PHYSICAL REVIEW B VOLUME 50, NUMBER 21 1 DECEMBER 1994-I

Surface tension and kinetic coefficient for the normal/superconducting interface: Numerical results versus asymptotic analysis

James C. Osborn* and Alan T. Dorsey†
Department of Physics, University of Virginia, McCormick Road, Charlottesville, Virginia 22901
(Received 26 April 1994)

The dynamics of the normal/superconducting interface in type-I superconductors has recently been derived from the time-dependent Ginzburg-Landau theory of superconductivity. In a suitable limit these equations are mapped onto a "free-boundary" problem, in which the interfacial dynamics are determined by the diffusion of magnetic flux in the normal phase. The magnetic field at the interface satisfies a modified Gibbs-Thomson boundary condition which involves both the surface tension of the interface and a kinetic coefficient for motion of the interface. In this paper we calculate the surface tension and kinetic coefficient numerically by solving the one-dimensional equilibrium Ginzburg-Landau equations for a wide range of κ values. We compare our numerical results to asymptotic expansions valid for $\kappa \ll 1$, $\kappa \approx 1/\sqrt{2}$, and $\kappa \gg 1$, in order to determine the accuracy of these expansions.

I. INTRODUCTION

When a type-I superconductor in a magnetic field is subjected to either a sudden temperature or magnetic field quench which takes it from the normal phase into the Meissner phase, the approach to equilibrium will be determined by the rate at which superconducting islands are nucleated in the background normal phase, and the subsequent dynamics of the superconducting/normal interfaces. References 1 and 2 suggested that the essential features of the interface motion could be understood in terms of a free-boundary model for the magnetic field in the normal phase; this model is almost identical to a free-boundary model which is used to study the growth of a solid into its supercooled liquid phase. The interface motion in the latter case is known to be unstable, and leads to highly ramified solidification patterns (dendrites, for instance; see Ref. 3 for an overview). By analogy, the growth of the superconducting phase into the normal phase should be dynamically unstable. Numerical solutions of the time-dependent Ginzburg-Landau (TDGL) equations of superconductivity confirmed these expectations.[1,2] However, the precise connection between the TDGL equations and the free-boundary model was not made.

More recently, the free-boundary model has been *derived* from the TDGL equations using the method of matched asymptotic expansions.[4,5] The free-boundary model consists of a diffusion equation for the magnetic field $\mathbf{h}$ in the normal phase,

$$\partial_t \mathbf{h} = D_H \nabla^2 \mathbf{h}, \tag{1.1}$$

where $D_H = 1/4\pi\sigma^{(n)}$ is the diffusion constant for the magnetic flux, with $\sigma^{(n)}$ the normal state conductivity; a continuity equation for the magnetic field at the normal/superconducting interface,

$$(\nabla \times \mathbf{h}) \times \hat{\mathbf{n}}|_i = -D_H v_n \mathbf{h}_i, \tag{1.2}$$

with v_n the interface velocity normal to the interface (with $\hat{\mathbf{n}} \cdot \mathbf{h}_i = 0$); and a modified Gibbs-Thomson boundary condition for the magnetic field at the interface,

$$h_i = H_c \left[1 - \frac{4\pi}{H_c^2} \left(\sigma_{\rm ns} \mathcal{K} + \Gamma^{-1} v_n \right) \right], \tag{1.3}$$

where H_c is the thermodynamic critical field for the superconductor, $\sigma_{\rm ns}$ is the surface tension of the normal/superconducting interface, $\mathcal{K}$ is the curvature of the interface, and Γ^{-1} is a kinetic coefficient for motion of the interface (here we ignore thermal fluctuations[5]). It is convenient to introduce the dimensionless surface tension $\bar{\sigma}_{\rm ns}$ and kinetic coefficient $\bar{\Gamma}^{-1}$, which are defined through

$$\sigma_{\rm ns} = \frac{H_c^2 \lambda}{4\pi} \bar{\sigma}_{\rm ns}, \quad \Gamma^{-1} = \frac{H_c^2 \lambda}{4\pi} \frac{2m\gamma}{\hbar \kappa^2} \bar{\Gamma}^{-1}, \tag{1.4}$$

where λ is the magnetic penetration depth, κ is the Ginzburg-Landau parameter (the ratio of the penetration depth to the coherence length ξ), m is the mass of a Cooper pair, and γ is a dimensionless order parameter relaxation time. The dimensionless surface tension and the kinetic coefficient can be expressed in terms of the solutions to the one-dimensional *equilibrium* Ginzburg-Landau equations, which are

$$\frac{1}{\kappa^2} F'' - Q^2 F + F - F^3 = 0, \tag{1.5}$$

$$Q'' - F^2 Q = 0, \tag{1.6}$$

where $F(x)$ is the dimensionless order parameter amplitude and $Q(x)$ is the dimensionless magnetic vector potential [the magnetic field is $H(x) = Q'(x)$]. For an inter-

0163-1829/94/50(21)/15961(6)/$06.00

face between the normal and superconducting phases, we have in the superconducting phase $x \to -\infty$, $F(x) \to 1$ and $Q(x) \to 0$, while in the normal phase $x \to \infty$, $F(x) \to 0$ and $Q(x) \sim x/\sqrt{2}$. The surface tension and kinetic coefficient are then[4,5]

$$\bar{\sigma}_{\rm ns} = \frac{1}{\kappa^2}\left[I_1(\kappa) - I_2(\kappa)\right], \tag{1.7}$$

$$\bar{\Gamma}^{-1} = I_1(\kappa) - \frac{2\pi\hbar\sigma^{(n)}}{m\gamma} I_2(\kappa), \tag{1.8}$$

where the integrals I_1 and I_2 are defined by

$$\begin{aligned} I_1(\kappa) &= 2\int_{-\infty}^{\infty} dx\,(F')^2 \\ &= 2\kappa^2 \int_{-\infty}^{\infty} dx\,\left[F^2 - F^4 - F^2Q^2\right], \end{aligned} \tag{1.9}$$

$$\begin{aligned} I_2(\kappa) &= 2\kappa^2 \int_{-\infty}^{\infty} dx\,\left[\frac{1}{\sqrt{2}}Q' - (Q')^2\right] \\ &= 2\kappa^2 \int_{-\infty}^{\infty} dx\,F^2Q^2, \end{aligned} \tag{1.10}$$

where the second set of expressions are obtained by integrating by parts and using Eqs. (1.5) and (1.6). The Ginzburg-Landau equations have analytical solutions only in certain limiting cases, to be discussed below. Somewhat surprisingly, there appear to be few numerical calculations of the surface tension, even though the fundamental importance of this quantity in distinguishing type-I and type-II superconductors was recognized by Ginzburg and Landau in 1950.[6,7]

In order to complete the derivation of the free-boundary model from the TDGL equations, in this paper we solve the equilibrium Ginzburg-Landau equations numerically for a wide range of κ values, and use the solutions to calculate the surface tension and kinetic coefficient. Our paper is organized as follows. In Sec. II we review our numerical methods for solving the Ginzburg-Landau equations. In Sec. III we discuss the surface tension, and derive the asymptotic form of the surface tension for small κ, which agrees well with the numerical results for a large range of κ values. Our results also show that an asymptotic expansion for the surface tension near $\kappa = 1/\sqrt{2}$, which was derived in Appendix B of Ref. 5, has a larger range of validity than expected. In Sec. IV we discuss our results for the kinetic coefficient. The kinetic coefficient can be either positive or negative depending upon the ratio of relaxation times which appear in the TDGL equations;[4,5] from our results we are able to determine the range of parameter values which result in a positive kinetic coefficient. Sec. V is a discussion section in which we briefly summarize our results.

II. NUMERICAL METHODS

Numerical computation is necessary to solve Eqs. (1.5) and (1.6) in general. The method chosen was standard relaxation using Newton's method.[8] Since the relative size of the solution scales as $1/\kappa$ we rescaled the GL equations by substituting $x' = \kappa x$ which results in the rescaled equations

$$F'' - Q^2F + F - F^3 = 0, \tag{2.1}$$

$$\kappa^2 Q'' - F^2 Q = 0. \tag{2.2}$$

The rescaled differential equations can then be written as the following set of first order finite difference equations:

$$\begin{aligned} \Delta y_{1,k}/\Delta x'_k &= \overline{y}_{3,k}, \\ \Delta y_{2,k}/\Delta x'_k &= \overline{y}_{4,k}, \\ \Delta y_{3,k}/\Delta x'_k &= \overline{y}_{1,k}\left[\left(\overline{y}_{2,k}\right)^2 + \left(\overline{y}_{1,k}\right)^2 - 1\right], \\ \Delta y_{4,k}/\Delta x'_k &= \left(\overline{y}_{1,k}\right)^2 \overline{y}_{2,k}/\kappa^2, \end{aligned} \tag{2.3}$$

with $\mathbf{y}_1 = F$, $\mathbf{y}_2 = Q$, $\mathbf{y}_3 = F'$, $\mathbf{y}_4 = Q'$, $\Delta y_{n,k} = y_{n,k} - y_{n,k-1}$, $\Delta x'_k = x'_k - x'_{k-1}$, and $\overline{y}_{n,k} = \frac{1}{2}(y_{n,k} + y_{n,k-1})$. Each of the four equations is to be solved for $k = 2, \ldots, M$, with M the number of mesh points. We also have four boundary conditions as follows:

$$\begin{aligned} y_{1,1} &= 1, \\ y_{4,1} &= 0, \\ y_{1,M} &= 0, \\ y_{4,M} &= 1/(\kappa\sqrt{2}), \end{aligned} \tag{2.4}$$

giving a total of $4M$ equations of the $4M$ $y_{n,k}$'s. If we move the right-hand side of Eqs. (2.3) to the left-hand side and multiply by $\Delta x'_k$ we are then left with a set of homogeneous equations. Labeling these equations $E_{n,k}$ for $n = 1, \ldots, 4$, $k = 2, \ldots, M$, the problem is now to solve $E_{n,k} = 0$ and Eq. (2.4) simultaneously. Given an initial guess $\mathbf{y}_k$ we can improve the solution using the expansion

$$\mathbf{E}_k(\mathbf{y}_k + \Delta\mathbf{y}_k, \mathbf{y}_{k-1} + \Delta\mathbf{y}_{k-1}) \approx \mathbf{E}_k(\mathbf{y}_k, \mathbf{y}_{k-1}) + \sum_{n=1}^{4} \frac{\partial \mathbf{E}_k}{\partial y_{n,k-1}} \Delta y_{n,k-1} + \sum_{n=1}^{4} \frac{\partial \mathbf{E}_k}{\partial y_{n,k}} \Delta y_{n,k}, \tag{2.5}$$

where we want the left hand side to equal zero. This gives a system of linear equations to solve for the $\Delta\mathbf{y}_k$'s. We then add the $\Delta\mathbf{y}_k$ to the $\mathbf{y}_k$ to obtain a closer approximation. This process is repeated until the maximum value of $|E_{n,k}|$ is less than 10^{-6}.

The mesh of points x'_k are chosen at the start of the relaxation. We want the range to be large enough so that at the end points the functions are already close to their values at infinity. By trying different values for the end points we have found that at $x' = \pm 100$ the functions are

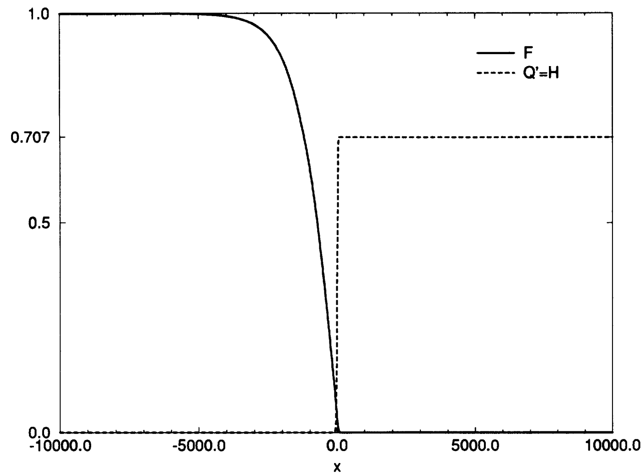

FIG. 1. Magnitude of the order parameter F and the magnetic field H for $\kappa = 10^{-3}$, which corresponds to a type-I superconductor. Lengths are in units of the penetration depth.

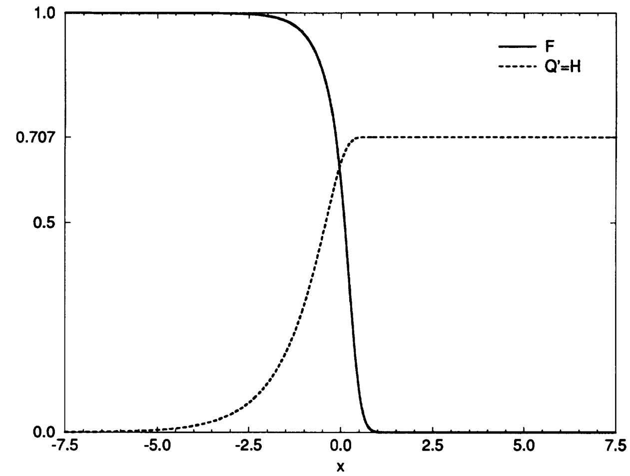

FIG. 2. Magnitude of the order parameter F and the magnetic field H for $\kappa = 10$, which corresponds to a type-II superconductor. Lengths are in units of the penetration depth.

all sufficiently close to their boundary values at infinity that imposing the conditions (2.4) here do not significantly affect the solutions. We also want the mesh to be fine enough to accurately pick up the detail of rapidly varying areas. Knowing that the solutions all show the most change in a small area (taken to be around 0) and are relatively linear outside, we have manufactured a grid with more closely spaced points in the center than the edges using a total of 2001 points. The spacing was chosen so as to make the change in y_1 (i.e., F) from one mesh point to the other roughly constant.

Convergence for this algorithm depends greatly upon the initial guess. For $0.1 < \kappa < 10$ the solution is obtained within about 30 iterations from our initial guess. Since we are interested in finding solutions for a large range of κ it is advantageous to use the previous solution to start a new solution changing κ slightly each time. In going from κ =0.1 to 0.001 by 0.001 each subsequent solution was obtained in only three iterations.

The results of our computations for large and small κ are shown in Figs. 1 and 2. In Fig. 1, $\kappa = 10^{-3}$, and we see that the magnetic field is essentially a step function, as suggested by the analysis in the next section. In this case the field only penetrates a short distance into the superconducting region, and therefore the full positive energy of flux expulsion is obtained, resulting in a positive surface tension (type-I superconductor). In Fig. 2 $\kappa = 10$; we see that the magnetic field penetrates far into the superconducting region, so that the positive energy of flux expulsion is reduced. However, the negative condensation energy is very large here, resulting in a net negative surface tension (type-II superconductor). From the numerical solutions we computed the surface tension and kinetic coefficient using Eqs. (1.7), (1.8), (1.9), and (1.10). Since our mesh spacing is already adapted to rapidly varying parts of the solution, we used a basic trapezoidal rule to calculate the integrals I_1 and I_2. These results are tabulated in Table I. As expected, the surface tension passes through zero at $\kappa = 1/\sqrt{2}$, separating type-I from type-II superconductors. In the following sections we will compare these numerical results against asymptotic solutions of the Ginzburg-Landau equations.

TABLE I. Representative numerical results from the solution of the Ginzburg-Landau equations.

κ	$I_1(\kappa)$	$I_2(\kappa)$	$\bar{\sigma}_{ns}$	$\bar{\Gamma}^{-1}$ [a]
0.001	0.000926	0.0000161	910	0.000910
0.01	0.00891	0.000516	84.0	0.00840
0.05	0.0414	0.00586	14.2	0.0356
0.1	0.0782	0.0169	6.13	0.0613
0.2	0.144	0.0495	2.36	0.0942
0.3	0.202	0.0943	1.19	0.107
0.4	0.254	0.150	0.648	0.104
0.5	0.301	0.217	0.338	0.0845
0.6	0.345	0.294	0.142	0.0511
$1/\sqrt{2}$	0.388	0.388	0.000219	0.000109
1.0	0.490	0.706	-0.216	-0.216
5.5	1.16	17.2	-0.529	-16.0
10.0	1.43	55.8	-0.544	-54.4

[a]We have chosen $(2\pi\hbar\sigma^{(n)}/m\gamma) = 1$ for the purposes of illustration.

III. SURFACE TENSION

The surface tension is the excess free energy per unit area due to the presence of the interface. As shown by Ginzburg and Landau[6] (see also Ref. 9), for $\kappa \ll 1/\sqrt{2}$, $\bar{\sigma}_{\rm ns} = 2\sqrt{2}/3\kappa + O(\kappa^{-1/2})$. Unfortunately, we have found that the lowest order expansion provides a very poor approximation to the surface tension, except for very small values of κ (less than 10^{-3}); this fact was also noted by Ginzburg and Landau.[6] Therefore, in this section we will first generalize their result somewhat by calculating the next order term in the expansion.

In the small-κ limit it is convenient to work with the rescaled Ginzburg-Landau equations, Eqs. (2.1) and (2.2). The lowest order approximation is obtained by setting the first term in the second Ginzburg-Landau equation, Eq. (2.2), equal to zero so that $F^2Q = 0$. In the superconducting phase $Q = 0$ with F in the superconducting phase determined by Eq. (2.1) with $Q = 0$; the solution to this equation is $F(x') = -\tanh(x'/\sqrt{2})$, for $x' > 0$. When this solution is substituted into the expression for the surface tension it produces the lowest order expansion derived by Ginzburg and Landau.[6] To calculate the next order term, we need to take this expression for the order parameter and substitute it back into Eq. (2.2), and then solve for Q. For $x' > 0$ (the normal phase), we have $Q''_> = 0$, which integrates to $Q_>(x') = x'/\kappa\sqrt{2} + C_1$, with C_1 a constant to be determined by matching onto the $x' < 0$ solution. For $x' < 0$ (the superconducting phase), the vector potential satisfies

$$\kappa^2 Q''_< - \tanh^2(x'/\sqrt{2})Q_< = 0. \tag{3.1}$$

Although this equation does not appear to have an explicit analytical solution, for $\kappa \ll 1$ we can use the WKB method to obtain an asymptotic solution. Some care is necessary as this equation has a second-order turning point at $x' = 0$. The uniformly valid asymptotic solution (i.e., a solution valid both near and away from the turning point) is[10]

$$Q_<(x') = C_2 \frac{2^{5/8}\Gamma(3/4)}{\pi} \frac{1}{\kappa^{1/4}} \left[\frac{\ln\cosh(x'/\sqrt{2})}{-\tanh(x'/\sqrt{2})}\right]^{1/2} \times K_{-1/4}\left[\frac{\sqrt{2}}{\kappa}\ln\cosh(x'/\sqrt{2})\right], \tag{3.2}$$

where C_2 is a second constant of integration and $K_{-1/4}(z)$ is the modified Bessel function of order $-1/4$. The integration constants are determined by matching the solutions and their derivatives at $x' = 0$, with the result

$$C_1 = C_2 = -\frac{\pi}{2^{3/4}\Gamma(3/4)^2}\frac{1}{\sqrt{\kappa}}, \tag{3.3}$$

so that

$$Q_<(x') = -\frac{1}{2^{1/8}\Gamma(3/4)\kappa^{3/4}} \left[\frac{\ln\cosh(x'/\sqrt{2})}{-\tanh(x'/\sqrt{2})}\right]^{1/2} \times K_{-1/4}\left[\frac{\sqrt{2}}{\kappa}\ln\cosh(x'/\sqrt{2})\right]. \tag{3.4}$$

To calculate the surface tension, we substitute our solution into our expression for the surface tension, Eq. (1.7). For I_1 we have

$$I_1(\kappa) = 2\kappa\int_{-\infty}^{0} dx' \left[F^2 - F^4 - F^2Q^2\right] = \frac{2\sqrt{2}\kappa}{3} - \frac{2^{3/4}\pi}{8\Gamma(3/4)^2}\kappa^{3/2}, \tag{3.5}$$

and for I_2,

$$I_2(\kappa) = 2\kappa\int_{-\infty}^{0} dx'\, F^2Q^2 = \frac{2^{3/4}\pi}{8\Gamma(3/4)^2}\kappa^{3/2}. \tag{3.6}$$

Therefore, from Eq. (1.7) we find that the surface tension in the small-κ limit is

$$\bar{\sigma}_{\rm ns} = \frac{2\sqrt{2}}{3}\frac{1}{\kappa} - \frac{2^{3/4}\pi}{4\Gamma(3/4)^2}\frac{1}{\sqrt{\kappa}} + O(1). \tag{3.7}$$

The first term in the expansion was previously obtained by Ginzburg and Landau,[6] and the second term is the new result. This calculation can also be formulated as a variational calculation, with $F(x') = -\tanh(x'/\xi_v\sqrt{2})$ a trial solution for the order parameter; the WKB calcu-

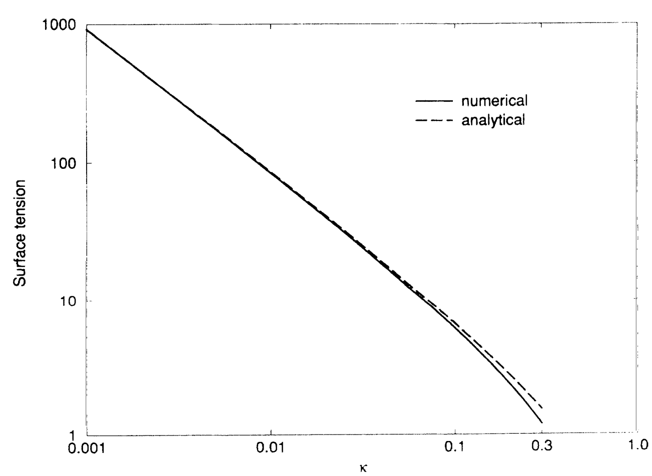

FIG. 3. Dimensionless surface tension $\bar{\sigma}_{\rm ns}$ as a function of the Ginzburg-Landau parameter κ for $10^{-3} < \kappa < 0.3$. The solid line is the numerical result, and the dashed line is the asymptotic expansion for $\kappa \ll 1$ given in Eq. (3.8).

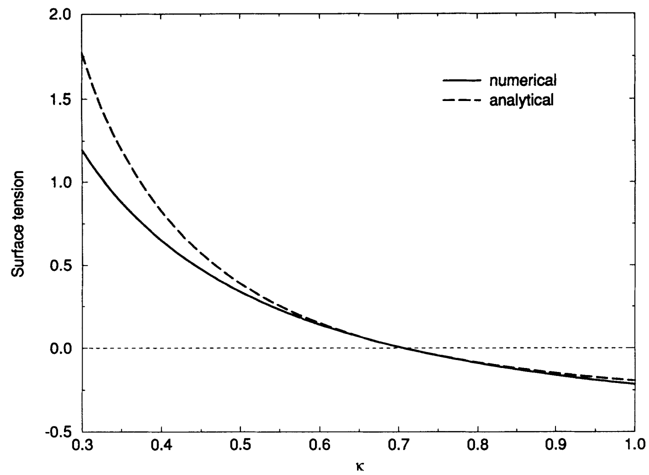

FIG. 4. Dimensionless surface tension $\bar{\sigma}_{ns}$ as a function of the Ginzburg-Landau parameter κ for $0.3 < \kappa < 1.0$. The solid line is the numerical result, and the dashed line is the asymptotic expansion about $\kappa = 1/\sqrt{2}$ given in Eq. (3.8).

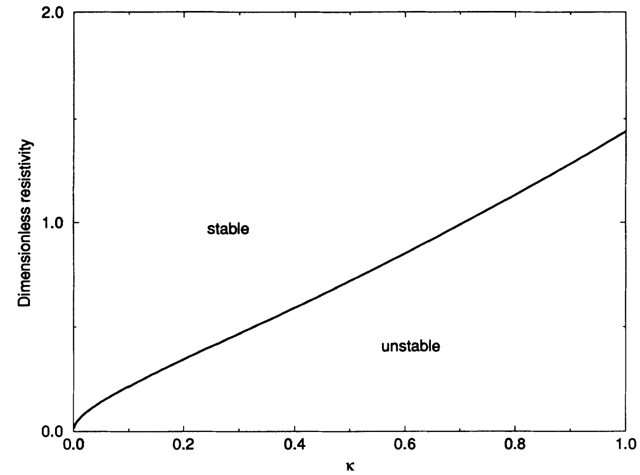

FIG. 6. Stability diagram for the kinetic coefficient, determined by Eq. (4.1) in the text. The y-axis is $m\gamma/2\pi\hbar\sigma^{(n)}$, the inverse of the dimensionless conductivity. For parameters in the region above the line the kinetic coefficient is positive, while the region below the line corresponds to a negative kinetic coefficient.

lation may be repeated, and the resulting solution used to calculate the surface tension as a function of the variational parameter ξ_v. Minimizing this expression and taking the small-κ limit, we obtain Eq. (3.7).[11]

For $\kappa \gg 1/\sqrt{2}$ the second derivative term in the first Ginzburg-Landau equation, Eq. (1.5), may be neglected, and the resulting algebraic equation solved for F as a function of Q. This expression is then substituted into the second Ginzburg-Landau equation, Eq. (1.6), and the resulting nonlinear differential equation may also be solved.[9] The surface tension in this limit is $\bar{\sigma}_{ns} = -4(\sqrt{2}-1)/3$.

The surface tension is zero at $\kappa = 1/\sqrt{2}$;[6] at this point the Ginzburg-Landau equations become integrable.[5] The

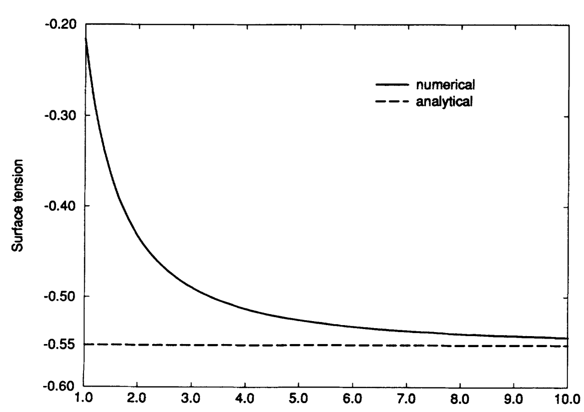

FIG. 5. Dimensionless surface tension $\bar{\sigma}_{ns}$ as a function of the Ginzburg-Landau parameter κ for $1.0 < \kappa < 10.0$. The solid line is the numerical result, and the dashed line is the limiting value for $\kappa \gg 1$ given in Eq. (3.8).

solutions may be used to carry out a local analysis of the surface tension about $\kappa = 1/\sqrt{2}$,[5] with the result $\bar{\sigma}_{ns} = 0.388(1/2\kappa^2 - 1)$.

Summarizing, we have

$$\bar{\sigma}_{ns} \approx \begin{cases} 0.943\kappa^{-1} - 0.880\kappa^{-1/2}, & \kappa \ll 1, \\ 0.388(1/2\kappa^2 - 1), & \kappa \approx 1/\sqrt{2}, \\ -0.552, & \kappa \gg 1. \end{cases} \quad (3.8)$$

In Fig. 3 the numerical results for the surface tension are compared to the asymptotic expressions for $\kappa \ll 1$. The asymptotic result is accurate for $\kappa < 0.2$; the $\kappa^{-1/2}$ correction in Eq. (3.7) is important for values of κ which are greater that 10^{-3}. In Fig. 4 we compare the numerical results against the asymptotic expansion derived in Ref. 5 for $\kappa \approx 1/\sqrt{2}$. The asymptotic expansion is reasonably accurate for $0.5 < \kappa < 1.0$. If we identify the small parameter in this expansion to be $\epsilon = 1/(2\kappa^2) - 1$, then this would imply that the expansion is accurate for values of ϵ as large as $\epsilon = 1$, a somewhat surprising result. In this figure we also see that the surface tension changes sign at $\kappa = 1/\sqrt{2}$, as expected. Figure 5 shows the surface tension in the range $1.0 < \kappa < 10.0$; for large κ we see that the surface tension is approaching the limiting value of -0.55.

IV. KINETIC COEFFICIENT

The kinetic coefficient $\bar{\Gamma}^{-1}$ is a function of κ, which is a ratio of length scales, as well as $2\pi\hbar\sigma^{(n)}/m\gamma$, which is the ratio of the diffusion constant for the order parameter, $D_\psi = \hbar/2m\gamma$, to the diffusion constant for the magnetic field, $D_H = 1/4\pi\sigma^{(n)}$. If this latter ratio is sufficiently large, the kinetic coefficient may also change sign (resulting in some sort of dynamic instability). By setting $\bar{\Gamma}^{-1} = 0$ in Eq. (1.8), we obtain an expression for the neutral stability curve :

$$\left[\frac{m\gamma}{2\pi\hbar\sigma^{(n)}}\right]_{\text{neutral}} = \frac{I_2(\kappa)}{I_1(\kappa)}. \tag{4.1}$$

The numerical result is plotted in Fig. (6).

In the limit of small κ we may use the previously derived expansions for I_1 and I_2 in Eqs. (3.5) and (3.6) to obtain

$$\bar{\Gamma}^{-1} = \frac{2\sqrt{2}}{3}\kappa - \left(1 + \frac{2\pi\hbar\sigma^{(n)}}{m\gamma}\right)\frac{2^{3/4}\pi}{8\Gamma(3/4)^2}\kappa^{3/2} + O(\kappa^2). \tag{4.2}$$

By setting $\bar{\Gamma}^{-1} = 0$, we see that for small κ the stability curve should behave as $\kappa^{1/2}$ for small κ, which is confirmed by the numerical results shown in Fig. 6.

V. DISCUSSION

We have investigated in some detail the behavior of the solutions to the one-dimensional Ginzburg-Landau equations using both numerical methods and asymptotic expansions. The numerical results for the surface tension of the normal-superconducting interface agree well with the small-κ asymptotic expansion developed in this paper. In addition, a recently developed asymptotic expansion of the surface tension for values of κ near $1/\sqrt{2}$ (Ref. 5) agrees with the numerical results over a surprisingly large range of κ values. We have also calculated the neutral stability curve for the kinetic coefficient, which will be important in studies of the dynamics of the normal/superconducting interface.[4,5]

Note added in proof. After these calculations were completed we realized that Eq. (3.1) does have an exact solution. By changing variables to $\xi = \tanh(x'/\sqrt{2})$ the resulting equation can be solved in terms of the associated Legendre function $P_\nu^\mu(\xi)$. The values of C_1 and C_2 calculated in this manner agree with the results in Eq. (3.3) for $\kappa \to 0$. However, we have not succeeded in calculating I_1 and I_2 using the exact solutions.

ACKNOWLEDGMENTS

This work was supported by NSF Grant No. DMR 92-23586, and by the Alfred P. Sloan Foundation.

* Present address: Department of Physics, S.U.N.Y. at Stony Brook, Stony Brook, NY 11794-3800. Electronic address: josborn@ic.sunysb.edu

† Electronic address: atd2h@virginia.edu

[1] H. Frahm, S. Ullah, and A. T. Dorsey, Phys. Rev. Lett. **66**, 3067 (1991).

[2] F. Liu, M. Mondello, and N. D. Goldenfeld, Phys. Rev. Lett. **66**, 3071 (1991).

[3] D. A. Kessler, J. Koplik, and H. Levine, Adv. Phys. **37**, 255 (1988).

[4] S. J. Chapman, Q. Appl. Math. (to be published).

[5] A. T. Dorsey, Ann. Phys. **233**, 248 (1994).

[6] V. L. Ginzburg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1064 (1950); for an English translation, see *The Collected Papers of L. D. Landau*, edited by D. ter Haar (Gordon and Breach Science Publishers, Inc., New York, 1969), p. 546.

[7] There is reference to some numerical work in Ref. 6, where it was determined that for $\kappa = 0.165$, $\bar{\sigma}_{\text{ns}} \approx 3$.

[8] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in FORTRAN*, 2nd ed. (Cambridge University Press, Cambridge, UK, 1992), Chap. 17.

[9] See D. Saint-James, G. Sarma, and E. J. Thomas, *Type-II Superconductivity* (Pergamon Press, Oxford, 1969), pp. 31–37.

[10] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978), p. 541, Problem 10.26. Our form of the result is obtained by using $D_{-1/2}(z) = (z/2\pi)^{1/2}K_{-1/4}(z^2/4)$, where $D_{-1/2}(z)$ is the parabolic cylinder function of order $-1/2$; see I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products, Corrected and Enlarged Edition* (Academic Press, New York, 1980), pp. 1062–1064.

[11] A similar variational calculation for a piecewise linear trial order parameter was carried out by H. W. Lewis, Phys. Rev. **104**, 942 (1956). The surface tension calculated in this manner will not extrapolate to the asymptotically exact small-κ result.