

Transport properties of a one-dimensional two-component quantum liquid with hyperbolic interactions

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We present an investigation of the sinh-cosh (SC) interaction model with twisted boundary conditions. We argue that, when unlike particles repel, the SC model may be usefully viewed as a Heisenberg-Ising fluid with moving Heisenberg-Ising spins. We derive the Luttinger-liquid relation for the stiffness and the susceptibility, both from conformal arguments, and directly from the integral equations. Finally, we investigate the opening and closing of the ground-state gaps for both SC and Heisenberg-Ising models, as the interaction strength is varied.

In Refs. 1 and 2, we solved the integrable one-dimensional (1D) sinh-cosh (SC) model defined by the Hamiltonian

$$H = -\frac{1}{2} \sum_{1 \leq j \leq N} \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq j < k \leq N} v_{jk}(x_j - x_k). \quad (1)$$

The pair potential is given as

$$v_{jk}(x) = s(s+1) \left[\frac{1 + \sigma_j \sigma_k}{2 \sinh^2(x)} - \frac{1 - \sigma_j \sigma_k}{2 \cosh^2(x)} \right], \quad s > -1, \quad (2)$$

and the quantum number $\sigma = \pm 1$ distinguishes the two kinds of particles. We may usefully think of it as either representing charge or spin. For values of the interaction strength s in the range $-1 < s < 0$, the system exhibits two gapless excitation branches with different Fermi velocities as does the repulsive 1D Hubbard model,³ and thus may be classified as a typical two-component 1D Luttinger liquid.⁴ The asymptotic behavior of the correlation functions is given by finite-size arguments of conformal field theory. A Wiener-Hopf type calculation² shows that the spin-spin part of the dressed charge matrix is essentially identical to the dressed charge scalar in the Heisenberg-Ising (HI) model.⁵

In this paper, we will further explore the connection of the SC model with the HI model by examining the response of the system to a flux Φ . The addition of a flux is compatible with integrability and allows the study of the transport properties by an adiabatic variation of Φ . For the HI model, this has already been done^{6,7} for the interaction strength range $-1 \leq \Delta \leq 1$. We will show that the spin degrees of freedom of the SC model for $0 > s > -1$ may be usefully viewed as a HI model with moving HI spins. The presence of the translational degrees of freedom will simply renormalize the spin-spin

coupling.

We thus restrict ourselves in what follows to the unbound case $-1 < s < 0$, such that there are two gapless excitations corresponding to a particle hole and a two spin-wave continuum with excitation velocities v and v_s , respectively. Let us then modify the Bethe ansatz equations of Eq. (7) of Ref. 2 by threading them with a flux Φ . We have two coupled equations for N particles with pseudomomenta $\mathbf{k} = (k_1, \dots, k_N)$ and M spin waves with rapidities $\lambda = (\lambda_1, \dots, \lambda_M)$ on a ring of length L . The energy of a given state is $E(\mathbf{k}) = \frac{1}{2} \sum_{j=1}^N k_j^2$ and the total momentum is $P(\mathbf{k}) = \sum_{j=1}^N k_j$. Boosting the system by Φ will accelerate the two kinds of particles in opposite directions due to the two components being of equal but *opposite* charge. Therefore, we have no center-of-mass motion and $P = 0$. The energy of a given state will change as a function of Φ , and the energy shift of the ground state may be written as $\Delta E_0(\Phi) \equiv E_0(\Phi) - E_0(0) \equiv D\Phi^2/2L + O(\Phi^4)$, where D is called the stiffness constant and can be specified by perturbation arguments for Φ up to π .⁶ Note that since we do not have any center-of-mass motion, we can call D either spin or charge stiffness depending on what interpretation of σ we adopt. We choose the spin language for comparison with the HI model. However, the charge interpretation is probably more natural to describe transport properties. We, furthermore, caution the reader that the term charge stiffness has been previously used in lattice models to describe center-of-mass motion.

The twisted Bethe ansatz equations are given by

$$-Lk_j = 2\pi I_j(k_j) - \frac{M}{N} \Phi + \sum_{a=1}^M \theta_{0,-1}(k_j - \lambda_a) + \sum_{l=1}^N \theta_{0,0}(k_j - k_l), \quad (3a)$$

$$0 = 2\pi J_a(\lambda_a) + \Phi + \sum_{b=1}^M \theta_{-1,-1}(\lambda_a - \lambda_b) + \sum_{j=1}^N \theta_{0,-1}(\lambda_a - k_j). \quad (3b)$$

The two-body phase shifts for particle-particle, particle-spin wave, and spin wave-spin wave scattering, $\theta_{0,0}(k)$, $\theta_{0,-1}(k)$, and $\theta_{-1,-1}(k)$, respectively, have been given in Ref. 1. The particle quantum numbers I_j and the spin-wave quantum numbers J_a are integers or half-odd integers depending on the parities of N , M as well as on the particle statistics.² For simplicity, we use bosonic selection rules, although a purely fermionic or a mixed bose-fermi system may be studied along similar lines. In the ground state of the bosonic system, we have

$$I_1, I_2, \dots, I_N = -\frac{(N-1)}{2}, -\frac{(N-3)}{2}, \dots, \frac{(N-1)}{2},$$

$$J_1, J_2, \dots, J_M = -\frac{(M-1)}{2}, -\frac{(M-3)}{2}, \dots, \frac{(M-1)}{2}, \quad (4)$$

for both N and M even.

We start with some general considerations. Let us denote by $E_{\{\mathbf{I}, \mathbf{J}\}}(\Phi)$ the energy of a state specified by the $\Phi = 0$ set of quantum numbers $\{\mathbf{I}, \mathbf{J}\}$. We then adiabatically turn on the flux until we return to our initial state. The energy will also return to its initial value, although, it may return sooner; so, the period of the wave function will be an integer multiple of the period of the energy. We can define a topological winding number n to be the number of times the flux Φ increases by 2π before the state returns to its initial value. As Sutherland and Shastry have shown, the ground-state winding number of the HI model with $S_z = 0$ in the parameter range $-1 < -\cos(\mu) \equiv \Delta < 1$ is 2, implying charge carriers with one half the quantum of charge, *except* at isolated points $\Delta = \cos(\pi/Q)$, where $M \geq Q \geq 2$ is an integer. In particular, at $\Delta = 0$, the free particle wave function has periodicity $2\pi N_{\text{HI}}$, where N_{HI} is the number of HI sites, implying free acceleration in the thermodynamic limit.

We now note the following important fact: Choosing $\mu \equiv -\pi s$, the spin-wave-spin-wave phase shift $\theta_{-1,-1}$ is identical to the spin-spin phase shift in the HI model, and we may rewrite the equation for the rapidities as

$$N\bar{\theta}_{0,-1}(\lambda_a, \mu) \equiv N \sum_{j=1}^N \theta_{0,-1}(\lambda_a - k_j, \mu)/N$$

$$= 2\pi J_a(\lambda_a) + \Phi + \sum_{b=1}^M \theta_{-1,-1}(\lambda_a - \lambda_b, \mu), \quad (5)$$

which nearly is identical to the Bethe Ansatz equation of the HI model, as can be readily seen when we use the standard transformation for the HI momenta $p = f(\alpha, \mu)$. We then merely have to identify $\alpha \equiv \pi\lambda$. The sole effect of the pseudomomenta \mathbf{k} is an averaging on the left-hand side. Let us now restrict ourselves in what follows to the neutral (spin zero) sector such that we

have M particles with $\sigma = -1$ and M particles with $\sigma = +1$ for a total of $N = 2M$. Then, a discussion of the behavior of the rapidities λ for varying Φ exactly mimics the discussion of the HI momenta p in Ref. 6 at $S_z = 0$: As long as $|\Phi| \leq 2\pi(s+1)$, all λ stay on the real axis. At $\Phi = 2\pi(s+1)$, the largest root λ_M goes to infinity. For Φ increasing beyond this point, λ_M will reappear from infinity as $i\pi + \gamma_1$ until exactly at $\Phi = 2\pi$, $\lambda_M = i\pi$ ($\gamma_1 = 0$) and the remaining $M-1$ rapidities have redistributed themselves symmetrically around 0 on the real axis. However, as mentioned above, this behavior is different at the threshold values $s = (1-Q)/Q$. The momenta \mathbf{k} are always real and distributed about the origin. Equation (3b) simplifies at $\Phi = 2\pi(s+1)$ (and thus $\lambda_M = \infty$) and is in fact just the equation for $M-1$ rapidities in the ground state. So as in Ref. 6 using simple thermodynamical arguments, we may write

$$\Delta E_0[2\pi(s+1)] = E_0(N, M-1) - E_0(N, M) = 1/2L\chi^{-1}, \quad (6)$$

where χ is the susceptibility. Comparing with the definition of the stiffness constant D , we find $D = \chi^{-1}/4\pi^2(s+1)^2$.

On the other hand, we can read off the finite-size energy corrections for the SC model, and then finite-size arguments of conformal field theory give an expression for $\Delta E_0[2\pi(s+1)]$ in terms of the conformal weights, the dressed charge matrix Ξ and the spin-wave velocity v_s . The neutral sector dressed charge matrix is given in Equation (35) of Ref. 2 and thus we have $\chi^{-1} = 2\pi v_s(s+1)$. We may, therefore, express the stiffness D in terms of the spin-wave velocity as

$$D = v_s/2\pi(s+1). \quad (7)$$

We emphasize that this formula for D is true also for a system of purely fermionic particles. Shastry and Sutherland⁶ have given an exact formula for the stiffness constant in the HI model, by using the known expression for the H-I model spin-wave velocity $v_s = \pi \sin(\mu)/\mu$.⁸ No such expression is known for the SC model and we can only give v_s as

$$v_s = \frac{1}{2\pi} \frac{\int_{-B}^B e^{-\pi k/2s} \epsilon'(k) dk}{\int_{-B}^B e^{-\pi k/2s} \rho(k) dk}. \quad (8)$$

Here, we use the definitions of Ref. 2, Sec. II. However, written in terms of spin velocities the stiffness formulas are identical and only the values of the respective spin-wave velocities are different. Thus the presence of the translational degrees of freedom in the SC model simply renormalizes the spin-wave velocity.

We have iterated the Bethe Ansatz equations (3) in the neutral sector for reasonably large systems and density $d \equiv N/L = 1/2$ as a function of Φ . By our correspondence between the HI model, and the spin-wave part of the SC model, we expect *free spin waves* at $s = -1/2$. In the thermodynamic limit, we would thus expect the periodicity of the ground-state energy to be infinite. For a finite system, this will be reduced to a periodicity that scales with the system size. For the SC model we have

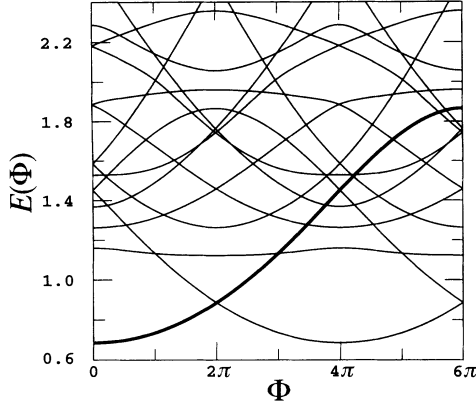


FIG. 1. The low-lying states for the SC model at $L = 12$, $N = 6$, and $M = 3$. The bold curve corresponds to the ground state and the winding number is $n = 6 = N$. Note the various level crossing in this free spin-wave case, especially the crossing of the ground state and the first excited state at $\Phi = 2\pi$.

indeed found that at $s = -1/2$ the periodicity of the ground-state energy is given as $2\pi N$. We may then speak of $s \rightarrow -1^+$ as the ferromagnetic critical point and $s \rightarrow 0^-$ as the antiferromagnetic critical point of the SC model. In Fig. 1 we show the full spectrum of low-lying states with zero momentum at $s = -1/2$ for $L = 12$, $N = 6$, and $M = 3$. The ground-state curve is emphasized and its periodicity is $6 \times 2\pi$.

Note that at $\Phi = 2\pi$ there is a level crossing between the ground state and the first excited state in Fig. 1. When the interaction strength changes from $s = -1/2$, immediately a gap opens between the ground state and the first excited state. Just as in the H-I model the periodicity is reduced to 4π . Note that a perturbation theory argument cannot describe this behavior. Figure 2 shows the behavior of the ground-state energy variation $L[1 - E(\Phi)/E(2\pi)]$ for $s = -1/3$ near $\Phi = 2\pi$ for different lattice sizes. The rounding is well pronounced and does not vanish as we increase the size.

Thus the behavior of the low-lying states in the SC and HI models is qualitatively the same, up to the renormalization of quantities such as the spin-wave velocity v_s . Let us briefly describe the behavior of the gaps in the HI model, keeping in mind the correspondence $\mu = -\pi s$. Increasing μ beyond $\pi/2$ ($\Delta = 0$), we see that the gap continues to widen up to a maximum value at $\mu \sim 7\pi/12$

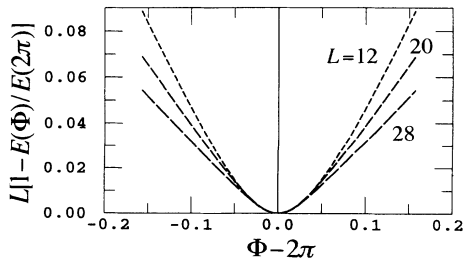


FIG. 2. Plot of the ground-state energy variation $L[1 - E(\Phi)/E(2\pi)]$ for the SC model at $s = -1/3$ for $L = 12, 20$, and 28 .

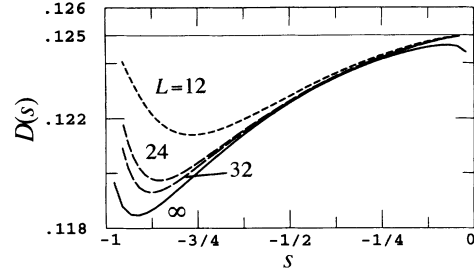


FIG. 3. The charge stiffness $D(s)$ for the SC model. The dashed curves correspond to $L = 12, 24$, and 32 and converge to $D(0) = 1/8$ at $s \rightarrow 0^-$. The solid curve comes from Eq. (7), which can be derived by conformal methods or from the Luttinger relation. (Note that as $s \rightarrow 0^-$, the solid curve does not converge to $1/8$. This is due to a buildup of numerical errors in the integration routine.)

($\Delta \sim 0.26$). It then closes up again exactly at $\mu = 2\pi/3$ ($\Delta = 1/2$). As has been noted before, this value of μ coincides with the appearance of a $Q = 3$ string in the ground state.⁵ Further increase of μ again opens, and then closes the gap at the threshold for the next-longer $Q = 4$ string. This behavior continues, and the threshold values accumulate as $\mu \rightarrow \pi$ ($\Delta \rightarrow 1$). In Fig. 3, we show the ground state and the first excited state of the HI model on a ring of $N_{\text{HI}} = 12$. Note that due to the finite size of the ring, we can only observe strings up to length $Q = 6$. We will present a more detailed finite-size study of the behavior of the gaps in the HI and SC model in another publication. We only mention that for fixed μ the gap scales with the system size as a negative power of N_{HI} , with variable exponent depending on the coupling constant μ .

The stiffness constant D is the curvature of the ground-state energy $E_0(\Phi)$ as a function of Φ . In Fig. 4, we show D for systems of 12, 24, and 32 lattice sites. We also show the behavior of D as given by Eq. (7). As $s \rightarrow 0^-$, the spin-wave velocity approaches the velocity of a noninteracting single-component model, i.e., $v_s \rightarrow \pi d/2$.² Thus D approaches the nonzero value $1/8$ which is compatible with the result of Ref. 6. Furthermore, the SC model exhibits a gap for $s > 0$ and so D is zero. Thus D exhibits a jump discontinuity at $s = 0$ just as in the HI model for $\Delta = -1$.

Note that Eq. (7) may also be written as $D\chi^{-1} = v_s^2$. This is nothing but the Luttinger relation for the spin-

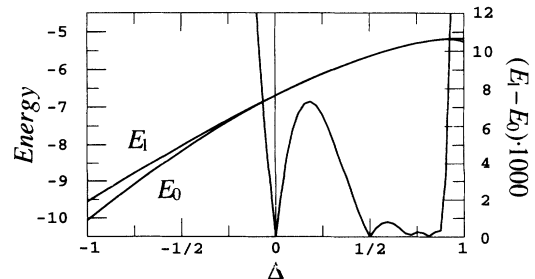


FIG. 4. Energy of the ground state and the first excited state and their difference in the HI model for $N_{\text{HI}} = 12$. Note the closing of the gap at $\Delta = \cos(\pi/Q)$ for $Q = 2, 3, 4, 5$.

wave excitations.⁴ Let us briefly explain how to derive this formula without using arguments of conformal field theory. In the thermodynamic limit, we convert Eq. (3) into a set of coupled integral equations as in Eq. (10) of Ref. 2. Here $\rho(k)$ and $\sigma(\lambda)$ are the distribution functions of particles and down spins, respectively. The density d and the magnetization \mathcal{M} are then given parametrically in terms of the integral limits B and C . We now use an iteration scheme, i.e., first, for $B = \infty$ and $\mathcal{M} = 0$, i.e. at half filling and zero magnetization, we calculate $\rho(k) \equiv \rho_0(k)$. We then use this $\rho_0(k)$ in the equation for $\sigma(\lambda)$ with C finite and B nearly ∞ . Finally, we use this $\sigma(\lambda)$ to calculate $\rho(k)$ and thus the effect on the momenta and the energy. Since we are only interested in the leading order correction terms, we may stop. Then the corrections to the energy are

$$\begin{aligned} \frac{\Delta E}{L} &\equiv \frac{1}{2} \left[D \left(\frac{\Phi}{L} \right)^2 + \chi^{-1} \mathcal{M}^2 \right] \\ &= \frac{v_s}{4\pi(1+s)} \left[\left(\frac{\Phi}{L} \right)^2 + [2\pi(1+s)]^2 \mathcal{M}^2 \right], \quad (9) \end{aligned}$$

where $\mathcal{M} = \frac{1}{2}d(1 - 2M/N)$. A complete account of this calculation can be found in Ref. 9.

The derivation of the Luttinger relation uses integral equations and as such is valid in the thermodynamical limit. Most of the other results given above have been de-

rived using Eq. (3). These equations, however, have been derived by the asymptotic Bethe Ansatz (ABA). This method is only correct in the thermodynamic limit.¹⁰ Thus all our finite-size results should exhibit correction terms. From the hyperbolic form of the pair potential (2), we may expect these corrections to be exponentially small in L . Indeed, a log-log plot of the ground-state energy versus L at fixed interaction strength shows a simple power law behavior already for $L \geq 6$. Thus the $L \rightarrow \infty$ behavior of the finite-size Bethe-Ansatz equations for the SC model does not seem to differ in any significant respect from usual finite size behavior for short ranged models. This further supports our use of the ABA in the present study.

In conclusion, we have shown that the SC model exhibits all the rich structure of the HI model for $-1 < s < 0$. In particular, there is a Luttinger relation for the spin waves just as in the HI model, that can be derived from (i) conformal arguments, (ii) an exact calculation in the thermodynamic limit, and (iii) is furthermore supported by numerical results for finite systems. Thus this yields credibility to both the conformal and the Luttinger approach in models solved by the ABA. Finally, we have reported an interesting behavior of the gaps in the HI and SC models.

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