which, along with their quasiparticle-quasihole conjugates, include all of the experimentally well-established fractional quantized Hall states within the first Landau level. On the finite spherical system, however, Eq. (28) predicts that the quantized Hall states occur when

$$2S(N,m,p) = [\nu(m,p)]^{-1}N - (2m+p). \tag{30}$$

The deviation $(2m + p)$ from the infinite system filling fraction is a finite-size effect which has been called the "shift" of the state.[9] The shifts predicted by this mean-field composite fermion model are in accordance with those predicted by Haldane's formula for the hierarchy of stable states on the sphere[10–12] and also agree with Jain's composite fermion results on the sphere.[5]

III. THE RESPONSE FUNCTION

A. Spherical basis

In order to exploit the rotational symmetry of this system, we will want to work in a basis that transforms in a simple way under rotation. Specifically, we will expand functions in angular momentum eigenstates. For scalar functions this can be done using the usual[13,14] spherical harmonics Y_{lm}. However, for vector valued functions, we will have to use a more complicated vector function basis. Furthermore, since our system is restricted to the surface of a sphere, we will want to work with a basis that naturally decouples the radial degree of freedom. We define a vector function basis[14] as

$$\mathbf{T}_{1lm} = \frac{r\nabla Y_{lm}}{\sqrt{l(l+1)}}, \quad l \geq 1 \tag{31}$$

$$\mathbf{T}_{2lm} = \frac{-\mathbf{r}\times\nabla Y_{lm}}{\sqrt{l(l+1)}}, \quad l \geq 1 \tag{32}$$

$$\mathbf{T}_{3lm} = \hat{\mathbf{r}} Y_{lm}, \quad l \geq 0 \tag{33}$$

where $\mathbf{r}$ is the radial vector of magnitude r and direction (unit vector) $\hat{\mathbf{r}}$. Note that the vector $\mathbf{T}_3$ is radial, whereas $\mathbf{T}_1$ and $\mathbf{T}_2$ are tangential to the sphere. Thus we will be able to completely decouple the $\mathbf{T}_3$ direction from our problem.

This $\mathbf{T}$ basis is orthonormal and complete on the space of vector valued functions on the surface of a sphere. That is to say,

$$\int d\mathbf{\Omega}\, [\mathbf{T}_{ilm}(\mathbf{\Omega})]^* \cdot [\mathbf{T}_{jkn}(\mathbf{\Omega})] = \delta_{ij}\delta_{lk}\delta_{mn} , \tag{34}$$

and given a vector valued function $\mathbf{f}(\mathbf{\Omega})$, we have

$$\mathbf{f}(\mathbf{\Omega}) = \sum_{ilm} f_{ilm}\mathbf{T}_{ilm}(\mathbf{\Omega}), \tag{35}$$

where

$$f_{ilm} = \int d\mathbf{\Omega}\, [\mathbf{T}_{ilm}(\mathbf{\Omega})]^* \cdot \mathbf{f}(\mathbf{\Omega}). \tag{36}$$

Sometimes we will find it more convenient to work with "four-vectors." To this end, we define a unit vector $\hat{\mathbf{e}}_w$ which is orthogonal to all three elements of the usual three-dimensional Cartesian basis $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$, where the metric is always taken to be Euclidean. Any four-vector can then be expressed in terms of the Cartesian basis and the unit vector $\hat{\mathbf{e}}_w$. In order to represent four-vector valued functions on the sphere, we define a 0th component of the $\mathbf{T}$ basis as

$$\mathbf{T}_{0lm}(\mathbf{\Omega}) = Y_{lm}(\mathbf{\Omega})\hat{\mathbf{e}}_w , \tag{37}$$

which, combined with the other three $\mathbf{T}$ basis vectors, form an orthonormal and complete basis set for functions on the surface of the sphere with values in four-dimensional space. We will use the Greek indices μ and ν exclusively for such four-vectors, while the Latin indices $i, j, \ldots$ will be used for the indices of a three-vector.

B. Definition of the response function

We are concerned with finding the response of our system to electromagnetic perturbations. The angular current operator is defined by

$$\mathbf{\Gamma}^{\boldsymbol{\mathcal{A}}}(\mathbf{\Omega}) = \frac{1}{e}\frac{\delta H}{\delta \boldsymbol{\mathcal{A}}(\mathbf{\Omega})} = \frac{1}{m_b}\psi_e^\dagger(\mathbf{\Omega})\left[\hat{\boldsymbol{\mathcal{P}}} + e\boldsymbol{\mathcal{A}}\right]\psi_e(\mathbf{\Omega}) . \tag{38}$$

Sometimes we will leave out the superscript $\boldsymbol{\mathcal{A}}$ when it is clear what the vector potential is in the system. Note that there is no radial component of the angular current. We would like to think of the fluctuation in the density as the 0th element of this vector, thus forming a four-vector. We define (with $\boldsymbol{\mathcal{A}}$ a dummy index here)

$$\Gamma_0^{\boldsymbol{\mathcal{A}}}(\mathbf{\Omega}) = n_e(\mathbf{\Omega}) - \bar{n}_e = \psi_e^\dagger(\mathbf{\Omega})\psi_e(\mathbf{\Omega}) - \bar{n}_e , \tag{39}$$

where

$$\bar{n}_e = \frac{N}{4\pi} . \tag{40}$$

Similarly, we would like to think of the scalar potential

$-\Phi$ as the 0th element of the (angular) vector potential [see Eq. (10)]

$$\mathcal{A}_0 = -\Phi. \tag{41}$$

If we now subject the system to an external time dependent electromagnetic perturbation $\mathcal{A}_\mu^{\text{ext}}$ the resulting linear response in the angular current and density can be given in terms of a response function $K_{\alpha\beta}(\mathbf{\Omega}, t; \mathbf{\Omega}', t')$ by the equation

$$\langle \Gamma_\alpha^{\mathcal{A}}(\mathbf{\Omega}, t) \rangle = e \int_{-\infty}^{\infty} dt' \int d\mathbf{\Omega}' \; K_{\alpha\beta}(\mathbf{\Omega}, t; \mathbf{\Omega}', t') \times \mathcal{A}_\beta^{\text{ext}}(\mathbf{\Omega}', t'), \tag{42}$$

where the indices α and β can take on the value 0 (corresponding to the density and the scalar potential) or 1, 2, 3 corresponding to the vector components of the current or vector potential for some orthonormal vector basis. Note that here $\mathcal{A}$ is the sum of the background vector potential from the monopole and the perturbation $\mathcal{A}_\mu^{\text{ext}}$.

We now use our $\mathbf{T}$ vector function basis to expand the quantities in this equation:

$$\Gamma_{\mu l m}^{\mathcal{A}}(\omega) = \int dt\, e^{i\omega t} \int d\mathbf{\Omega}\, [\mathbf{T}_{\mu l m}(\mathbf{\Omega})]^* \cdot \mathbf{\Gamma}(\mathbf{\Omega}) , \tag{43}$$

$$\mathcal{A}_{\mu l m}^{\text{ext}}(\omega) = \int dt\, e^{i\omega t} \int d\mathbf{\Omega}\, [\mathbf{T}_{\mu l m}(\mathbf{\Omega})]^* \cdot \boldsymbol{\mathcal{A}}^{\text{ext}}(\mathbf{\Omega}) , \tag{44}$$

where $\mathbf{T}, \mathbf{\Gamma}$, and $\boldsymbol{\mathcal{A}}$ are considered four-vectors in these equations. Note that $\Gamma_3 = \mathcal{A}_3 = 0$, since $\mathbf{\Gamma}$ and $\boldsymbol{\mathcal{A}}$ are tangential vectors and $\mathbf{T}_3$ is radial. Similarly, we can define

$$K_{\mu l m;\nu l' m'}(\omega) = \int d(t-t') e^{i\omega(t-t')} \int d\mathbf{\Omega} \int d\mathbf{\Omega}' [\mathbf{T}_{\mu l m}(\mathbf{\Omega})]^* \cdot K_{(\alpha,\beta)}(\mathbf{\Omega}, \mathbf{\Omega}', t-t') \cdot [\mathbf{T}_{\nu l' m'}(\mathbf{\Omega}')] , \tag{45}$$

where the first dot product is with respect to the α index of K and the second dot product is with respect to the β index of K. In this equation we have explicitly used the time translational invariance of the response function (i.e., that K is a function of $t - t'$ only). In terms of these transformed quantities, the response equation (42) is written in the simple form

$$\langle \Gamma_{\mu l m}^{\mathcal{A}}(\omega) \rangle = e \sum_{\nu, l', m'} K_{\mu l m;\nu l' m'}(\omega) \mathcal{A}_{\nu l' m'}^{\text{ext}}(\omega). \tag{46}$$

The advantage of having made this transformation is that the spherical symmetry of the system is now reflected in the simplifying relationship (proven in Appendix A)

$$K_{\mu l m;\nu l' m'}(\omega) = \delta_{l l'} \delta_{m m'} K_{\mu\nu}(l, \omega) \tag{47}$$

for $m = -l, -l+1, \ldots, l-1, l$. In Appendix B we use gauge invariance and current conservation to derive further restrictions on the form of the response function $K_{\mu\nu}(l, \omega)$.

C. Bare mean-field response

As a starting point, we will need to calculate the "bare" electromagnetic response function $K_{\mu\nu}^0$ for the mean-field system defined by the Hamiltonian H_0 given in Eq. (24). This is simply the problem of finding the response function for a system of N noninteracting quasiparticles of charge $-e$ on a unit sphere around a monopole of charge $2\Delta S$ which creates a vector potential $\Delta\boldsymbol{\mathcal{A}}$ such that an integer number of quasiparticle Landau levels are filled. If we add an external perturbing (angular) vector potential $\boldsymbol{\mathcal{A}}^{\text{ext}}$, the perturbation Hamiltonian is given by

$$H_0^{\text{ext}} = \int d\mathbf{\Omega} \left[e\mathbf{\Gamma}^{\Delta\mathcal{A}}(\mathbf{\Omega}) \cdot \boldsymbol{\mathcal{A}}^{\text{ext}}(\mathbf{\Omega}) + \frac{e^2}{2m_b} n_e(\mathbf{\Omega}) \boldsymbol{\mathcal{A}}^{\text{ext}}(\mathbf{\Omega}) \cdot \boldsymbol{\mathcal{A}}^{\text{ext}}(\mathbf{\Omega}) \right] , \tag{48}$$

where the dot product in the first term (the paramagnetic term) is a four-vector dot product whereas the dot product in the second term (the diamagnetic term) is a three-vector dot product. One must now include the contribution of the perturbation Hamiltonian in the full physical current:

$$\mathbf{\Gamma}^{\Delta\mathcal{A}+\mathcal{A}^{\text{ext}}}(\mathbf{\Omega}) = \frac{1}{e} \frac{\delta(H_0 + H_0^{\text{ext}})}{\delta\boldsymbol{\mathcal{A}}(\mathbf{\Omega})} = \mathbf{\Gamma}^{\Delta\mathcal{A}} + \frac{e}{m_b} n_e(\mathbf{\Omega}) \boldsymbol{\mathcal{A}}^{\text{ext}}(\mathbf{\Omega}). \tag{49}$$

Using standard linear response theory,[15] we now have

$$\langle \Gamma_\alpha^{\Delta\mathcal{A}}(\mathbf{\Omega}, t) \rangle_{H_0 + H_0^{\text{ext}}} = \langle \Gamma_\alpha^{\Delta\mathcal{A}}(\mathbf{\Omega}, t) \rangle_{H_0} - i \int_{-\infty}^{t} dt' \int d\mathbf{\Omega}' \times \left\langle \left[\Gamma_\alpha^{\Delta\mathcal{A}}(\mathbf{\Omega}, t), H_0^{\text{ext}} \right] \right\rangle . \tag{50}$$

In the unperturbed state, there is no current, so the first term on the right hand side of this equation vanishes. Furthermore, since we are working to linear order, the commutator of the diamagnetic term of the perturbation Hamiltonian is neglected. Thus we rewrite this equation as

$$\left\langle \Gamma_\alpha^{\Delta\mathcal{A}+\mathcal{A}^{\text{ext}}}(\mathbf{\Omega}, t) \right\rangle = \frac{e\bar{n}_e}{m_b} [1 - \delta_{\alpha 0}] \mathcal{A}_\alpha^{\text{ext}}(\mathbf{\Omega}, t) \tag{51}$$

$$- ie \int_{-\infty}^{t} dt' \int d\mathbf{\Omega}' \langle [\Gamma_\alpha^{\Delta\mathcal{A}}(\mathbf{\Omega}, t), \mathbf{\Gamma}^{\Delta\mathcal{A}}(\mathbf{\Omega}', t')] \rangle \cdot \boldsymbol{\mathcal{A}}^{\text{ext}}(\mathbf{\Omega}', t') , \tag{52}$$

where we have used Eq. (49) to obtain the first term on the right hand side. Thus the response function is written as the sum of diamagnetic and paramagnetic parts as

$$K^0_{\alpha\beta}(\mathbf{\Omega},t;\mathbf{\Omega}',t') = \frac{e\bar{n}_e}{m_b}\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(t-t')[1-\delta_{\beta 0}]\delta_{\alpha\beta} + D^0_{\alpha\beta}(\mathbf{\Omega},t;\mathbf{\Omega}',t') \ , \tag{53}$$

where the superscript 0 indicates that it is a mean-field quantity, any diamagnetic current in the radial direction is projected out, and the paramagnetic piece is just the retarded current correlation function given by

$$D^0_{\alpha\beta}(\mathbf{\Omega},t;\mathbf{\Omega}',t') = -i\theta(t-t')\left\langle\left[\Gamma^{\Delta\mathcal{A}}_\alpha(\mathbf{\Omega},t),\Gamma^{\Delta\mathcal{A}}_\beta(\mathbf{\Omega}',t')\right]\right\rangle \tag{54}$$

In terms of angular momentum and frequency components, we have

$$K^0_{\mu lm;\nu l'm'}(\omega) = D^0_{\mu lm;\nu l'm'}(\omega) + \frac{e\bar{n}_e}{m_b}\delta_{\mu\nu}\delta_{ll'}\delta_{mm'}[1-\delta_{\mu 0}] \ , \tag{55}$$

where

$$D^0_{\mu lm;\nu l'm'}(\omega) = \int d(t-t')\int d\mathbf{\Omega}\int d\mathbf{\Omega}' e^{i\omega(t-t')}[\mathbf{T}_{\mu lm}(\mathbf{\Omega})]^* \cdot D^0_{(\alpha\beta)}(\mathbf{\Omega},\mathbf{\Omega}',t-t')\cdot[\mathbf{T}_{\nu l'm'}(\mathbf{\Omega}')], \tag{56}$$

where the first dot product is with respect to α and the second dot product is with respect to β, and all the vectors are four-vectors. Once again, we expect that the spherical symmetry (see Appendix A) will allow us to write the retarded correlation function as

$$D^0_{\mu lm;\nu l'm'}(\omega) = \delta_{ll'}\delta_{mm'}D^0_{\mu\nu}(l,\omega) \tag{57}$$

so that the bare response function may be written as

$$K^0_{\mu\nu}(l,\omega) = D^0_{\mu\nu}(l,\omega) + \frac{e\bar{n}_e}{m_b}\delta_{\mu\nu}[1-\delta_{\mu 0}] \ . \tag{58}$$

The actual calculation of the retarded correlation function D^0 is relegated to Appendix C.

IV. THE RPA

A. Self-consistent RPA

The RPA on a plane can be defined by saying that the quasiparticles respond via the bare response function K^0 to an effective scalar and vector potential $A^{\rm eff}_\mu$ given by

$$A^{\rm eff}_\mu = A^{\rm ext}_\mu + A^{\rm in}_\mu \ , \tag{59}$$

where $A^{\rm in}_\mu$ is an induced vector potential which includes a contribution from a self-consistently calculated Coulomb potential due to the induced variation of the electron density and a contribution arising from the self-consistent Chern-Simons magnetic and electric fields given by[3]

$$\mathbf{b}^{\rm in}(\mathbf{r},t) = 2\pi\tilde{\phi}\left[\langle n_e(\mathbf{\Omega},t)\rangle - \bar{n}_e\right]\hat{\mathbf{z}} \ , \tag{60}$$

$$\mathbf{e}^{\rm in}(\mathbf{r},t) = 2\pi\tilde{\phi}\,\hat{\mathbf{z}}\times\langle\mathbf{J}\rangle \ , \tag{61}$$

where $\langle n_e\rangle$ and $\langle\mathbf{J}\rangle$ are the density and current calculated self-consistently in the RPA. For the present case of N electrons on a sphere, we write the analogous expressions

$$\begin{aligned}\mathbf{b}^{\rm in}(\mathbf{\Omega},t) &= \left(\tfrac{N-1}{N}\right)2\pi\tilde{\phi}\,\langle\Gamma_0(\mathbf{\Omega},t)\rangle\,\mathbf{\Omega}\\ &= \left(\tfrac{N-1}{N}\right)2\pi\tilde{\phi}\left[\langle n_e(\mathbf{\Omega},t)\rangle-\bar{n}_e\right]\mathbf{\Omega} \ ,\end{aligned} \tag{62}$$

$$\mathbf{e}^{\rm in}(\mathbf{\Omega},t) = \left(\tfrac{N-1}{N}\right)2\pi\tilde{\phi}\,\langle\mathbf{\Gamma}(\mathbf{\Omega},t)\rangle \ , \tag{63}$$

where $\langle\Gamma_0\rangle$ and $\langle\mathbf{\Gamma}\rangle$ are the induced density fluctuation and angular current calculated self-consistently in the RPA. The factor of $\frac{N-1}{N}$ is introduced here to correct for the fact that the quasiparticle is not affected by its own field. This factor disappears in the limit of $N\to\infty$, but is clearly necessary if we wish to recover the correct response function for a single electron when $N=1$. Similarly, the induced Coulomb potential is taken to be that arising from a charge density

$$\begin{aligned}\delta n^{\rm in}_e(\mathbf{\Omega},t) &= \left(\tfrac{N-1}{N}\right)\langle\Gamma_0(\mathbf{\Omega},t)\rangle\\ &= \left(\tfrac{N-1}{N}\right)\left[\langle n_e(\mathbf{\Omega},t)\rangle-\bar{n}_e\right]\end{aligned} \tag{64}$$

so that an induced electric field $\mathbf{E}^{\rm in}$ can be defined whose three-dimensional divergence satisfies

$$4\pi\epsilon\nabla\cdot\mathbf{E}^{\rm in}(\mathbf{r}) = \delta n^{\rm in}_e(\mathbf{r}/|\mathbf{r}|)\,\delta(|\mathbf{r}|-1). \tag{65}$$

The required induced Chern-Simons fields and Coulomb potential can be obtained on the sphere from an induced angular vector potential $\boldsymbol{\mathcal{A}}^{\rm in} = \mathbf{\Omega}\times\mathbf{A}^{\rm in}$ and scalar potential $\mathcal{A}^{\rm in}_0 = A^{\rm in}_0$ whose angular momentum components are given by

$$e\mathcal{A}^{\rm in}_{\mu lm} = \left(\tfrac{N-1}{N}\right)\sum_\nu U_{\mu\nu}(l)\,\langle\Gamma_{\nu lm}\rangle \ , \tag{66}$$

where

$$U_{\mu\nu}(l) = v(l)\delta_{\mu 0}\delta_{\nu 0} + w(l)\left[\delta_{\mu 1}\delta_{\nu 0}+\delta_{\nu 1}\delta_{\mu 0}\right] \tag{67}$$

and the coefficients v and w are given by

$$v(l) = \frac{e^2}{\epsilon}\frac{4\pi}{2l+1} \ , \tag{68}$$

$$w(l) = \tilde{\phi}\frac{2\pi}{\sqrt{l(l+1)}} \ . \tag{69}$$

To derive Eqs. (66) and (67) we calculate the scalar potential and angular vector potential created by the N quasiparticles (charge $-e$ particles attached to $\tilde{\phi}$ flux quanta) when their density and current is given by $\langle\Gamma_\mu\rangle$. These potentials are then multiplied by the above mentioned factor of $\frac{N-1}{N}$ to account for the fact that a quasiparticle is not affected by its own field. With this prescription, the Coulomb contribution to the scalar potential from induced charge fluctuation is given by

$$e\mathcal{A}_0^{\text{in}}(\mathbf{\Omega})_{\text{Coulomb}} = \left(\tfrac{N-1}{N}\right)\int d\mathbf{\Omega}'\, v(\mathbf{\Omega},\mathbf{\Omega}')\langle\Gamma_0(\mathbf{\Omega}')\rangle \ , \tag{70}$$

whereas the scalar potential from the motion of the Chern-Simons flux quanta is given by

$$\mathcal{A}_0^{\text{in}}(\mathbf{\Omega})_{\text{Chern-Simons}} = \left(\tfrac{N-1}{N}\right)\int d\mathbf{\Omega}'\, \mathbf{W}(\mathbf{\Omega},\mathbf{\Omega}')\cdot\langle\mathbf{\Gamma}(\mathbf{\Omega}')\rangle \tag{71}$$

so that

$$\mathcal{A}_0^{\text{in}}(\mathbf{\Omega}) = \mathcal{A}_0^{\text{in}}(\mathbf{\Omega})_{\text{Coulomb}} + \mathcal{A}_0^{\text{in}}(\mathbf{\Omega})_{\text{Chern-Simons}} \tag{72}$$

and the angular vector potential from the Chern-Simons flux quanta is given by

$$\boldsymbol{\mathcal{A}}^{\text{in}}(\mathbf{\Omega}) = \left(\tfrac{N-1}{N}\right)\tilde{\phi}\int d\mathbf{\Omega}'\, \mathbf{W}(\mathbf{\Omega},\mathbf{\Omega}')\langle\Gamma_0(\mathbf{\Omega}')\rangle \ , \tag{73}$$

where

$$\mathbf{W}(\mathbf{\Omega},\mathbf{\Omega}') = -\frac{\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{\Omega}')}{|\mathbf{\Omega}-\mathbf{\Omega}'|^2} \tag{74}$$

is the flux quanta Green's function [see Eq. (16)]. Finally, we use Eqs. (E3) and (E14) to evaluate the angular momentum components of v and $\mathbf{W}$, and thus establish the given form [Eqs. (66) and (67)] of the induced vector potential.

We can verify that this prescription [described by Eq. (66)] does in fact correspond to the induced Chern-Simons electric and magnetic fields given by Eqs. (62) and (63) as well as the induced Coulombic electric field described by Eq. (65) by reconstructing the induced vector potential

$$A_0^{\text{in}} = \mathcal{A}_0^{\text{in}} = \sum_{lm}\mathcal{A}_{0lm}^{\text{in}}\mathbf{T}_{0lm} \ , \tag{75}$$

$$\mathbf{A}^{\text{in}} = -\mathbf{\Omega}\times\boldsymbol{\mathcal{A}}^{\text{in}} = \sum_{ilm}\mathcal{A}_{ilm}^{\text{in}}\left(-\mathbf{\Omega}\times\mathbf{T}_{ilm}\right) \ , \tag{76}$$

and then differentiating to find

$$\nabla A_0^{\text{in}} - \frac{\partial}{\partial t}\mathbf{A}^{\text{in}} = \mathbf{E}^{\text{in}} + \mathbf{e}^{\text{in}} \ , \tag{77}$$

$$\mathbf{\Omega}\cdot(\nabla\times\mathbf{A}^{\text{in}}) = \mathbf{b}^{\text{in}} \ . \tag{78}$$

These equalities are demonstrated explicitly in Appendix F.

Once we have derived Eqs. (66) and (67) we can make the RPA approximation that the quasiparticles respond via the bare response to the effective angular vector and scalar potential

$$\langle\Gamma_{\mu lm}\rangle = e\sum_\nu K_{\mu\nu}^0(l)\mathcal{A}_{\nu lm}^{\text{eff}}, \tag{79}$$

where the effective angular vector and scalar potential is given by the sum of external and induced contributions

$$\mathcal{A}_\mu^{\text{eff}} = \mathcal{A}_\mu^{\text{ext}} + \mathcal{A}_\mu^{\text{in}} \ . \tag{80}$$

On the other hand, the full response is defined via

$$\langle\Gamma_{\mu lm}\rangle = e\sum_\nu K_{\mu\nu}(l)\mathcal{A}_{\nu lm}^{\text{ext}}. \tag{81}$$

Using Eq. (66) and eliminating $\langle\mathbf{\Gamma}\rangle$ we solve to find the RPA result

$$K(l) = K^0(l)\left[1-\left(\tfrac{N-1}{N}\right)U(l)K^0(l)\right]^{-1} \ , \tag{82}$$

where all of these quantities are 3×3 matrices ($\mu,\nu = 0,1,2$).

Although this self-consistent calculation has been performed in terms of the response function K and the full physical current $\mathbf{\Gamma}$ it is also possible[16] to perform a self-consistent calculation in terms of the retarded correlation function D and the mean-field current $\mathbf{\Gamma}^{\Delta\mathcal{A}}$. The result of such an approximation must then be corrected to account for the fluctuations in the diamagnetic current. However, once this correction has been properly taken into account to find the response K, the end result will be exactly the same as the RPA equation we have given here.

At this point we note that there are restrictions on the response function from gauge invariance [Eq. (B7)] and current conservation [Eq. (B15)] derived in Appendix B. With these restrictions, along with the manifest symmetries of the response matrix, there are only three independent entries in the 3×3 response matrix, and we can write

$$K(l,\omega) = \begin{pmatrix} K_{00}(l,\omega) & K_{01}(l,\omega) & \frac{-i\omega}{\sqrt{l(l+1)}}K_{00}(l,\omega) \\ K_{01}(l,\omega) & K_{11}(l,\omega) & \frac{-i\omega}{\sqrt{l(l+1)}}K_{01}(l,\omega) \\ \frac{i\omega}{\sqrt{l(l+1)}}K_{00}(l,\omega) & \frac{i\omega}{\sqrt{l(l+1)}}K_{01}(l,\omega) & \frac{\omega^2}{l(l+1)}K_{00}(l,\omega) \end{pmatrix} . \tag{83}$$

Furthermore, the Hermitian interaction matrix U, considered as a 3×3 matrix, only has entries in the (0,0), (0,1), and (1,0) places. If we choose the gauge of our external magnetic field so that $\mathcal{A}^{\rm ext}_{2lm} = 0$, we can eliminate redundancies and reduce the RPA to a 2×2 matrix equation. In this gauge, the RPA equation (82) remains the same, but now the matrices are 2×2 and are given by

$$K(l,\omega) = \begin{pmatrix} K_{00}(l,\omega) & K_{01}(l,\omega) \\ K_{01}(l,\omega) & K_{11}(l,\omega) \end{pmatrix} \quad (84)$$

and

$$U(l) = V(l) + W(l) \ , \quad (85)$$

where

$$V(l) = \begin{pmatrix} v(l) & 0 \\ 0 & 0 \end{pmatrix} \ , \quad (86)$$

$$W(l) = \begin{pmatrix} 0 & w(l) \\ w(l) & 0 \end{pmatrix} \ . \quad (87)$$

It is sometimes convenient to think in terms of the conductivity rather than the electromagnetic response.[2,1] The conductivity σ is defined as the response to the total electromagnetic field $\mathcal{A}_{\mu lm}$ whereas the electromagnetic response K is the response to the external electromagnetic field $\mathcal{A}^{\rm ext}_{\mu lm}$. The magnetic field generated by the quantum Hall system is small, so there is essentially no difference between $\boldsymbol{\mathcal{A}}_{lm}$ and $\boldsymbol{\mathcal{A}}^{\rm ext}_{lm}$. On the other hand, the scalar potentials $\mathcal{A}_{0lm}$ and $\mathcal{A}^{\rm ext}_{0lm}$ differ by the Coulomb potential $v(l)\Gamma_{0lm}$ generated by density fluctuations. Thus we define a 2×2 matrix $\Pi(l,\omega)$ which is more closely related to the conductivity to be the electromagnetic response without the Coulomb contribution:

$$K^{-1}(l,\omega) = \Pi^{-1}(l,\omega) - (\tfrac{N-1}{N})V(l,\omega). \quad (88)$$

In other words, Π is the sum of all diagrams that are irreducible with respect to the Coulomb interaction.

We now can define the 2×2 conductivity matrix σ as

$$\sigma = iT\Pi T^*, \quad (89)$$

where T is the conversion matrix

$$T = e \begin{pmatrix} i\sqrt{\frac{\omega}{l(l+1)}} & 0 \\ 0 & \sqrt{\frac{1}{\omega}} \end{pmatrix} \ . \quad (90)$$

This conductivity matrix has been constructed to satisfy Ohm's law

$$-e \begin{pmatrix} \Gamma_{2lm} \\ \Gamma_{1lm} \end{pmatrix} = \sigma \begin{pmatrix} \mathcal{E}_{2lm} \\ \mathcal{E}_{1lm} \end{pmatrix} \ , \quad (91)$$

where $\boldsymbol{\mathcal{E}}_{ilm}$ are the angular momentum components in the $\mathbf{T}$ basis of the angular electric field $\boldsymbol{\mathcal{E}}$ which is given in terms of the true electric field $\mathbf{E}$ as

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\Omega} \times \mathbf{E}. \quad (92)$$

In terms of angular momentum and frequency components (in our chosen gauge) we can easily derive

$$\mathcal{E}_{1lm} = i\omega\mathcal{A}_{1lm} \ , \quad (93)$$

$$\mathcal{E}_{2lm} = -\mathcal{A}_{0lm}\sqrt{l(l+1)} \ , \quad (94)$$

which is then used, along with the gauge invariance (B7) and current conservation (B14), to derive the form of the conversion equation (89).

Similarly, we can define a bare quasiparticle conductivity $\tilde{\sigma}$ which in the RPA we approximate as

$$\tilde{\sigma} = iTK^0T^*. \quad (95)$$

It should be noted that these definitions of the conductivity in some sense do not give the proper $\omega \to 0$ limit.[2,1] However, we will not be concerned with this limit in the present paper, and we expect that over the range of frequencies where we expect the RPA to be a good approximation, these definitions are appropriate.

In terms of these conductivities the RPA equation is written as

$$K^{-1} = iT^*\rho T - (\tfrac{N-1}{N})V \ , \quad (96)$$

$$\rho = \tilde{\rho} + \rho_{cs} \ , \quad (97)$$

$$\rho_{cs} = i[T^{-1}]^*W[T^{-1}] = \frac{2\pi\tilde{\phi}}{e^2}(\tfrac{N-1}{N})\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \ , \quad (98)$$

where

$$\rho = \sigma^{-1} \ , \quad (99)$$

$$\tilde{\rho} = \tilde{\sigma}^{-1}. \quad (100)$$

B. Mass renormalization: Modified RPA

As pointed out in previous work,[2,1] within the RPA theory considered thus far, the quasiparticle effective mass m^* is just the bare band mass m_b. In this theory, we perturb around the reference Hamiltonian (24) that describes quasiparticles with the bare band mass m_b. We expect, however, that interactions may renormalize the quasiparticle mass significantly. In the limit where the Coulomb interaction is turned off, all of the states in a Landau level become degenerate, and the effective mass diverges. In order to estimate the importance of the mass renormalization when the Coulomb interaction is turned back on, we follow Halperin, Lee, and Read[2] and make a crude estimate for the value of the effective mass. If we assume that the characteristic Coulomb interaction energy $(e^2/\epsilon l_0)$ is small compared to the characteristic energy spacing between Landau levels $(\hbar\omega_c)$, then level mixing can be neglected and all energies of interaction should scale as $e^2/\epsilon l_0$. Thus, if a finite effective mass m^* exists, we should have

$$\frac{\hbar^2}{m^* l_0^2} \propto \frac{e^2}{\epsilon l_0} \quad (101)$$

or

$$\frac{\epsilon}{e^2 l_0} = Cm^* \quad (102)$$

for some proportionality constant C where we have set $\hbar = 1$ again. The estimate $C \approx 0.3$ has been made by Halperin, Lee, and Read[2] by examining results from exact diagonalization of small systems. Using the experimentally relevant dielectric constant $\epsilon = 12.6$ appropriate for GaAs, a magnetic field B=10 T, and a filling fraction $\nu = \frac{1}{2}$, this estimate yields

$$m^* \approx 4m_b. \tag{103}$$

Using a self-consistent analysis of a selected set of diagrams for the self-energy of the transformed fermions which describes the interaction with long wavelength fluctuations in the Chern-Simons vector potential, Halperin, Lee, and Read[2] conclude that for the case of the Coulomb interaction between the electrons, the effective mass m^* should actually exhibit a logarithmic divergence for energies near the Fermi energy, and for $p \to \infty$ (i.e., for $\nu \to \frac{1}{2}$). The coefficient in front of the logarithm obtained by Ref. 2 is relatively small, however, and the resulting values of the effective mass, in practice, will not be very different from those given by Eq. (102).

The important thing to note here is that the effective mass can be renormalized considerably. Therefore our RPA procedure of expanding around a Hamiltonian that describes particles of mass m_b is likely to be a poor approximation. As noted in Ref. 1, if one simply inserts the effective mass m^* in place of the band mass m_b in the reference Hamiltonian (24), one obtains a "naive response" function that should reasonably represent the low energy excitations of the system[1] but which violates the f-sum rule. In this section we will follow Ref. 1 and propose a modified RPA, generalized to the sphere, that reasonably represents the low energy excitations while satisfying the f-sum rule.[17]

As described in Ref. 1, the f-sum rule requires that the high frequency response of our system must be determined by the bare band mass m_b and not the effective mass m^*. More specifically, in the high frequency limit, the response of N interacting electrons on a sphere around a monopole of charge $2S$ flux quanta must be the same as the response of a system of N noninteracting electrons on the same sphere. In terms of the resistivity matrix ρ, this limit is given as

$$\rho \sim \frac{m_b}{e^2\bar{n}_e}\begin{bmatrix} -i\omega & \frac{-S}{m_b} \\ \frac{S}{m_b} & -i\omega \end{bmatrix}, \tag{104}$$

which can be verified by numerically examining the result obtained in Appendix C where we derive the response for a system of noninteracting electrons (In the Appendix, the total field is called $2\Delta S$.) Since at high frequency the particles essentially oscillate in place, it is not surprising that this high frequency limit yields exactly the same high frequency resistivity as the analogous planar system.[1]

Now, in our Chern-Simons approach, we actually begin by calculating the response of a system of quaisiparticles in the effective field of a monopole of charge $2\Delta S$ flux quanta. In the RPA, the high frequency limit of the quasiparticle resistance is given by

$$\tilde{\rho} \sim \frac{m_b}{e^2\bar{n}_e}\begin{bmatrix} -i\omega & \frac{-\Delta S}{m_b} \\ \frac{\Delta S}{m_b} & -i\omega \end{bmatrix} \tag{105}$$

as required by the f-sum rule for the effective quasiparticle system. This is then converted into a total resistance using Eq. (97) to yield

$$\rho \sim \frac{m_b}{e^2\bar{n}_e}\begin{bmatrix} -i\omega & \frac{-\Delta S}{m_b} \\ \frac{\Delta S}{m_b} & -i\omega \end{bmatrix} + \frac{2\pi\tilde{\phi}}{e^2}\left(\tfrac{N-1}{N}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{106}$$

which can be shown to be equivalent to Eq. (104) by using relation (23) to relate S to ΔS. More generally, we see that our original electron system will satisfy the f-sum rule with respect to the monopole charge S, (104), if and only if the quasiparticle system satisfies the f-sum rule with respect to the effective monopole charge ΔS, (105).

As we mentioned above, we know that the quasiparticle mass is renormalized to some new effective mass m^*. In the naive approach we simply replace the band mass m_b by the effective quasiparticle mass m^* everywhere it occurs in the calculation of $\tilde{\rho}$, which amounts to simply replacing the band mass by the effective mass in the Hamiltonian (24). The result of such a substitution is what we will call the naive quasiparticle resistivity tensor

$$\tilde{\rho}^n \sim \frac{m^*}{e^2\bar{n}_e}\begin{bmatrix} -i\omega & \frac{-\Delta S}{m^*} \\ \frac{\Delta S}{m^*} & -i\omega \end{bmatrix}, \tag{107}$$

which clearly violates the f-sum rule. In order to repair this naive approximation, we will simply adopt the result of the Landau-Fermi-liquid theory[18] approach developed in Ref. 1. In that paper, an analogous theory is developed in detail for a planar geometry. It is found that Fermi-liquid theory can be used to self-consistently account for an arbitrary quasiparticle mass renormalization. The full resistivity is then given in terms of the naive resistivity and the mass renormalization as

$$\tilde{\rho} = \tilde{\rho}^n + \frac{i\omega(m^* - m_b)}{\bar{n}_e e^2}\mathbf{1}. \tag{108}$$

If we assume that the same prescription works on the sphere, we find that the resulting resistivity satisfies the f-sum rule as desired. One can in fact formally derive this result on a sphere, following the exact same procedure as given in Ref. 1. The validity of such an approach for small systems is, however, questionable. Instead, we simply take Eq. (108) as a motivated ansatz for modifying our naive results such that they satisfy the f-sum rule and give the correct energy scale for low energy excitations. In addition, this approach clearly agrees with Ref. 1 in the limit where the sphere is taken to be large.

The complete prescription for the modified RPA for quantized Hall states on a sphere is to calculate the unperturbed response function for the quasiparticle system K^0 for quasiparticles of mass m^* as described in Eqs. (58) and (C31) where all occurrences of the band mass m_b are replaced with the effective mass m^*, then convert to the naive resistivity $\tilde{\rho}^n$ using Eqs. (95) and (100), and then add the off diagonal Chern-Simons term and the diagonal mass renormalization term to yield the total resistivity

$$\rho = \tilde{\rho}^n - \frac{i\omega(m_b - m^*)}{n_e e^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2\pi\tilde{\phi}}{e^2}\left(\tfrac{N-1}{N}\right)\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tag{109}$$

which can be converted back to a response function using Eq. (96).

V. SUMMARY

In this paper we have used the fermion Chern-Simons approach to model a fractional quantized Hall state on a sphere as an integer quantized Hall state of transformed fermions interacting with a Chern-Simons field. Linear response theory is used to find the bare mean-field response function. We then use the RPA to account for fluctuations in the Chern-Simons field as well as the direct Coulomb interaction. The RPA result, however, cannot properly account for the effects of mass renormalization. Using results derived in Ref. 1 we propose a "modified RPA" prescription that more appropriately accounts for these mass renormalization effects. A quantitative comparison between the response functions derived in this paper and response functions given by exact numerical diagonalizations will be made in a following paper.[4] We remark that the modified RPA can also be used to describe excitations of a fractional quantum Hall state of *bosons* on a sphere at $\nu = p/(\tilde{\phi}p+1)$, if we simply choose $\tilde{\phi}$ to be an odd integer.

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APPENDIX A: ROTATIONAL INVARIANCE IN THE SPHERICAL BASIS

Given a function $K_{\alpha\beta}(\mathbf{\Omega}, \mathbf{\Omega}')$ (such as our response function) that is rotationally invariant, for any rotation $\tilde{R}$ around the origin of our three-dimensional space, we have

$$K_{(\alpha,\beta)}(\mathbf{\Omega}, \mathbf{\Omega}', t-t') = K_{(\tilde{R}\alpha,\tilde{R}\beta)}(\tilde{R}\mathbf{\Omega}, \tilde{R}\mathbf{\Omega}', t-t'), \tag{A1}$$

where the $\tilde{R}$ applied to the α and β indices rotates the three-vector valued part of the K function. We now transform into the $\mathbf{T}$ basis

$$K_{\mu l m;\nu l' m'} = \int d\mathbf{\Omega} \int d\mathbf{\Omega}' [\mathbf{T}_{\mu l m}(\mathbf{\Omega})]^* \cdot K_{(\alpha,\beta)}(\mathbf{\Omega}, \mathbf{\Omega}', t-t') \cdot [\mathbf{T}_{\nu l' m'}(\mathbf{\Omega}')] \tag{A2}$$

$$= \int d\mathbf{\Omega} \int d\mathbf{\Omega}' [\mathbf{T}_{\mu l m}(\mathbf{\Omega})]^* \cdot K_{(\tilde{R}\alpha,\tilde{R}\beta)}(\tilde{R}\mathbf{\Omega}, \tilde{R}\mathbf{\Omega}', t-t') \cdot [\mathbf{T}_{\nu l' m'}(\mathbf{\Omega}')] \tag{A3}$$

or equivalently

$$K_{\mu l m;\nu l' m'} = \int d\mathbf{\Omega} \int d\mathbf{\Omega}' \left[\tilde{R}^{-1}\mathbf{T}_{\mu l m}(\tilde{R}^{-1}\mathbf{\Omega})^*\right] \cdot K_{(\alpha,\beta)}(\mathbf{\Omega}, \mathbf{\Omega}', t-t') \cdot \left[\tilde{R}^{-1}\mathbf{T}_{\nu l' m'}(\tilde{R}^{-1}\mathbf{\Omega}')\right], \tag{A4}$$

where we have used the fact that the measure $d\mathbf{\Omega}$ is invariant under rotation. Now, since the basis vectors $\mathbf{T}_{ilm}$ are spherical tensors[13,14] of rank l, under an arbitrary rotation $\tilde{R}$, we have the rotation law

$$\tilde{R}\mathbf{T}_{ilm}(\tilde{R}\mathbf{\Omega}) = \sum_n D^l_{nm}(\tilde{R})\mathbf{T}_{iln}, \tag{A5}$$

where D is the rotation matrix as defined in Ref. 14. This then results in the identity

$$K_{\mu l m;\nu l' m'} = \sum_{n\,n'} [D^l_{nm}(\tilde{R})]^* [D^{l'}_{n'm'}(\tilde{R})] K_{\mu l n;\nu l' n'}, \tag{A6}$$

which must be true for all rotations $\tilde{R}$. Thus we can integrate over all possible rotations and use the orthogonality relation[13,14]

$$\int d\tilde{R}\, [D^l_{nm}(\tilde{R})]^* [D^{l'}_{n'm'}(\tilde{R})] = \frac{8\pi^2}{2l+1}\delta_{ll'}\delta_{mm'}\delta_{nn'}, \tag{A7}$$

which immediately gives us

$$K_{\mu l m;\nu l' m'} = \delta_{ll'}\delta_{mm'}K_{\mu\nu}(l). \tag{A8}$$

APPENDIX B: GAUGE INVARIANCE AND CURRENT CONSERVATION

The response of the system must be invariant under a gauge transformation of the perturbing electromagnetic field. Given an arbitrary function $\chi(\mathbf{\Omega}, t)$, a gauge transformation is given by

$$\Phi \longrightarrow \Phi - \frac{\partial \chi}{\partial t}, \tag{B1}$$

$$\mathbf{A} \longrightarrow \mathbf{A} + \nabla\chi. \tag{B2}$$

If we expand χ into its frequency and angular momentum components

$$\chi(\mathbf{\Omega}, t) = \frac{1}{2\pi}\int d\omega e^{-i\omega t}\sum_{lm} Y_{lm}(\mathbf{\Omega})\chi_{lm}(\omega) \tag{B3}$$

and use the definitions of $\mathcal{A}$ [Eqs. (10) and (41)] as well as Eq. (44) and the definitions of the basis $\mathbf{T}$ [Eqs. (31) and (32)] we find that the gauge transformation results in a transformation of the angular vector potential $\mathcal{A}$ given by

$$\mathcal{A}_{0lm} \longrightarrow \mathcal{A}_{0lm} - i\omega\chi_{lm}(\omega) \ , \tag{B4}$$

$$\mathcal{A}_{1lm} \longrightarrow \mathcal{A}_{1lm} \ , \tag{B5}$$

$$\mathcal{A}_{2lm} \longrightarrow \mathcal{A}_{2lm} - \sqrt{l(l+1)}\,\chi_{lm}(\omega). \tag{B6}$$

Since the current Γ_μ must be unchanged under this arbitrary gauge transformation, we have from Eqs. (46) and (47)

$$i\omega K_{\mu 0}(l,\omega) + \sqrt{l(l+1)}K_{\mu 2}(l,\omega) = 0. \tag{B7}$$

On the other hand, current conservation demands that

$$\nabla\cdot\mathbf{J} + \frac{\partial n_e}{\partial t} = 0 \ , \tag{B8}$$

where the current $\mathbf{J}$ is given by

$$\mathbf{J} = -\mathbf{\Omega}\times\mathbf{\Gamma} = \frac{1}{m_b}\psi_e^\dagger(\mathbf{\Omega})\left[-i\nabla + e\mathbf{A}\right]\psi_e(\mathbf{\Omega}). \tag{B9}$$

Recalling that Γ_0 is just the fluctuation in density, the current conservation equation becomes

$$\nabla\cdot(\mathbf{\Omega}\times\mathbf{\Gamma}) - \frac{\partial\Gamma_0}{\partial t} = 0. \tag{B10}$$

We can rewrite this in terms of the angular momentum and frequency components

$$i\omega\Gamma_{0lm}(\omega) + \sum_j \nabla\cdot(\mathbf{\Omega}\times\mathbf{T}_{jlm})\Gamma_{jlm}(\omega) = 0 \ , \tag{B11}$$

where the index j is summed over the values 1 and 2. Using the definition of the $\mathbf{T}$ basis [Eqs. (31) and (32)] we can derive[13,14]

$$\nabla\cdot\mathbf{T}_{1lm} = \frac{-\sqrt{l(l+1)}}{r}Y_{lm} \ , \tag{B12}$$

$$\nabla\cdot\mathbf{T}_{2lm} = 0 \ , \tag{B13}$$

which then can be used with the current conservation equation (B11) to yield

$$i\omega\Gamma_{0lm}(\omega) - \sqrt{l(l+1)}\Gamma_{2lm}(\omega) = 0 \tag{B14}$$

and hence

$$i\omega K_{0\nu}(l,\omega) - \sqrt{l(l+1)}K_{2\nu}(l,\omega) = 0. \tag{B15}$$

APPENDIX C: THE RETARDED CORRELATION FUNCTION

Here we calculate the retarded correlation function for a system of N noninteracting quasiparticles of mass m_b on a sphere around a monopole of magnetic charge $2\Delta S$ flux quanta. It should be noted that in the calculation of the naive response $\tilde{\sigma}^n = [\tilde{\rho}^n]^{-1}$, we must replace m_b by m^* everywhere in this calculation. Since the eigenstates of this system are given by the monopole spherical harmonics of Wu and Yang[7] (see Sec. II D) we can expand the quasiparticle operator in this set of eigenstates[16]

$$\psi^\dagger(\mathbf{\Omega},t) = \sum_{lm}[Y_{lm}^{(\Delta S)}(\mathbf{\Omega})]^* c_{lm}^\dagger(t) \ , \tag{C1}$$

where $c_{lm}^\dagger$ creates a quasiparticle in the l,m state, and the sum is over all quasiparticle eigenstates. We also define

$$\hat{\mathbf{\Gamma}}^{\Delta\mathcal{A}} = \frac{1}{m_b}\left[\hat{\boldsymbol{\mathcal{P}}} + e\Delta\boldsymbol{\mathcal{A}}\right] \tag{C2}$$

such that

$$\mathbf{\Gamma}^{\Delta\mathcal{A}}(\mathbf{\Omega}) = \psi^\dagger(\mathbf{\Omega})\hat{\mathbf{\Gamma}}^{\Delta\mathcal{A}}\psi(\mathbf{\Omega}). \tag{C3}$$

In addition, we define

$$\hat{\Gamma}_0^{\Delta\mathcal{A}} = 1 \tag{C4}$$

for this calculation which yields

$$\Gamma_0^{\Delta\mathcal{A}} = \psi^\dagger(\mathbf{\Omega})\psi(\mathbf{\Omega}) = n_e(\mathbf{\Omega}) \ , \tag{C5}$$

which differs from the proper definition (39) by a constant. This difference can be neglected here since the commutator [Eq. 54] of a constant with anything is zero.

With these definitions, we can now write the retarded correlation function (54) as

$$\begin{aligned} D^0_{\alpha\beta}(\mathbf{\Omega},t;\mathbf{\Omega}',t') = -i\theta(t-t') \sum_{\substack{pqp'q'\\ rsr's'}} &[Y_{pr}^{(\Delta S)*}(\mathbf{\Omega})\hat{\Gamma}_\alpha^{\Delta\mathcal{A}}Y_{qs}^{(\Delta S)}(\mathbf{\Omega})Y_{p'r'}^{(\Delta S)*}(\mathbf{\Omega}')\hat{\Gamma}_\beta^{\Delta\mathcal{A}}Y_{q's'}^{(\Delta S)}(\mathbf{\Omega}') \\ &\times\langle F|c_{pr}^\dagger(t)c_{qs}(t)c_{p'r'}^\dagger(t')c_{q's'}(t')|F\rangle \\ &-Y_{pr}^{(\Delta S)*}(\mathbf{\Omega}')\hat{\Gamma}_\beta^{\Delta\mathcal{A}}Y_{qs}^{(\Delta S)}(\mathbf{\Omega}')Y_{p'r'}^{(\Delta S)*}(\mathbf{\Omega})\hat{\Gamma}_\alpha^{\Delta\mathcal{A}}Y_{q's'}^{(\Delta S)}(\mathbf{\Omega})\langle F|c_{pr}^\dagger(t')c_{qs}(t')c_{p'r'}^\dagger(t)c_{q's'}(t)|F\rangle] \ , \end{aligned} \tag{C6}$$

where F represents the full Fermi sea (i.e., the ground state of quasiparticles filled up to the Fermi level). The matrix element is clearly zero unless $(pr) = (q's')$ are states below the Fermi level, and $(qs) = (p'r')$ are states above the

Fermi level. Thus we have

$$D^0_{\alpha\beta}(\mathbf{\Omega},t;\mathbf{\Omega}',t') = -i\theta(t-t')\sum_{pr}^{\text{below}}\sum_{qs}^{\text{above}}[e^{i(t-t')(\omega_p-\omega_q)}Y^{(\Delta S)*}_{pr}(\mathbf{\Omega})\hat{\Gamma}^{\Delta\mathcal{A}}_{\alpha}Y^{(\Delta S)}_{qs}(\mathbf{\Omega})Y^{(\Delta S)*}_{qs}(\mathbf{\Omega}')\hat{\Gamma}^{\Delta\mathcal{A}}_{\beta}Y^{(\Delta S)}_{pr}(\mathbf{\Omega}')$$
$$-e^{i(t'-t)(\omega_p-\omega_q)}Y^{(\Delta S)*}_{pr}(\mathbf{\Omega}')\hat{\Gamma}^{\Delta\mathcal{A}}_{\beta}Y^{(\Delta S)}_{qs}(\mathbf{\Omega}')Y^{(\Delta S)*}_{qs}(\mathbf{\Omega})\hat{\Gamma}^{\Delta\mathcal{A}}_{\alpha}Y^{(\Delta S)}_{pr}(\mathbf{\Omega})] \,, \tag{C7}$$

where we have also used the law for the time propagation of creation and annihilation operators. Transforming into angular momentum and frequency components as defined in Eq. (56) now yields

$$D^0_{\mu lm,\nu l'm'}(\omega) = \sum_{p}^{\text{below}}\sum_{q}^{\text{above}}\left[\frac{M_{\mu lm,\nu l'm'}(p,q)}{(\omega+i0^+)-(\omega_q-\omega_p)} - \frac{M_{\mu lm,\nu l'm'}(q,p)}{(\omega+i0^+)+(\omega_q-\omega_p)}\right] \,, \tag{C8}$$

where ω_p is given by Eq. (25),

$$M_{\mu lm,\nu l'm'}(p,q) = \sum_{rs} N_{\mu lm}(p,r,q,s)N^*_{\nu l'm'}(p,r,q,s) \,, \tag{C9}$$

and

$$N_{\mu lm}(p,r,q,s) = \int d\mathbf{\Omega}\,[\mathbf{T}_{\mu lm}(\mathbf{\Omega})]^* \cdot [Y^{(\Delta S)*}_{pr}(\mathbf{\Omega})\,\hat{\mathbf{\Gamma}}^{\Delta\mathcal{A}} \times Y^{(\Delta S)}_{qs}(\mathbf{\Omega})] \,, \tag{C10}$$

where the $\mathbf{T}$ and $\hat{\mathbf{\Gamma}}$ are four-vectors. The evaluation of the $\mu = 0$ element of this vector is quite simple using the identity derived in Appendix D:

$$N_{0lm}(p,r,q,s) = \int d\mathbf{\Omega}\,Y^*_{lm}(\mathbf{\Omega})Y^{(\Delta S)*}_{pr}(\mathbf{\Omega})Y^{(\Delta S)}_{qs}(\mathbf{\Omega}) \tag{C11}$$

$$= \left[\frac{(2l+1)(2p+1)(2q+1)}{4\pi}\right]^{\frac{1}{2}} \times(-1)^{s+\Delta S}\begin{pmatrix} l & p & q \\ -m & -r & s\end{pmatrix} \times\begin{pmatrix} l & p & q \\ 0 & \Delta S & -\Delta S\end{pmatrix} \,, \tag{C12}$$

where the large parentheses are Wigner $3j$ symbols.[13,14] Now in order to calculate the other elements N_i, we consider the natural angular momentum operator for the monopole system[7,19]

$$\hat{\mathbf{L}} = \mathbf{r}\times(\mathbf{p}+e\Delta\mathbf{A}) - \hat{\mathbf{r}}\Delta S \tag{C13}$$

$$= m_b\hat{\mathbf{\Gamma}}^{\Delta\mathcal{A}} + \text{radial component.} \tag{C14}$$

This operator acts on the monopole spherical harmonic $Y^{(\Delta S)}_{qs}$ to give the monopole vector harmonics of Olsen, Osland, and Wu:[19]

$$\hat{\mathbf{L}}Y^{(\Delta S)}_{qs}(\mathbf{\Omega}) = \sqrt{q(q+1)}\,\mathbf{Y}^{(\Delta S)}_{qqs}(\mathbf{\Omega}) \,, \tag{C15}$$

where the monopole vector harmonic is defined as

$$\mathbf{Y}^{(\Delta S)}_{jlm}(\mathbf{\Omega}) = \sum_{n,\alpha}\langle l1n\alpha|jm\rangle Y^{(\Delta S)}_{ln}(\mathbf{\Omega})\hat{\mathbf{e}}_{\alpha} \,. \tag{C16}$$

Here, the Clebsch-Gordan coefficients are defined as in Ref. 14, and the spherical tensor vector basis $\hat{\mathbf{e}}_{\alpha}$ is given by

$$\hat{\mathbf{e}}_{\pm 1} = \mp\frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_x \pm i\hat{\mathbf{e}}_y) \,, \tag{C17}$$

$$\hat{\mathbf{e}}_0 = \hat{\mathbf{e}}_z \,, \tag{C18}$$

where $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$ are the usual three-dimensional Cartesian basis vectors. In the case of $\Delta S = 0$, Eq. (C16) defines the usual[13,14] spherical vector harmonics $\mathbf{Y}_{jlm} \equiv \mathbf{Y}^{(0)}_{jlm}$ in terms of the usual spherical harmonics $Y_{ln} \equiv Y^{(0)}_{ln}$.

Using the relation (C14) we now have

$$Y^{(\Delta S)*}_{pr}(\mathbf{\Omega})\,\hat{\mathbf{\Gamma}}^{\Delta\mathcal{A}}\,Y^{(\Delta S)}_{qs}(\mathbf{\Omega}) = \frac{1}{2m_b}[\sqrt{q(q+1)}Y^{(\Delta S)*}_{pr}(\mathbf{\Omega})\mathbf{Y}^{(\Delta S)}_{qqs}(\mathbf{\Omega}) + \sqrt{p(p+1)}\mathbf{Y}^{(\Delta S)*}_{ppr}(\mathbf{\Omega})Y^{(\Delta S)}_{qs}(\mathbf{\Omega})]$$
$$+\text{radial component} \,, \tag{C19}$$

where we have left-right symmetrized the operator. Since the unknown radial component cannot contribute to the response of a system on the sphere, we can safely drop this piece [as long as we do not try to calculate a radial ($\mu = 3$) response].

We now note that the **T** functional vector basis can be written in terms of the usual spherical vector harmonics as[14]

$$\mathbf{T}_{ilm} = \sum_k C_{ik} \mathbf{Y}_{l(l+k)m} \ , \tag{C20}$$

where $i = 1, 2, 3$, the index k is summed over the values $-1, 0, 1$, and

$$\begin{array}{c} k = \quad -1 \qquad 0 \qquad +1 \ , \\ C_{ik} = \begin{pmatrix} \sqrt{\frac{l+1}{2l+1}} & 0 & \sqrt{\frac{l}{2l+1}} \\ 0 & -i & 0 \\ \sqrt{\frac{l}{2l+1}} & 0 & -\sqrt{\frac{l+1}{2l+1}} \end{pmatrix} . \end{array} \tag{C21}$$

Now using this relation, the definition (C16) of the monopole vector harmonics, and the same identity (C12) derived in Appendix D, then converting Clebsch-Gordan coefficients to $3j$ coefficients,[13,14] we find that Eq. (C10) can be rewritten for $i = 1, 2$ as

$$N_{ilm}(p, r, q, s) = \sum_k C_{ik} Q_{l,l+k,m}(p, r, q, s) \ , \tag{C22}$$

where k is summed over the values $-1, 0, 1$, and

$$\begin{aligned} Q_{l,j,m}(p, r, q, s) = {} & \frac{1}{2m_b} \left[\frac{(2l+1)(2j+1)(2p+1)(2q+1)}{4\pi} \right]^{\frac{1}{2}} \begin{pmatrix} j & p & q \\ 0 & \Delta S & -\Delta S \end{pmatrix} \begin{pmatrix} l & p & q \\ -m & -r & s \end{pmatrix} \\ & \times \left[\sqrt{p(p+1)(2p+1)} (-1)^{(p+q+j+s+\Delta S)} \begin{Bmatrix} l & p & q \\ p & j & 1 \end{Bmatrix} \right. \\ & \left. + \sqrt{q(q+1)(2q+1)} (-1)^{(3q+1+\Delta S-s+l+p)} \begin{Bmatrix} l & q & p \\ q & j & 1 \end{Bmatrix} \right] . \end{aligned} \tag{C23}$$

The curly brackets here denote Wigner $6j$ symbols, and we have used the orthogonality identity[13]

$$\sum_{n_1, n_2, n_3} (-1)^{(l_1+l_2+l_3+n_1+n_2+n_3)} \begin{pmatrix} j_1 & l_2 & l_3 \\ m_1 & n_2 & -n_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -n_1 & m_2 & n_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3 \end{Bmatrix} \tag{C24}$$

and the symmetry properties of the $3j$ symbols[13,14] to derive Eq. (C23). We now note that each N_μ contains a Wigner $3j$ coefficient with indices r and s that are then summed over in Eq. (C9). Using the orthogonality relation[13,14]

$$\sum_{r,s} \begin{pmatrix} l & p & q \\ -m & -r & s \end{pmatrix} \begin{pmatrix} l' & p & q \\ -m' & -r & s \end{pmatrix} = \frac{\delta_{ll'} \delta_{mm'}}{2l+1} \tag{C25}$$

we perform the sum in (C9) such that

$$M_{\mu lm, \nu l'm'}(p, q) = M_{\mu,\nu}(l, p, q) \delta_{ll'} \delta_{mm'} \tag{C26}$$

and

$$M_{\mu,\nu}(l, p, q) = \left[\frac{(2p+1)(2q+1)}{4\pi} \right] \tilde{N}_\mu(p, q) \tilde{N}^*_\nu(p, q) \ , \tag{C27}$$

where

$$\tilde{N}_0(p, q) = \begin{pmatrix} l & p & q \\ 0 & \Delta S & -\Delta S \end{pmatrix} \ , \tag{C28}$$

$$\tilde{N}_i(p, q) = \sum_k C_{ik} \tilde{Q}_{l,l+k}(p, q). \tag{C29}$$

Once again, k is summed over $-1, 0, 1$, the matrix C is the conversion matrix from Eq. (C21), i takes on the values $1, 2$, and now

$$\tilde{Q}_{l,j}(p,q) = \frac{1}{2m_b}\sqrt{2j+1}\begin{pmatrix} j & p & q \\ 0 & \Delta S & -\Delta S \end{pmatrix}\Bigg[(-1)^{p+q+j}\sqrt{p(p+1)(2p+1)}\begin{Bmatrix} l & p & q \\ p & j & 1 \end{Bmatrix} + (-1)^{q-p+l+2\Delta S+1}\sqrt{q(q+1)(2q+1)}\begin{Bmatrix} l & q & p \\ q & j & 1 \end{Bmatrix}\Bigg] . \quad \text{(C30)}$$

Finally, collecting our result, we have

$$D^0_{\mu,\nu}(l,\omega) = \sum_p^{\text{below}}\sum_q^{\text{above}}\left[\frac{M_{\mu\nu}(l,p,q)}{(\omega+i0^+)-(\omega_q-\omega_p)} - \frac{M_{\mu\nu}(l,q,p)}{(\omega+i0^+)+(\omega_q-\omega_p)}\right] , \quad \text{(C31)}$$

where the p sum is over states such that ω_p is less than or equal to the Fermi energy, and the q sum is over states such that ω_q is greater than the Fermi energy.

APPENDIX D: COUPLING OF THREE MONOPOLE HARMONICS

The monopole spherical harmonics $Y^{(q)}_{lm}(\theta,\phi)$ of Wu and Yang[7] can be written in terms of the rotation matrices $D^l_{m'm}(\phi,\theta,\psi)$ of ordinary quantum mechanics.[14] The relation between the two is given by

$$Y^{(q)}_{lm}(\theta,\phi) = \left[\frac{2l+1}{4\pi}\right]^{\frac{1}{2}}[D^l_{m,-q}(\phi,\theta,-\phi)]^* . \quad \text{(D1)}$$

Now, using the coupling relation for three rotation matrices[13,14]

$$\frac{1}{8\pi^2}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\psi\, D^{j_1}_{m_1m_1'}(\phi,\theta,\psi)D^{j_2}_{m_2m_2'}(\phi,\theta,\psi)D^{j_3}_{m_3m_3'}(\phi,\theta,\psi) = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{pmatrix} \quad \text{(D2)}$$

and the decomposition of the rotation matrix,

$$D^l_{m'm}(\phi,\theta,\psi) = e^{-i\phi m'}d^j_{m'm}(\theta)e^{-i\psi m} , \quad \text{(D3)}$$

we can easily derive the corresponding law for monopole harmonics

$$\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, Y^{(q_1)}_{l_1m_1}(\theta,\phi)Y^{(q_2)}_{l_2m_2}(\theta,\phi)Y^{(q_3)}_{l_3m_3}(\theta,\phi)$$

$$= \left[\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}\right]^{\frac{1}{2}}\begin{pmatrix} l_1 & l_2 & l_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ q_1 & q_2 & q_3 \end{pmatrix} . \quad \text{(D4)}$$

Finally, we use the relation[7]

$$[Y^{(q)}_{lm}(\theta,\phi)]^* = (-1)^{q+m}Y^{(-q)}_{l,-m}(\theta,\phi) \quad \text{(D5)}$$

and the fact that $Y^{(0)}_{lm} \equiv Y_{lm}$ is just the usual spherical harmonic to trivially derive Eq. (C12).

APPENDIX E: INTERACTION COEFFICIENTS

An arbitrary rotationally symmetric function $v(|\boldsymbol{\Omega}-\boldsymbol{\Omega}'|)$ on the unit sphere can be expanded in terms of spherical harmonics as[13,14]

$$v(\boldsymbol{\Omega},\boldsymbol{\Omega}') = 4\pi\sum_{lm} Y^*_{lm}(\boldsymbol{\Omega})Y_{lm}(\boldsymbol{\Omega}')f_l , \quad \text{(E1)}$$

where

$$f_l = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\, v([2-2\cos\theta]^{\frac{1}{2}}) \quad \text{(E2)}$$

and P_l is the Legendre polynomial.[20] For the Coulomb interaction $v = 1/|\boldsymbol{\Omega}-\boldsymbol{\Omega}'|$, for example, it is a well-known result that[13]

$$v(l) \equiv 4\pi f_l^{\text{Coulomb}} = \frac{4\pi}{2l+1}. \quad \text{(E3)}$$

In order to calculate the Chern-Simons interaction coefficient, we begin by considering the function

$$\frac{1}{2}\ln(|\boldsymbol{\Omega}-\boldsymbol{\Omega}'|^2) = 2\pi\sum_{lm} Y^*_{lm}(\boldsymbol{\Omega})Y_{lm}(\boldsymbol{\Omega}')f_l , \quad \text{(E4)}$$

where

$$f_l = \frac{1}{2}\int_{-1}^{1} dz\, P_l(z)\ [\ln 2 + \ln(1-z)]\,. \tag{E5}$$

Now, since P_0 is a constant, and the Legendre polynomials form an orthogonal set,[20] the first term vanishes except when $l = 0$, which is a case with which we will not be concerned. To evaluate this integral, we first change the integration variable to $-z$, using the fact that $P_l(-z) = (-1)^l P_l(z)$. Next, we use the well-known identity

$$\ln x = \lim_{\alpha\to 0} \frac{x^\alpha - 1}{\alpha} \tag{E6}$$

to rewrite the integral as the following limit for $l \neq 0$:

$$f_l = \frac{(-1)^l}{2}\int_{-1}^{1} dz\, P_l(z) \lim_{\alpha\to 0}\left[\frac{(1+z)^\alpha - 1}{\alpha}\right]\,. \tag{E7}$$

The second term is a constant with respect to z and, as discussed above, integrates to zero except in the $l = 0$ term. The first term is an integral that can be found in a standard table[20] to yield

$$f_l = (-1)^l \lim_{\alpha\to 0}\left[\frac{2^{\alpha+1}\,[\Gamma(\alpha+1)]^2}{\alpha\Gamma(\alpha+l+2)\Gamma(\alpha+1-l)}\right]\,. \tag{E8}$$

The limit can now be taken by using the reflection principle that $\Gamma(z)\Gamma(1-z)\sin(\pi z) = \pi$ to give

$$f_l = \frac{1}{l(l+1)} \tag{E9}$$

for $l \neq 0$. Using this result along with Eq. (E4), we then derive the useful identity

$$\frac{1}{2}\nabla_{\mathbf{\Omega}} \ln(|\mathbf{\Omega}-\mathbf{\Omega}'|^2) = \frac{\mathbf{\Omega}-\mathbf{\Omega}'}{|\mathbf{\Omega}-\mathbf{\Omega}'|^2}$$

$$= 2\pi\sum_{lm} \mathbf{T}^*_{1lm} Y_{lm}(\mathbf{\Omega}')\frac{1}{\sqrt{l(l+1)}}\,, \tag{E10}$$

where we have used Eq. (31) to take the gradient of the spherical harmonic.

We are now interested in the angular momentum components of the quantity

$$\mathbf{W}(\mathbf{\Omega},\mathbf{\Omega}') = -\frac{\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{\Omega}')}{|\mathbf{\Omega}-\mathbf{\Omega}'|^2} = \frac{(\mathbf{\Omega}'-\mathbf{\Omega})}{|\mathbf{\Omega}-\mathbf{\Omega}'|^2} + \frac{\mathbf{\Omega}}{2}\,, \tag{E11}$$

where we have used various simple vector identities[20] to obtain this form. Since the second term is purely radial, it will only contribute to the $\mathbf{T}_3$ component which we have already completely decoupled. Thus we can safely drop this piece. So, converting into angular momentum components yields, for $i = 1, 2$, and $l \neq 0$,

$$W_{ilm;0l'm'} = \int d\mathbf{\Omega}\int d\mathbf{\Omega}'\mathbf{T}_{ilm}(\mathbf{\Omega})\cdot\mathbf{W}(\mathbf{\Omega},\mathbf{\Omega}')Y^*_{lm}(\mathbf{\Omega}') \tag{E12}$$

$$= \int d\mathbf{\Omega}\int d\mathbf{\Omega}'\mathbf{T}_{ilm}(\mathbf{\Omega})\cdot\left[\sum_{kn}\mathbf{T}^*_{1kn}Y_{kn}(\mathbf{\Omega}')\times\frac{-2\pi}{\sqrt{k(k+1)}}\right]Y^*_{l'm'}(\mathbf{\Omega}') \tag{E13}$$

$$= -\delta_{1i}\delta_{ll'}\delta_{mm'}\frac{2\pi}{\sqrt{l(l+1)}}\,. \tag{E14}$$

APPENDIX F: EFFECTIVE ELECTRIC AND MAGNETIC FIELDS

Here we write the induced vector potential in terms of its angular momentum components as described in Eqs. (66), (75), and (76):

$$A_0^{\rm in}(\mathbf{\Omega})_{\rm Coulomb} = \left(\tfrac{N-1}{N}\right)\sum_{lm} Y_{lm}(\mathbf{\Omega})v(l)\Gamma_{0lm}\,, \tag{F1}$$

$$A_0^{\rm in}(\mathbf{\Omega})_{\text{Chern-Simons}} = \left(\tfrac{N-1}{N}\right)\sum_{lm} Y_{lm}(\mathbf{\Omega})w(l)\Gamma_{1lm}\,, \tag{F2}$$

$$\mathbf{A}^{\rm in}(\mathbf{\Omega}) = \left(\tfrac{N-1}{N}\right)\sum_{lm} \mathbf{T}_{2lm}(\mathbf{\Omega})w(l)\Gamma_{0lm}\,, \tag{F3}$$

where here $\Gamma_{\mu lm}$ represents the expectation $\langle\Gamma_{\mu lm}\rangle$ and we have used the definition of the $\mathbf{T}$ basis. We will now establish that associated electric and magnetic fields are those given by Eqs. (63), (62), and (65)

We calculate the Chern-Simons magnetic field

$$\mathbf{b}^{\rm in}(\mathbf{\Omega}) = \mathbf{\Omega}\cdot[\nabla\times\mathbf{A}^{\rm in}(\mathbf{\Omega})] \tag{F4}$$

$$= \left(\tfrac{N-1}{N}\right)\sum_{lm} w(l)\Gamma_{0lm}\mathbf{\Omega}\cdot[\nabla\times\mathbf{T}_{2lm}(\mathbf{\Omega})]. \tag{F5}$$

Using a vector identity[14] of the $\mathbf{T}$ basis this becomes

$$\mathbf{b}^{\rm in}(\mathbf{\Omega}) = \left(\tfrac{N-1}{N}\right)\sum_{lm}\tilde{\phi}\frac{2\pi}{\sqrt{l(l+1)}}\Gamma_{0lm}\times\sqrt{l(l+1)}Y_{lm}(\mathbf{\Omega})\mathbf{\Omega} \tag{F6}$$

$$= \left(\tfrac{N-1}{N}\right)2\pi\tilde{\phi}\,\langle\Gamma_0(\mathbf{\Omega})\rangle\,\mathbf{\Omega}\,, \tag{F7}$$

which agrees with Eq. (62). Similarly we can calculate the Chern-Simons electric field

$$\mathbf{e}^{\rm in}(\mathbf{\Omega}) = \nabla A_0^{\rm in}(\mathbf{\Omega})_{\text{Chern-Simons}} - \frac{\partial}{\partial t}\mathbf{A}^{\rm in}(\mathbf{\Omega}) \tag{F8}$$

$$= \left(\tfrac{N-1}{N}\right)\sum_{lm} w(l)\left[\nabla Y_{lm}(\mathbf{\Omega})\Gamma_{1lm} - \mathbf{T}_{2lm}(\mathbf{\Omega})\frac{\partial}{\partial t}\Gamma_{0lm}\right]. \tag{F9}$$

Using the definition (31) of the $\mathbf{T}$ basis and the current conservation equation (B14) this can be rewritten as

$$\mathbf{e}^{\text{in}}(\mathbf{\Omega}) = \left(\tfrac{N-1}{N}\right) \sum_{lm} \tilde{\phi} \frac{2\pi}{\sqrt{l(l+1)}} \times \left[\mathbf{T}_{1lm}(\mathbf{\Omega})\Gamma_{1lm} + \mathbf{T}_{2lm}(\mathbf{\Omega})\Gamma_{2lm}\right] \sqrt{l(l+1)} \quad \text{(F10)}$$

$$= \left(\tfrac{N-1}{N}\right) 2\pi\tilde{\phi} \, \langle \mathbf{\Gamma}(\mathbf{\Omega}) \rangle \quad \text{(F11)}$$

in accordance with Eq. (63). Finally we have the Coulombic electric field

$$\mathbf{E}^{\text{in}}(\mathbf{\Omega}) = \left(\tfrac{N-1}{N}\right) \sum_{lm} \nabla Y_{lm}(\mathbf{\Omega}) v(l) \Gamma_{0lm}. \quad \text{(F12)}$$

We cannot take the divergence of this expression directly to verify Eq. (65) since there is some component of the electric field which is normal to the surface of the sphere that is not included in (F12) (we do not care about this component since it does not couple to our problem). However, it is a trivial application of electrodynamics to use Eqs. (E1), (E3), and (70) to establish that this electric field does indeed correspond to the charge density $\langle \Gamma_0 \rangle$ and thus satisfies Eq. (65).

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[1] S. H. Simon and B. I. Halperin, Phys. Rev. B **48**, 17 386 (1993).
[2] B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
[3] A. Lopez and E. Fradkin, Phys. Rev. B **44**, 5246 (1991); **47**, 7080 (1993); see also E. Fradkin, *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, MA, 1991), pp. 324–338. It should be noted that Lopez and Fradkin call the RPA a "semiclassical" approximation.
[4] S. He, S. H. Simon, and B. I. Halperin, following paper, Phys. Rev. B **50**, 1823 (1994).
[5] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989); Phys. Rev. B **40**, 8079 (1989); **41**, 7653 (1990); Adv. Phys. **41**, 105 (1992); G. Dev and J. K. Jain, Phys. Rev. B **45**, 1223 (1992).
[6] P. A. M. Dirac, Proc. R. Soc. London Ser. A **133**, 60 (1931).
[7] T. T. Wu and C. N. Yang, Nucl. Phys. **B107**, 365 (1976); note that we have written $Y_{lm}^{(q)}$ for the notation $Y_{q,l,m}$ used in this reference.
[8] I. Tamm, Z. Phys. **71**, 141 (1941); M. Fierz, Helv. Phys. Acta **17**, 27 (1944).
[9] X. G. Wen and A. Zee, Phys. Rev. Lett. **69**, 953 (1992).
[10] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[11] N. d'Ambrumenil and R. Morf, Phys. Rev. B **40**, 6108 (1989), and references therein.
[12] C. Gros and A. H. MacDonald, Phys. Rev. B **42**, 9514 (1990), and references therein.
[13] A. R. Edmonds, *Angular Momentum in Quantum Mechanics*, 2nd ed. (Princeton University Press, Princeton, NJ, 1974).
[14] J.-M. Normand, *A Lie Group: Rotations in Quantum Mechanics* (North-Holland, Amsterdam, 1980); we use the convention that $c = i$ on page 305. Note also that we use the notation $\mathbf{T}_{3l}$ in place of R^l, the Clebsch-Gordan coefficient $\langle j_1 j_2 m_1 m_2 | jm \rangle$ as defined in this reference corresponds to the notation $\langle j_1 m_1 j_2 m_2 | j_1 j_2 jm \rangle$ of Ref. 13, and the rotation matrix $D^j_{mm'}(\phi, \theta, \psi)$ corresponds to $D^j_{m'm}(-\phi, -\theta, -\psi)$ of Ref. 13.
[15] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, San Francisco, 1971).
[16] See, for example, A. L. Fetter, C. B. Hanna, and R. B. Laughlin, Phys. Rev. B **39**, 9679 (1989); Y.-H. Chen, F. Wilczek, E. Witten, and B. I. Halperin, Int. J. Mod. Phys. B **3**, 1001 (1989).
[17] In Ref. 1 we were also concerned with satisfying Kohn's theorem — another result of Galilean invariance — in the planar geometry. As far as we know, there is no simple analog to Kohn's theorem in the spherical geometry.
[18] D. Pines and P. Nozières, *Theory of Quantum Liquids* (Benjamin, New York, 1966), Vol. I.
[19] H. A. Olsen, P. Osland, and T. T. Wu, Phys. Rev. D **42**, 665 (1990); see also E. Weinberg, *ibid.* **49**, 1086 (1994).
[20] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, San Diego, 1980).