BCS-Bose model of exotic superconductors: Generalized coherence length

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Analytic expressions are derived for the root-mean-square (rms) radius of a pair of fermions in a BCS many-fermion state in one, two, and three dimensions, in terms of the BCS gap energy and the associated chemical potential. These expressions are valid for any coupling strength of *any* pair interaction model implying a momentum-independent gap energy. The latter holds, e.g., for an attractive δ pair potential examined in the one-dimensional (1D) case (whose N-fermion ground state can be determined exactly) or for the BCS (electron-phonon) model interaction in any dimension. Weak-coupling and/or high-density limits for the rms radius are identical in 1D, 2D, and 3D, and reduce to the familiar well-known Pippard result to within a factor of order unity. In contrast, strong-coupling and/or low-density limits coincide in 1D and 3D, but differ by a factor of order unity in the 2D limit, and in each case are essentially the size of a single, isolated pair. The 1D δ interaction McGuire-Yang-Gaudin many-fermion model is studied in detail. The interaction renormalization scheme of Miyake and of Randeria, Duan, and Shieh, and the BCS interaction model, both in 2D, are employed to analyze cuprate superconductor empirical results. Reasonable agreement between theoretical rms radii with experimental coherence lengths suggests that cuprates can be described moderately well as *weakly coupled* superconductors within the BCS-Bose formalism.

I. INTRODUCTION

The coherence length ξ_0 measuring the Cooper pair radius, and the magnetic field "penetration depth" Λ, allow categorizing conventional superconductors into two kinds: (a) type-I (or Pippard) superconductors, with $\Lambda \ll \xi_0$, are generally nontransition metals for which the London equation must be modified with Pippard corrections; and (b) type-II (or London) superconductors, with $\Lambda \gg \xi_0$, are usually transition metals or intermetallic compounds like Nb_3Sn, V_3Ga, etc., for which the London equation is accurate for weak fields.

Interest in these two characteristic lengths resides in classifying conventional superconductors (carrier density $\sim 10^{22}$ cm^{-3}), where ξ_0 is 3–4 orders of magnitude *larger* than the interparticle spacing. This interest has been renewed with the discovery of high-T_c superconductors[1] (densities $\sim 10^{21}$ cm^{-3}) with their extremely short coherence length (approaching an average interparticle spacing). Since typically k_F^{-1} is of the order of the average interparticle spacing, the dimensionless parameter $\xi_0 k_F$ for high-T_c materials is conjectured to be $\sim O(1)$, i.e., intermediate somewhere between the *BCS limit* ($\xi_0 k_F \gg 1$) of large, overlapping, weakly coupled Cooper pairs, and what one might call a *Bose limit* ($\xi_0 k_F \ll 1$) of well-separated, "local," bosonic pairs. This latter extreme is perhaps realized in the controversially small concentrations of $\sim 10^{15}$ cm^{-3} in, e.g., Zr-doped $SrTiO_3$.[2]

The penetration depth is a measurable quantity[3] and the coherence length can also be obtained indirectly from experiments. Theoretically, the coherence length most commonly used is the familiar (weak-coupling) Pippard expression $\xi_0 = \hbar v_F/\pi\Delta$ associated with BCS theory, where Δ is the gap energy parameter; this is very nearly the value $\hbar v_F/4\Delta$ estimated from uncertainty-principle arguments valid for small Δ/E_F.

In this paper, using a one-dimensional (1D) fermion fluid with pairwise attractive δ interactions (the McGuire-Yang-Gaudin model[4]) treated within the BCS theory[5] but *without* setting the chemical potential equal to the Fermi energy, we first obtain closed analytic expressions for the root-mean-square pair radius to track its evolution from the BCS limit (weak-coupling and/or high-density) to the Bose limit (strong-coupling and/or low-density). These results are then compared with similarly derived results in 2D and 3D using more realistic interaction models. In Sec. II we briefly sketch the (exactly soluble) 1D model solved within BCS theory in an effort to test[5] the latter; in Sec. III analytical expressions are derived for the root-mean-square pair radius in 1D and compared with results in 2D and in 3D with the BCS interaction model; in Sec. IV we use the 2D low-density interaction renormalization scheme of Miyake and of Randeria, Duan, and Shieh[6] (MRDS) as well as the BCS model interaction to analyze the coherence length experimental results of selected cuprate superconductors; and Sec. V contains our conclusions that cuprates are moderately well described as weakly coupled superconductors in the BCS-Bose picture.

II. THE MCGUIRE-YANG-GAUDIN MODEL IN BCS THEORY

Consider a system of $N \gg 1$ fermions of mass m and degeneracy 2 (say, spin up and spin down) in a box of length L interacting via a pairwise attractive δ potential. The Hamiltonian of this McGuire-Yang-Gaudin[4] (MYG) model system is

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{d^2}{dx_i^2} - v_0\sum_{i>j}\delta(x_i - x_j), \tag{1}$$

with v_0 a positive coupling constant. Introducing dimensionless coordinates $x'_i = \rho x_i$ where $\rho = N/L$ is the density, one can write the dimensionless Hamiltonian

$$H' \equiv \frac{mH}{\hbar^2\rho^2} = -1/2\sum_{i=1}^{N}\frac{d^2}{dx'^2_i} - \lambda\sum_{i>j}\delta(x'_i - x'_j), \tag{2}$$

where the dimensionless coupling constant $\lambda \equiv mv_0/\hbar^2\rho$ ranges between 0 (weak-coupling and/or high-density) and ∞ (strong-coupling and/or low-density). The Hamiltonian (2) is exactly solvable[4,5,8] for the lowest N-body quantum state for all values of λ, and is of interest in superconductivity as it possesses the *same dynamics* as the 3D "jellium" electron gas model in that coupling strength and density scale reciprocally to each other.

It is well known that the BCS theory for *any* (S-wave) interaction $V_{kk'}$ (in any dimensionality D) given by

$$V_{kk'} \equiv L^{-D}\int_{L^D} d^D r \int_{L^D} d^D r'\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, V(\mathbf{r},\mathbf{r}')\, e^{i\mathbf{k}'\cdot\mathbf{r}'} \tag{3}$$

implies the (at worst numerical) solution of the gap equation

$$\Delta_k = -\sum_{k'} V_{kk'} v_{k'}\sqrt{1 - v_{k'}^2}, \tag{4}$$

to be carried out *self-consistently*[9] with that of the number equation

$$N = 2\sum_k v_k^2. \tag{5}$$

This self-consistency *defines* the BCS-Bose model. Here, the BCS transformation coefficients v_k are given by

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\xi_k}{E_k}\right), \tag{6}$$

where E_k are the quasiparticle (bogolon) energies

$$E_k = \sqrt{\xi_k^2 + \Delta_k^2}, \tag{7}$$

while ξ_k can be the Hartree-Fock single-particle energies ϵ_k relative to the chemical potential μ, namely

$$\xi_k \equiv \epsilon_k - \mu. \tag{8}$$

For the pair interaction in (1) $V_{kk'}$ of (3) is simply v_0/L, since then $V(\mathbf{r},\mathbf{r}') \equiv -v_0\delta(\mathbf{r}-\mathbf{r}')\delta(\mathbf{r})$, so that the gap parameter (4) becomes *independent of* k. Thus, our two coupled equations (4) and (5) simplify to

$$1 = \frac{v_0}{4\pi}\int_{-\infty}^{\infty} dk \frac{1}{\sqrt{\xi_k^2 + \Delta^2}}\ , \tag{9}$$

$$N = \frac{L}{2\pi}\int_{-\infty}^{\infty} dk\left(1 - \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2}}\right), \tag{10}$$

where

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \frac{1}{2}\rho v_0 - \mu \equiv \frac{\hbar^2 k^2}{2m} - \mu'. \tag{11}$$

The self-consistent numerical solution[5] of (9) and (10) then determines μ and Δ for each value of the coupling λ. Only in the limit of weak coupling (BCS regime) does the chemical potential coincide with the Fermi energy, $\mu = E_F = \hbar^2 k_F^2/2m$, where $k_F = \pi\rho/2$ in 1D. In the strong-coupling (Bose) limit the chemical potential approaches[5,6] one-half the pair binding energy, $\mu = -E_0(2)/2$, with $E_0(2) = mv_0^2/4\hbar^2$. Thus, as coupling λ is increased, μ *decreases* and changes sign from positive to negative. As shown in Ref. 5, in the two extremes of λ the BCS many-fermion theory reproduces the exact (i.e., Schrödinger) results for the ground-state energy per particle of the MYG model. As expected, very good agreement is obtained between the BCS theory and the exact results in the weak-coupling limit ($\lambda^{-1} \geq 3$), but the agreement grows poorer[5] for intermediate values of λ ($0 < \lambda^{-1} < 1$). Nevertheless, the ground-state BCS energy agrees much better with the exact ground-state energy than the best Hartree-Fock result[8] for the same Hamiltonian (1). Finally, the energy gap has the same formal expression in 1D, 2D, and 3D Fermi gases, with a bifurcation[6] in its expression as a function of the coupling occurring precisely when $\mu = 0$, at which point the energy gap itself is a maximum. This point can thus be viewed as the *boundary* between BCS and Bose regimes.

III. ROOT-MEAN-SQUARE PAIR RADIUS

A. One dimension

We first calculate the root-mean-square radius $x_{\rm rms}^{\rm quant}$ of a single bound pair of fermions interacting with the pair potential in (1). Since the pair wave function $\psi(x)$ is just $e^{-mv_0|x|/2\hbar^2}$, with x the relative coordinate, then

$$(x_{\rm rms}^{\rm quant})^2 \equiv \frac{\int_{-\infty}^{\infty} dx\ x^2 e^{-mv_0|x|/\hbar^2}}{\int_{-\infty}^{\infty} dx\ e^{-mv_0|x|/\hbar^2}} = \sqrt{2}\frac{\hbar^2}{mv_0}. \quad (12)$$

Similarly, the root-mean-square radius $x_{\rm rms}^{\rm Coop}$ of a Cooper pair under the same interaction (1) will be similar to (12) *except* that the pair wave function $\psi(x) \equiv \sum_k C_k e^{ikx}$, with $C_k \equiv 0$ for $-k_F < k < k_F$, ensuring that states occupied by the $N-2$ background fermions are not occupied by either Cooper-pair partner fermion in accordance with the Pauli exclusion principle. Thus

$$(x_{\rm rms}^{\rm Coop})^2 \equiv \frac{\int_{-\infty}^{\infty} dx\ x^2 \mid \psi(x) \mid^2}{\int_{-\infty}^{\infty} dx \mid \psi(x) \mid^2} = -\hbar^2 \frac{\int dk\ C_k^* \frac{d^2}{dk^2} C_k}{\int dk\ C_k^* C_k} = \frac{4}{3}\left(\frac{\hbar v_F}{\Delta^{\rm Coop}}\right)^2, \quad (13)$$

where further details can be found in Ref. 10 and $\Delta^{\rm Coop}$ is the (positive) binding energy of a single Cooper pair.[7]

On the other hand, to measure the pair radius *inside* the nontrivial interacting BCS ground-state condensate we define

$$r_{\rm rms}^2 \equiv \frac{\int d^D k\ \psi_k^*\ r^2\ \psi_k}{\int d^D k\ \psi_k^*\ \psi_k}, \quad (14)$$

where D is the system dimension, and ψ_k is an appropriate pair wave function in the momentum representation so that $r^2 \to -\nabla_k^2$. After the early work of Eagles,[11] it was recognized by Leggett,[12] and further clarified by Nozières and Schmitt-Rink,[13] that if $\Delta_k/2E_k \equiv \psi_k$, the BCS gap equation (4) *at low density* reduces in leading order to

$$\left(\frac{\hbar^2 k^2}{m} - 2\mu\right)\psi_k = \sum_{k'} V_{kk'}\psi_{k'}, \quad (15)$$

which is the Schrödinger equation in momentum space for an isolated pair of fermions, where 2μ plays the role of the eigenvalue. This remarkable result holds in *any* dimension and for *any* interaction $V_{kk'}$, since (4) is easily seen to vanish as $\rho \equiv N/L^D$ tends to zero, by using (5)–(8).

For any interaction model for which $\Delta_k = \Delta$, [e.g., the δ function model (1), the 2D low-density renormalization MRDS scheme[6] for S waves, or the 3D BCS interaction model], the integrals of Eq. (14) *can be evaluated analytically*; this is the main point of this paper. In the 1D case (and for any pair interaction implying a momentum-independent gap energy) we obtain the closed expression (see the Appendix)

$$r_{\rm rms}^2 = \frac{\hbar^2}{4m\Delta^2(\mu^2+\Delta^2)}[\mu(\mu^2+\tfrac{3}{4}\Delta^2) + \tfrac{1}{2}\Delta(\mu^2+\tfrac{3}{2}\Delta^2)\tan(\phi/2)], \quad 1D, \quad (16)$$

where

$$\phi = \tan^{-1}\left(\frac{\Delta}{\mu}\right), \quad \mu > 0,$$
$$\phi = \pi + \tan^{-1}\left(\frac{\Delta}{\mu}\right), \quad \mu < 0. \quad (17)$$

In particular, this result holds for the 1D model (1) if μ is replaced by μ' as defined in (11). We examine the two extreme limits of this result. In weak coupling, $v_0 \to 0$ and (4) implies that $\Delta \to 0$, so that

$$\mu' \to E_F > 0 \quad (\text{as } \lambda \to 0\ ,\ v_0 \to 0,\ \rho \text{ finite}). \quad (18)$$

Hence,

$$\tan(\phi/2) \to \frac{\Delta}{2\mu'} \quad (19)$$

and (16) reduces to

$$r_{\rm rms}^2 \to \frac{\hbar^2 E_F}{4m\Delta^2} = \frac{1}{8}\left(\frac{\hbar v_F}{\Delta}\right)^2 \equiv (r_{\rm rms}^{\rm BCS})^2. \quad (20)$$

We have thus defined *three* distinct lengths $r_{\rm rms}^{\rm quant}$, $r_{\rm rms}^{\rm Coop}$, $r_{\rm rms}^{\rm BCS}$ in (12), (13), and (20), respectively. (Note that $r_{\rm rms}^{\rm BCS} \equiv \hbar v_F/\sqrt{8}\Delta \simeq \hbar v_F/\pi\Delta \equiv \xi_0$, the Pippard coherence length mentioned earlier.) We now show that in the weak-coupling limit these three lengths *differ drastically in magnitude.* It can be shown[5,7] that for weak coupling, $\lambda \equiv mv_0/\hbar^2\rho \to 0$, the 1D Cooper-pair binding energy

$$\Delta^{\rm Coop} \to 8E_F e^{-\pi^2/\lambda} \quad (\text{as } \lambda \to 0). \quad (21)$$

On the other hand, the BCS gap energy behaves as

$$\Delta \to 8E_F e^{-\pi^2/2\lambda} \quad (\text{as } \lambda \to 0). \quad (22)$$

Clearly, for $\lambda \to 0$, $\Delta \gg \Delta^{\rm Coop}$. Hence, comparing (12), (13), (21), and (22) gives

$$r_{\rm rms}^{\rm Coop} : r_{\rm rms}^{\rm BCS} : r_{\rm rms}^{\rm quant} = e^{-C/v_0} : e^{-C/2v_0} : D/v_0 \quad (\text{for } v_0 \to 0 \text{ or } \rho \to \infty), \quad (23)$$

where C and D are positive constants. Consequently

$$r_{\rm rms}^{\rm Coop} \gg r_{\rm rms}^{\rm BCS} \gg r_{\rm rms}^{\rm quant} \text{ (as } v_0 \to 0 \text{ or } \rho \to \infty). \quad (24)$$

In the opposite limit of very low density (5) and (4) again implies that $\Delta \to 0$, and (15) then means that

$$\mu' \to -\frac{1}{2}E_0(2) = -mv_0^2/8\hbar^2 < 0$$

$$\text{(as } \lambda \to \infty, \quad \rho \to 0, \quad v_0 \text{ finite).} \quad (25)$$

In this case, expanding $\tan^{-1}(\Delta/\mu')$ in (17) gives

$$\tan(\phi/2) \to -\frac{2\mu'}{\Delta} - \frac{\Delta}{2\mu'} + O\left(\frac{\Delta}{\mu'}\right)^3, \quad (26)$$

so that Eq. (16) then reduces to

$$r_{\rm rms}^2 \to \frac{\hbar^2}{2mE_0(2)} = 2\left(\frac{\hbar^2}{mv_0}\right)^2 \equiv (r_{\rm rms}^{\rm quant})^2. \quad (27)$$

Thus, the rms radius is precisely equal to the pair radius of an isolated pair, and by (24) is much smaller than the average interparticle spacing (or $r_{\rm rms}^{\rm quant}k_F \ll 1$) as appropriate to tightly bound local pairs.

As an illustration of how the coherence length evolves between the BCS limit of large overlapping Cooper pairs and that of the Bose regime, we display in Figs. 1 and 2 the ratios $r_{\rm rms}/r_{\rm rms}^{\rm BCS}$ and $r_{\rm rms}/r_{\rm rms}^{\rm quant}$ as a function of λ^{-1} for the 1D δ-potential many-fermion BCS system. This was done by numerically eliminating Δ and μ in (16) in favor of λ, by solving (9) and (10) simultaneously.[5] We plot $r_{\rm rms}/r_{\rm rms}^{\rm BCS}$ in Fig. 1 for $\tilde{\mu}' > 0$, using a dimensionless gap parameter $\tilde{\Delta}$ and chemical potential $\tilde{\mu}'$ defined by

$$\tilde{\Delta} \equiv \frac{\hbar^2}{mv_0^2}\Delta \text{ and } \tilde{\mu}' \equiv \frac{\hbar^2}{mv_0^2}\mu'. \quad (28)$$

It is easy to see, using (16), (20), and (28), that in the weak-coupling limit

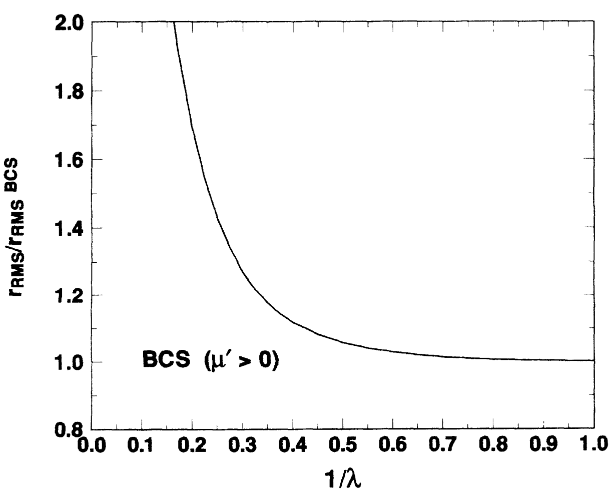

FIG. 1. Evolution of the rms radius $r_{\rm rms}$ for 1D many-fermion system (1) and (2) as given by (16), in units of $r_{\rm rms}^{\rm BCS}$ as defined by (20), for values of $\tilde{\mu}' > 0$ (BCS regime) as a function of λ^{-1}.

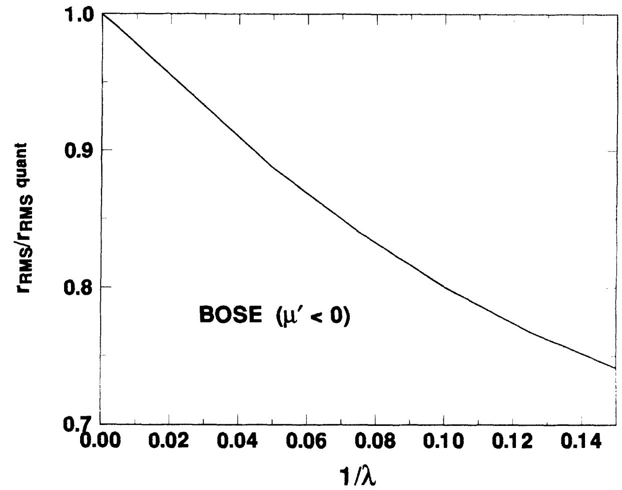

FIG. 2. Evolution of the rms radius $r_{\rm rms}$ for 1D many-fermion system as given by (16), in units of $r_{\rm rms}^{\rm quant}$ as defined by (12), for values of $\tilde{\mu}' < 0$ (Bose regime) as a function of λ^{-1}.

$$r_{\rm rms}/r_{\rm rms}^{\rm BCS} = \frac{2\lambda}{\pi}(2\mu')^{1/2} \to 1 \text{ (as } \lambda \to 0), \quad (29)$$

since $\tilde{\mu}' \simeq \tilde{E}_F \equiv \hbar^2 E_F/mv_0^2 = \frac{\pi^2}{8\lambda^2}$; this is displayed in Fig. 1. At first glance this limit is a rather reasonable approximation for values of $\lambda^{-1} \geq 0.5$. Nevertheless, for $\tilde{\mu}' = 0$ ($\lambda^{-1} = 0.16$), the ratio (29) $r_{\rm rms}/r_{\rm rms}^{\rm BCS} = \frac{\lambda}{\pi}(6\tilde{\Delta})^{1/2} = 2.04$. It follows that using of $r_{\rm rms}^{\rm BCS}$ for values of λ^{-1} between 0.16 and 0.5 can introduce large errors. For $\tilde{\mu}' < 0$, we have plotted $r_{\rm rms}/r_{\rm rms}^{\rm quant}$ in Fig. 2. The strong-coupling limit (27) here is a reasonable approximation only for values of $\lambda^{-1} \simeq 0$, and in the interval, say $0 < \lambda^{-1} < 0.16$ (which corresponds to $\tilde{\mu}' < 0$) the full expression (16) for $r_{\rm rms}$ must be used.

We emphasize that the coherence length given by (16) is formally identical to that obtained in 1D with *any* interaction model such that $\Delta_k \equiv \Delta$, e.g., the standard BCS interaction model. In this model, pairing in D dimensions emerges from a two-electron Schrödinger equation with an attractive electron-phonon interaction, mimicked in momentum space by

$$V_{kk'} = \begin{cases} -V & \text{if} \mu - \hbar\omega_D < \epsilon_k,\ \epsilon_{k'} < \mu + \hbar\omega_D, \\ 0 & \text{otherwise,} \end{cases} \quad (30)$$

where V is a positive coupling constant, $\epsilon_k \equiv \hbar^2k^2/2m$, and $\hbar\omega_D$ is the Debye energy. As shown in Ref. 13, the "pair wave function" in the momentum representation is also given by $\psi_k = \Delta/2E_k$ so that the coherence length is correctly given by (16) for *any* such interaction model. We now analyze the *two-* and *three-*dimensional cases.

B. Two dimensions

In 2D the formal expectation value equation (14) can also be evaluated analytically for $\Delta_k = \Delta$, and one obtains

$$r_{\rm rms}^2 = \frac{\hbar^2}{4m}\frac{1}{\Delta}\left[\frac{\mu}{\Delta} + \frac{\mu^2+2\Delta^2}{\mu^2+\Delta^2}\left(\frac{\pi}{2}+\tan^{-1}\frac{\mu}{\Delta}\right)^{-1}\right], \quad 2D. \tag{31}$$

This result coincides with that reported by Randeria *et al.*,[6] for the 2D Fermi gas model in the MRDS scheme, which *also* implies $\Delta_k \equiv \Delta$, which assumes that the underlying generic pair interaction is of sufficiently short range. Again, we exhibit results in the two extremes. For weak coupling

$$\mu \to E_F \quad \text{and} \quad \frac{\Delta}{\mu} \to 0 \tag{32}$$

so that (31) gives

$$r_{\rm rms}^2 \to \frac{\hbar^2 E_F}{4m\Delta^2} = \frac{1}{8}\left(\frac{\hbar v_F}{\Delta}\right)^2, \tag{33}$$

which is identical to the 1D limit result (20). In the opposite limit of very strong coupling

$$\mu \simeq -\frac{1}{2}E_0(2) \quad \text{and} \quad \frac{\Delta}{\mu} \to 0, \tag{34}$$

where $E_0(2)$ is again the (positive) binding energy of an isolated, single pair (see Ref. 6). Expanding $\tan^{-1}(\mu/\Delta)$ up to order $(\Delta/\mu)^3$ one obtains from (31) the limit

$$r_{\rm rms}^2 \to \frac{\hbar^2}{3m|\mu|} = \frac{2}{3}\frac{\hbar^2}{mE_0(2)} \tag{35}$$

so that one recovers the extreme Bose regime of noninteracting bosons with a pair size much smaller than the interparticle spacing, or $r_{\rm rms}k_F \ll 1$.

C. Three dimensions

For the 3D Fermi gas the formal expectation value in (14) can again be evaluated analytically if $\Delta_k = \Delta$, and one obtains (see the Appendix)

$$r_{\rm rms}^2 = \frac{\hbar^2}{4m\Delta^2(\mu^2+\Delta^2)^{1/2}}\left[\left(\mu^2+\frac{5}{4}\Delta^2\right) + \frac{\Delta\mu}{2}\tan(\phi/2)\right], \quad 3D, \tag{36}$$

where ϕ is again defined by (17). In the weak-coupling limit, when $\Delta \to 0$ and $\mu \to E_F$, use of (19) leads to

$$r_{\rm rms}^2 \to \frac{\hbar^2 E_F}{4m\Delta^2} = \frac{1}{8}\left(\frac{\hbar v_F}{\Delta}\right)^2. \tag{37}$$

In 3D we thus recover *the same BCS limit* as in the 1D and 2D Fermi gases (20) and (33), respectively, of large, overlapping, weakly bound, Cooper pairs, namely, a pair size much larger than the interparticle spacing. Note that the weak-coupling limits given by Eqs. (20), (33), and (37) are *identical* in all dimensions one, two, and three, and agree *up to a factor of order unity* with the (weak-coupling) Pippard expression[14] $\xi_0 = \hbar v_F/\pi\Delta$, or with the value $\hbar v_F/4\Delta$ derived via the well-known uncertainty-principle estimate. In the strong-coupling limit, $\Delta \to 0$ and again by (15) $\mu = -E_0(2)/2 < 0$, where $E_0(2)$ is now the binding energy of a single pair in vacuum, $\tan(\phi/2)$ is once more given by (17), and from (36) one obtains the limit

$$r_{\rm rms}^2 \to \frac{\hbar^2}{4m|\mu|} = \frac{\hbar^2}{2mE_0(2)}. \tag{38}$$

This expression agrees with the 1D result (27), but differs by a factor of order unity from the result for the 2D Fermi gas (35), a fact probably associated with the anomalousness[15] of 2D compared with 1D or 3D.

IV. COHERENCE LENGTH IN 2D AND CUPRATE SUPERCONDUCTORS

In this section we analyze empirical coherence length data reported for cuprate superconductors in terms of the theoretical rms radius (31) in 2D for *any* pair interaction model such that $\Delta_k = \Delta$, e.g., the BCS model interaction.[16] We shall imagine, say, a short-ranged attractive plus shorter-ranged repulsive interaction model implied by the MRDS renormalization scheme,[6] which must be sufficiently attractive to support an S-wave bound state of (positive) binding energy $E_0(2)$. The repulsion represents the screened Coulomb repulsion between electron holes; the attraction mimics the electron (hole)-phonon interaction of longer range. The MRDS scheme leads[6] to *closed expressions* for both the BCS gap energy Δ and the chemical potential μ, given by

$$\Delta = \sqrt{2E_F E_0(2)}, \tag{39}$$

$$\mu = E_F - \frac{1}{2}E_0(2), \tag{40}$$

where $E_F \equiv \hbar^2 k_F^2/2m = \hbar^2\pi\rho/m$, with ρ the 2D carrier density. A finite temperature extension of the MRDS scheme is due to van der Marel,[17] while the 3D treatment of Haussmann[18] is exceptionally clear.

Instead of depending *separately* on μ and Δ as in (31), the 2D rms radius can be written in terms of a *single variable* $0 < \eta \equiv E_0(2)/E_F < \infty$ as follows. Using (39) and (40), express the ratio

$$\frac{\mu}{\Delta} = \frac{E_F - E_0(2)/2}{\sqrt{2E_F E_0(2)}} = \frac{1}{\sqrt{2\eta}} - \sqrt{\frac{\eta}{8}} \equiv f(\eta). \tag{41}$$

Consequently, (31) simplifies to

$$r_{\rm rms} = \frac{1}{k_F(8\eta)^{1/4}} \left[f(\eta) + \frac{f(\eta)^2+2}{f(\eta)^2+1} \times \left(\frac{\pi}{2} + \tan^{-1} f(\eta) \right)^{-1} \right]^{1/2}, \quad (42)$$

or, to a function of the single variable $\eta \equiv E_0(2)/E_F$. This suggests that the pair binding energy $E_0(2)$ of the associate pair interaction model used might be more useful variable than the actual coupling strength of the pair interaction being employed. In terms of η (40) becomes simply

$$\frac{\mu}{E_F} = 1 - \frac{1}{2}\eta. \quad (43)$$

For example, for the 1D MYG model, $\eta \equiv E_0(2)/E_F = 2\lambda^2/\pi^2$, where $\lambda \equiv mv_0/\hbar^2\rho$ is the dimensionless coupling constant introduced in (2), and where $E_0(2) = mv_0^2/4\hbar^2$ was used. Weak coupling automatically implies $\eta \ll 1$, and expanding (42) for small η reveals that

$$r_{\rm rms} \to \frac{\pi}{2\sqrt{2}}\xi_0 \simeq 1.11\xi_0 \quad (\eta \ll 1), \quad (44)$$

where $\xi_0 \equiv \hbar v_F/\pi\Delta$ is the Pippard coherence length.[14]

Measured values of the zero-temperature gap parameter $\Delta(0)$ for cuprates are currently highly controversial.[19] There seems to be some consensus,[20] however, that the dimensionless ratio $2\Delta(0)/k_BT_c$ lies between 5 and 8, as compared with the standard BCS value of 3.53. Assuming this range and the values of T_c and T_F cited in Ref. 21 for the three cuprates YBaCuO, BiSrCaCuO, and TlBaCaCuO, Eq. (42) allows one to determine the resulting range of values for η and for μ/E_F listed in Table I. The ensuing closeness of μ to E_F would suggest that in a 2D BCS-Bose description, with either the MRDS or BCS interaction models, these materials are *weakly coupled.*

As a further test of this conclusion, Eq. (42) can now be used to determine the range of values of the rms radius $r_{\rm rms}$ consistent with the range 5–8 of $2\Delta(0)/k_BT_c$. These results are also listed in the table, but to compare with experiment we assume the "clean" limit [meaning that the mean free path $l \gg \xi_{ab}(0)$, the coherence length in the ab plane], which implies[10] that $\xi_{ab}(0) = 0.74r_{\rm rms}$. Ranges for $\xi_{ab}(0)$ and $\xi_{ab}^{\rm Pipp}(0) = 0.74\hbar v_F/\pi\Delta$ are listed in the table, and are in moderate agreement with experimental values[22–24] based principally on upper critical field $H_{c2}(0) = (\hbar/2e)/2\pi r_{\rm rms}^2$ data, last column.

Finally, Fig. 3 displays $(r_{\rm rms}k_F)^{-1}$ and μ/E_F, as well as a portion of $[\mu/\Delta(0)]^2$ vs η^{-1}, where the left extreme corresponds to the Bose, and the right to the BCS, regimes. Horizontal "error bar" symbols mark off ranges of η^{-1} listed in Table I for the three cuprates considered. The inset in the upper-right-hand corner shows both $(r_{\rm rms}k_F)^{-1}$ and μ/E_F in the relevant range of η^{-1} values, in an amplified scale.

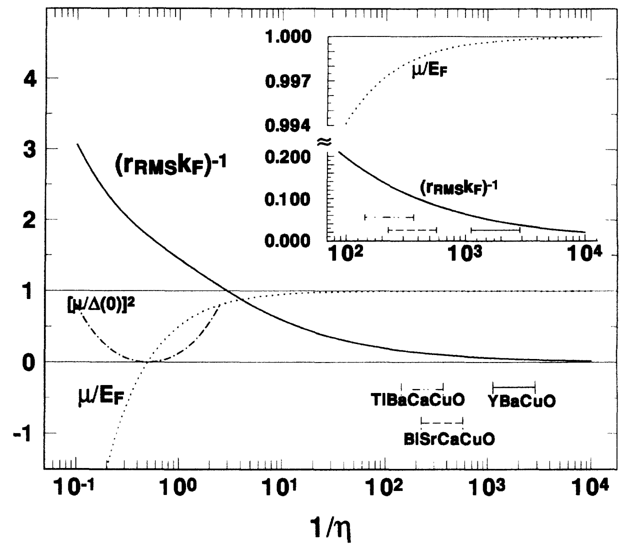

FIG. 3. Variations of $(r_{\rm rms}k_F)^{-1}$, $[\mu/\Delta(0)]^2$, and μ/E_F vs η^{-1} for either the MRDS or BCS interactions.

V. CONCLUSIONS

Analytic expressions have been obtained for root-mean-square pair radii $r_{\rm rms}$ in the BCS state in 1D, 2D, and 3D, for *any* interaction strength and *any* (S-wave) interaction model $V_{kk'}$ leading to a k -independent gap parameter, viz., $\Delta_k = \Delta$. All three cases reduce to the *same* expression for $r_{\rm rms}$ in the weak-coupling limit, which in turn differs only by a factor of order unity from the well-known Pippard value. In the opposite extreme of strong coupling, $r_{\rm rms}$ becomes the same in 1D and 3D, while for 2D it differs by a factor of order unity, and is essentially the root-mean-square radius of an isolated pair provided the N-fermion ground state is an ideal boson gas of pairs (which it indeed is in the 1D soluble MYG model treated).

The one-dimensional fermion fluid MYG model with pairwise δ attractive interaction solved in the BCS approximation allows one to track the evolution of the co-

TABLE I. Range of η, μ/E_F, $\xi_{ab}(0)$, and $\xi_{ab}^{\rm Pipp}(0)$ values as defined in text, for three cuprates and for the gap-to-T_c ratio $2\Delta(0)/k_BT_c$ ranging between 5 and 8, as explained in text, according to either the 2D MRDS or BCS interaction models. All lengths are in Å units.

Compound	T_c (K)	T_F (K)	k_F (Å^{-1})	η	μ/E_F	$\xi_{ab}(0)$	$\xi_{ab}^{\rm Pipp}(0)$	Expt.
YBaCuO	93	8807	0.4462	3.48–8.92$\times 10^{-4}$	0.9998–0.9995	45–28	40–25	66–12 (Ref. 22)
BiSrCaCuO	100	4234	0.3094	1.74–4.46$\times 10^{-3}$	0.9991–0.9978	29–18	26–16	35–18 (Ref. 23)
TlBaCaCuO	125	4234	0.3094	2.72–6.97$\times 10^{-3}$	0.9986–0.9965	23–15	21–13	31–18 (Ref. 24)

herence length in the regime intermediate between the two extremes of weak interaction ($\lambda \to 0$) and strong interaction ($\lambda \to \infty$).

The 2D MRDS and BCS interaction models are used to analyze cuprate data, from which it is concluded that they are moderately well described as weakly coupled superconductors, within the BCS regime ($\mu > 0$), in spite of their comparatively small pair sizes.

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APPENDIX

In this appendix we describe details in the calculation of the coherence length for the 1D model in the BCS-Bose approach. We start from expression (14) for $D = 1$, namely

$$r^2_{\rm rms} = \frac{\langle \psi_k \mid -\frac{d^2}{dk^2} \mid \psi_k \rangle}{\langle \psi_k \mid \psi_k \rangle}, \tag{A1}$$

where $\psi_k = \Delta/2E_k$. Recalling (7) and (11), this becomes

$$r^2_{\rm rms} = \frac{\hbar^2}{4m} \frac{\int_{-\infty}^{\infty} dk \, \frac{3\xi_k - \mu'}{[(\xi_k-\mu')^2+\Delta^2]^2}}{\int_{-\infty}^{\infty} \frac{dk}{[(\xi_k-\mu')^2+\Delta^2]}}. \tag{A2}$$

Factorize the polynomial in k as $(\xi_k - \mu')^2 + \Delta^2 = (\hbar^2/2m)^2(k-k_1)\cdots(k-k_4)$, with

$$k_1 = A - \left(\frac{\hbar^2}{2m}\right)^{1/2} \sqrt{\mu' - i\Delta}, \quad k_2 = \left(\frac{\hbar^2}{2m}\right)^{1/2} \sqrt{\mu' + i\Delta},$$
$$k_3 = \left(\frac{\hbar^2}{2m}\right)^{1/2} \sqrt{\mu' - i\Delta}, \quad k_4 = -\left(\frac{\hbar^2}{2m}\right)^{1/2} \sqrt{\mu' + i\Delta}. \tag{A3}$$

Thus, one can evaluate both integrals in the complex plane using the residue theorem. Choosing as the contour of integration the real axis and a semicircle in the upper half-plane, the two poles contributing are those corresponding to k_1 and k_2. For the first-order poles of the denominator of (A2) we find the residues

$$R_1 = \left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{1}{4i\Delta\sqrt{\mu' - i\Delta}},$$
$$R_2 = \left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{1}{4i\Delta\sqrt{\mu' + i\Delta}}. \tag{A4}$$

For the second-order poles of the numerator give

$$R_1 = \left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{4\mu'^2 - 3\Delta^2 - 6i\mu'\Delta}{16i\Delta^3(\mu' - i\Delta)^{3/2}},$$
$$R_2 = \left(\frac{2m}{\hbar^2}\right)^{1/2} \frac{4\mu'^2 - 3\Delta^2 + 6i\mu'\Delta}{16i\Delta^3(\mu' + i\Delta)^{3/2}}. \tag{A5}$$

One is then led to the final result

$$r^2_{\rm rms} = \left(\frac{\hbar^2}{2m}\right) \frac{1}{8\Delta^2} \sqrt{\mu' + \Delta^2} \frac{{\rm Re}\{\frac{4\mu'^2 - 3\Delta^2 + 6i\mu'\Delta}{(\mu'+i\Delta)^{3/2}}\}}{{\rm Re}\sqrt{\mu' + i\Delta}}, \tag{A6}$$

which is equivalent to (16). Similar methods were used in the 3D case to obtain (36); for $D = 2$ (14) can be directly computed on the real axis and gives (31), as reported by Randeria *et al.*[6]

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