# Brownian motion and magnetism

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We present an interesting connection between Brownian motion and magnetism. We use this to determine the distribution of areas enclosed by the path of a particle diffusing on a sphere. In addition, we find a bound on the free energy of an arbitrary system of spinless bosons in a magnetic field. The work presented here is expected to shed light on polymer entanglement, depolarized light scattering, and magnetic behavior of spinless bosons.

Consider a particle diffusing on a sphere. If the diffusing particle returns to its starting point at time $\beta$ its path subtends a solid angle $\Omega$ at the center of the sphere. We ask: *what is the probability distribution of* $\Omega$? This problem comes up if one considers a spin-$\frac{1}{2}$ system in a random magnetic field. As is well known, the state (up to a phase) of a spin-$\frac{1}{2}$ system can be represented as a point on the Poincaré sphere. Under the influence of a random Hamiltonian, the state of the system diffuses on the Poincaré sphere. From the work of Berry[1] and others, it is known that the system picks up a geometric phase $\gamma$ equal to half the solid angle swept out on the Poincaré sphere. To compute the distribution of geometric phases one is led to the question posed above. A closely related problem has already been studied in the context of polymer entanglement:[2] given that a Brownian path on the plane is closed at time $\beta$, what is the probability that it encloses a given area $A$?

In this paper we present a general method of solving these problems by using a connection between Brownian motion and magnetism. The qualitative idea is to use a magnetic field as a "counter," to measure the area enclosed in a Brownian motion. We derive a relation between the distribution of areas in a Brownian motion and the partition function of a magnetic system, which can be used to cast light on *both* subjects. Despite its apparent simplicity, this relation does not seem to have been noticed or exploited so far. Our main purpose here is to illustrate its usefulness. We first discuss the planar problem solved earlier. We then go on to solve the (as yet unsolved) problem of diffusion on the sphere. We also exploit the relation to learn about the magnetic properties of bosonic systems. Here we recover previously known results and arrive at some others. We conclude the paper with a few remarks.

Let a diffusing particle start from a point on a plane at time $\tau=0$. Given that the path is closed at time $\beta$ (not necessarily for the first time), what is the conditional probability that it encloses a given area $A$? By "area" we mean the algebraic area, including sign. The area enclosed to the left of the diffusing particle counts as positive and the area to the right as negative. This problem has been posed and solved[2] by polymer physicists, since it provides an idealized model for the entanglement of polymers. We present a method of solving this simple problem.

Let $\{\vec{x}(\tau), 0 \leq \tau \leq \beta, \vec{x}(0)=\vec{x}(\beta)\}$ be any realization of a closed Brownian path on the plane. As is well known, Brownian paths are distributed according to the Wiener measure:[3] if $f[\vec{x}(\tau)]$ is any functional on paths, the expectation value of $f$ is given by

$$\langle f[\vec{x}(\tau)]\rangle_{\mathscr{W}} \equiv \frac{\int \mathscr{D}[\vec{x}(\tau)] f[\vec{x}(\tau)] \exp\left[-\int_0^\beta \left\{\frac{1}{2}\frac{d\vec{x}}{d\tau}\cdot\frac{d\vec{x}}{d\tau}d\tau\right\}\right]}{\int \mathscr{D}[\vec{x}(\tau)] \exp\left[-\int_0^\beta \left\{\frac{1}{2}\frac{d\vec{x}}{d\tau}\cdot\frac{d\vec{x}}{d\tau}d\tau\right\}\right]}. \quad (1)$$

In Eq. (1) the functional integrals[4] are over all closed paths (the starting point is also integrated over). (We set the diffusion constant equal to half throughout this paper.) Let $\mathscr{A}[\vec{x}(\tau)]$ be the algebraic area enclosed by the path $\vec{x}(\tau)$. Clearly, the normalized probability distribution of areas $P(A)$ is given by

$$P(A) \equiv \langle \delta(\mathscr{A}[\vec{x}(\tau)]-A)\rangle_{\mathscr{W}}. \quad (2)$$

The expectation value $\bar{\phi}$ of any function $\phi(A)$ of the area is given by $\int P(A)\phi(A)dA$. As is usual in probability theory we focus on the generating function $\tilde{P}(B)$ of the distribution $P(A)$:

$$\tilde{P}(B) \equiv \overline{e^{ieBA}} = \int P(A)e^{ieBA}dA, \quad (3)$$

which is simply the Fourier transform of $P(A)$. For future convenience we write the Fourier transform variable as $eB$. The distribution $P(A)$ can be recovered from its generating function by an inverse Fourier transform. From Eqs. (2) and (3) above we find

$$\tilde{P}(B) = \langle e^{ieB\mathscr{A}}\rangle_{\mathscr{W}}. \quad (4)$$

Notice that $B\mathscr{A}$ can be expressed as

$$B\mathscr{A} = \int_0^\beta \vec{A}(\vec{x})\cdot\frac{d\vec{x}}{d\tau}d\tau, \quad (5)$$

where $\vec{A}(\vec{x})$ is any vector potential whose curl is a homogeneous magnetic field $B$. Equations (1), (4), and (5) yield

$$\tilde{P}(B) = \frac{\int \mathscr{D}[\vec{x}(\tau)] \exp\left[\int_0^\beta \left\{-\frac{1}{2}\frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} d\tau\right\} + ie\int_0^\beta \left\{\vec{A} \cdot \frac{d\vec{x}}{d\tau} d\tau\right\}\right]}{\int \mathscr{D}[\vec{x}(\tau)] \exp\left[\int_0^\beta \left\{-\frac{1}{2}\frac{d\vec{x}}{d\tau} \cdot \frac{d\vec{x}}{d\tau} d\tau\right\}\right]} . \tag{6}$$

By inspection of Eq.(6) we arrive at

$$\tilde{P}(B) = Z(B)/Z(0) , \tag{7}$$

where $Z(B)$ is the partition function ($Z(B)$ = Tr$\{\exp[-\beta H(B)]\}$) for a quantum particle of charge $e$ in a homogeneous magnetic field $B$ at an inverse temperature $\beta$. This is the central result of this paper and it relates Brownian motion and magnetism. As the reader can easily verify, the relation (7) holds even if there is an arbitrary biasing potential. The plane can also be replaced by a sphere or $(\mathbb{R}^3)^N$, the configuration space of $N$ particles in $(\mathbb{R}^3)$. In the last case, the area of interest is the sum of the weighted areas of the projections of the closed Brownian paths onto the $x$-$y$ plane. Now we demonstrate the utility of Eq. (7) by computing the distribution of areas for diffusion on a plane. The partition function $Z(B)$ for a particle of unit mass in a constant magnetic field, is easily computed from the energies $E_n = (n + \frac{1}{2})eB$ and degeneracy (or the number of states per unit area) $(eB/2\pi)$ of Landau levels[5] (throughout this paper we set $\hbar = c = 1$):

$$Z(B) = \sum_{n=0}^\infty \frac{eB}{2\pi} \exp[-(n + \frac{1}{2})\beta eB] = \frac{eB/4\pi}{\sinh(\beta eB/2)} .$$

From (7) we find $\tilde{P}(B) = (\beta eB/2)[\sinh(\beta eB/2)]^{-1}$. Taking the Fourier transform of $\tilde{P}(B)$ by contour integration we get the result $P(A) = (\pi/2\beta)[\cosh(\pi A/\beta)]^{-2}$ derived in Ref. 2. This provides a check on Eq. (7) and illustrates its use.

Let us now address the problem posed at the beginning of this paper: what is the distribution $P(\Omega)$ of solid angles enclosed by a diffusing particle on a unit sphere? Unlike the planar case, $P(\Omega)$ is a periodic[6] function with period $4\pi$. The generating function $\tilde{P}_g$ of the distribution of solid angles is given by

$$\tilde{P}_g = \int_0^{4\pi} d\Omega P(\Omega) e^{i\frac{g\Omega}{2}} \tag{8}$$

with $g$ an integer. $P(\Omega)$ is expressed in terms of $\tilde{P}_g$ by a Fourier series

$$P(\Omega) = \frac{1}{4\pi} \sum_{g=-\infty}^\infty e^{-i\frac{g\Omega}{2}} \tilde{P}_g \tag{9}$$

rather than an integral (3). Relation (7) now takes the form

$$\tilde{P}_g = Z_g/Z_0 , \tag{10}$$

where $Z_g$ is the partition function for a particle of charge $e$ on a sphere subject to a magnetic field created by a monopole of quantized strength $G = g/e$ (Ref. 7) at the center of the sphere. The energy levels of this system are easily computed:[8]

$$E_j = [j(j+1) - g^2]/2 ,$$

where $j$, the total angular momentum quantum number ranges from $|g|$ to infinity, and the $j$th level is $(2j+1)$-fold degenerate. The partition function is consequently given by

$$Z_g = \sum_{j=|g|}^\infty (2j+1) e^{-\frac{\beta\{j(j+1) - g^2\}}{2}} . \tag{11}$$

Combining (9),(10), and (11) and rearranging the summations we arrive at

$$P(\Omega) = \text{Re} \frac{1}{2\pi Z_0} \sum_{l=0}^\infty \left\{(2l+1)\frac{1+\zeta}{2(1-\zeta)} \right.$$

$$\left. + \left[\frac{2\zeta}{(1-\zeta)^2}\right]\right\} e^{-\frac{\beta}{2}l(l+1)} , \tag{12}$$

where $\zeta(l,\beta,\Omega) = \exp[-1/2\{\beta(2l+1) + i\Omega\}]$. The function (12) is plotted numerically for various values of $\beta$ in Fig. 1. The qualitative nature of these plots is easily understood. For small values of $\beta$ the particle tends to make small excursions and its path encloses solid angles close to 0 or $4\pi$ and consequently the plots are peaked around these two values. As the available time $\beta$ increases, other values of $\Omega$ are also probable and the peaks tend to spread and the curves to flatten out. Finally in the limit of $\beta \to \infty$ the particle has enough time to enclose all possible solid angles with equal probability. These plots give the answer to the question that was raised in the beginning of the paper.

Now we turn to the magnetic properties of spinless bosons. An $N$ particle system in three dimensions placed in a homogeneous external magnetic field which is along the $z$ direction has the Hamiltonian
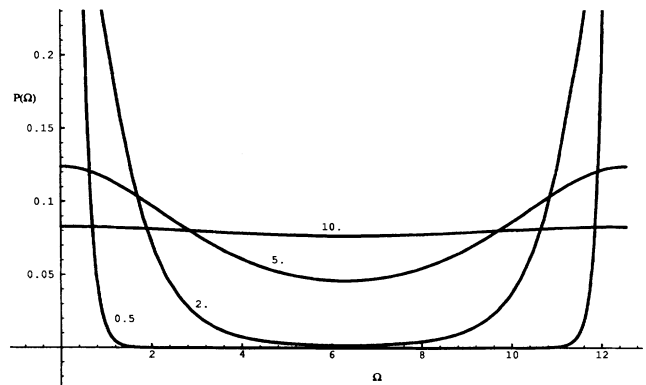


FIG. 1. The probability distribution $P(\Omega)$ of solid angles for closed random walks lasting a time $\beta$. $P(\Omega)$ is plotted above for four values of $\beta$: 0.5, 2, 5, and 10.

$$H(\vec{x}^a, \vec{p}_a) = \sum_{a=1}^{N} \frac{[\vec{p}_a - e_a \vec{A}(\vec{x}^a)]^2}{2m_a} + V(\vec{x}^a),$$

where $\vec{A}(\vec{x}^a)$ is the vector potential of the external magnetic field. $V(\vec{x}^a)$ includes an arbitrary interaction between the particles as well as an external potential, $m_a$ and $e_a$ are the masses and charges of the particles. $\{\vec{x}^a, a=1,2,\ldots,N\}$ are the position vectors of the $N$ particles. The configuration

space of the system is given by $Q = (\mathbb{R}^3)^N / \sim$, where $\sim$ means that we identify points in $(\mathbb{R}^3)^N$ which differ by an exchange of identical particles. For simplicity we give the argument for $N$ identical particles with unit mass and charge $e_a = e$. The argument is easily adapted to several species of particles of arbitrary charge and mass.

Now consider a diffusion on $Q$ biased by the potential $V(\vec{x}^a)$. The Wiener measure is now appropriately modified:

$$\langle f[\vec{x}(\tau)] \rangle_{\mathscr{W}(V)} \equiv \frac{\int \mathscr{D}[\vec{x}(\tau)] f[\vec{x}(\tau)] \exp\left[ -\int_0^\beta \left\{ \frac{1}{2}\left( \sum_a \frac{d\vec{x}_a}{d\tau} \cdot \frac{d\vec{x}_a}{d\tau} \right) + V(\vec{x}^a) \right\} d\tau \right]}{\int \mathscr{D}[\vec{x}(\tau)] \exp\left[ -\int_0^\beta \left\{ \frac{1}{2}\left( \sum_a \frac{d\vec{x}_a}{d\tau} \cdot \frac{d\vec{x}_a}{d\tau} \right) + V(\vec{x}^a) \right\} d\tau \right]}.$$

The area whose distribution we are interested in is defined as follows: Let $q(\tau)$ be a closed curve in $Q$. $q(\tau)$ determines trajectories of particles $\{\vec{x}^a(\tau), a=1,2,\ldots,N\}$ in $\mathbb{R}^3$. The area functional of interest is $\mathscr{A}[q(\tau)] \equiv \sum_a \int \vec{A}(\vec{x}^a) \cdot d\vec{x}^a$. The area functional has the following interpretation. If the final positions of the $N$ particles are the same as the initial ones (direct processes), $\mathscr{A}[q(\tau)]$ is simply the sum of the areas enclosed by the projection of the particle trajectories on the $(x\text{-}y)$ plane. If the final positions differ from the initial ones by a permutation (exchange processes), the projections of the particle trajectories still define closed curves on the $(x\text{-}y)$ plane. $\mathscr{A}[q(\tau)]$ is defined as the sum of areas enclosed by these closed curves.

As before we find that $\tilde{P}(B)$, which is the Fourier transform of the distribution $P(A) \equiv \langle \delta(\mathscr{A}[\vec{x}(\tau)] - A) \rangle_{\mathscr{W}(V)}$ of areas, is given by Eq. (7). It is crucial for our argument that the particles obey Bose statistics.[9] Since $P(A)$ is a probability distribution, $\tilde{P}(B) = Z(B)/Z(0)$ is the Fourier transform of a positive function. This places strong restrictions on the partition function $Z(B)$. Let $u_i, i=1,\ldots,n$ be $n$ real numbers. If one defines the $n \times n$ matrix $D_{ij}^{(n)} = \tilde{P}(u_i - u_j)$, the necessary and sufficient condition for $\tilde{P}(B)$ to be the Fourier transform of a positive function is[10,11]

$$\Delta^{(n)} \equiv \operatorname{Det} D^{(n)} \geq 0 \quad \text{for all } n. \tag{13}$$

This imposes restrictions on the free energy $F(B) = -(1/\beta)\ln Z(B)$ of the system in the presence of a magnetic field $B$.

For the simplest nontrivial case $n=2$, the inequality (13) with $B = u_1 - u_2$ leads to

$$Z(B) \leq Z(0) \tag{14}$$

or equivalently, $F(B) \geq F(0)$. Since the free energy of the system *increases* in the presence of a magnetic field, the material is diamagnetic. This universal diamagnetic behavior of spinless bosons at all temperatures is known in the mathematical physics literature.[11] However, our approach may be accessible to a wider community of physicists. Our approach relating Brownian motion to magnetism enriches *both* fields and provides each field with intuition derived from the

other. For instance, the zero-field susceptibility $\chi = -\partial^2 F(B)/\partial B^2|_{B=0}$ of the magnetic system is related to the variance of the distribution of areas in the diffusion problem:

$$\chi = \frac{1}{\beta}[\ln \tilde{P}(B)]''|_{B=0} = -\frac{e^2}{\beta}\overline{(A-\overline{A})^2} = -\frac{e^2}{\beta}\operatorname{Var}A \leq 0. \tag{15}$$

It is curious that the zero-field susceptibility can be interpreted as the variance of the distribution of areas. Since the variance cannot be negative, it follows that $\chi$, the zero-field susceptibility cannot be positive and so these systems are diamagnetic.

Next consider the case $n=3$. The $3 \times 3$ matrix $D_{ij}^{(3)}$ will then be a function of $u = u_1 - u_2$ and $v = u_2 - u_3$ ($u_1 - u_3$, being expressible in terms of $u$ and $v$). If we set $u=0$ (i.e., set $u_1 = u_2 = 0, u_3 = -v$), we find that $\Delta^{(3)}(u,v)|_{u=0} = 0$. It then follows from the inequality (13) that $\Delta^{(3)}(u,v)$ has a minimum at $u=0$ for all $v$. This implies that $\partial^2 \Delta^{(3)}/\partial u^2|_{u=0,v=-B} \geq 0$. Defining the function $\mathscr{U}(B) = \tilde{P}'[1 - \tilde{P}^2]^{-1}$, where the prime means derivative with respect to the magnetic field, we find

$$\mathscr{U}(B) \leq \mathscr{U}(0). \tag{16}$$

As can be seen by taking the limit $B \rightarrow 0$, $\mathscr{U}(0) = -\tilde{P}''(0) = -\beta\chi(0)$. We define a critical field $B_c = \pi/[2\sqrt{-\beta\chi(0)}]$. The inequality (16) implies a bound on the partition function. Notice that $\tilde{P}'$ lies in a cone defined by the lines of slope $-(\pi/2B_c)\sqrt{(1-\tilde{P}^2)}$ and $(\pi/2B_c)\sqrt{(1-\tilde{P}^2)}$. It follows that

$$\tilde{P}(B) \geq \cos(\pi B/2B_c) \quad \text{for } |B| \leq B_c. \tag{17}$$

The diamagnetic inequality due to Simon and Nelson[11] gives an upper bound on the partition function $Z(B)$ of a system of spinless bosons. The new inequality stated in (17) gives us a lower bound on $Z(B)$ (see Fig. 2) (or equivalently, an upper bound on the free energy).

As an explicit check on this new bound on the free energy we considered a simple system—a charged particle in a mag-
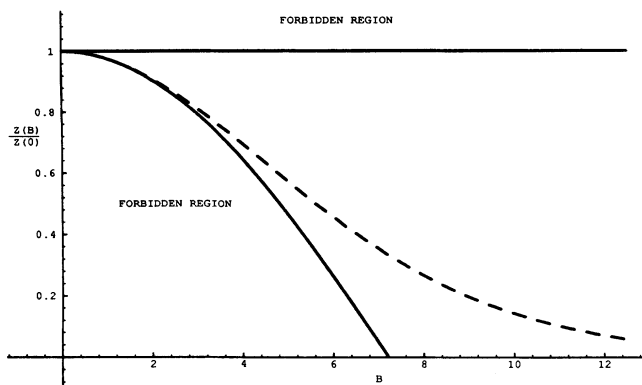
FIG. 2. The region forbidden by the bounds [inequalities (14) and (18)] on the partition function. These bounds are shown as solid lines. The dotted curve is the partition function for a charged simple harmonic oscillator in an external magnetic field. Notice that the dotted curve lies outside the forbidden region.

netic field subject to a harmonic oscillator potential. The calculated partition function of this system is close to, but above the lower bound set by (17). Needless to say, our bound is derived for an *arbitrary* interacting system of spinless bosons. The new bound presented here along with the earlier (14) diamagnetic inequality[11] places strong restrictions on the partition function of a bosonic system in the presence of a magnetic field. We find a curious and immediate consequence of these restrictions: if the zero-field susceptibility of the system vanishes, then Eqs. (14) and (17) imply that $Z(B) = Z(0)$, i.e., the system is *nonmagnetic* at *all* fields.

The key result of this paper is a connection between two apparently distinct classes of problems—Brownian motion and magnetism. This allows us to compute the distribution of solid angles enclosed in Brownian motion on a sphere. As mentioned earlier, this problem comes up when computing

the distribution of Berry phases in a random magnetic field. A more classical context is depolarized light scattering. As is well known, a light ray following a space curve picks up a geometric phase,[12,13] equal to the solid angle swept out by the direction vector. If a light ray inelastically scatters off a random medium, its direction vector does a random walk on the unit sphere of directions. The distribution $P(\Omega)$ computed here is relevant to the extent of depolarization in such an experiment.[14]

In the domain of magnetism we find an independent way of arriving at the diamagnetic inequality[11] which states that the free energy of a system of spinless bosons always increases in the presence of a magnetic field. Spinless charged bosonic systems occur in the context of superconductors (which are perfect diamagnets) and neutron stars.[15] We believe that the community of physicists working in these areas may not be aware of the general results available in the mathematical literature. For instance, the diamagnetism of bosons may be relevant[16] to the interpretation of recent experiments[17] on high-$T_c$ superconductivity.

Throughout this paper we have only discussed homogeneous magnetic fields. It is easy to generalize our discussion to take into account arbitrary inhomogeneous fields: all one does is consider the distribution of *weighted* areas. An obvious application of this is the computation of the probability of entanglement of a polymer with a background lattice of polymers. We expect the new method outlined here to shed light on open problems in polymer entanglement involving more complicated configurations of polymers than the simplest one solved so far. One can also use the relation (7) to compute the distribution of winding numbers in diffusion in a multiply connected space.

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[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon, Oxford, 1977), p. 457.

[6] A path that encloses a solid angle $\Omega$ to its left also encloses a solid angle of $4\pi - \Omega$ to its right. Using the symmetry $P(\Omega) = P(-\Omega)$ we conclude that $P(\Omega)$ is periodic with period $4\pi$.

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[9] The identity (10) is not true for Fermi particles since in this case exchange processes are weighted negatively relative to direct ones in the expression for the partition function. This ruins the probability interpretation that we give to the partition function.

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[16] In these experiments, diamagnetism is observed well above $T_c$. A possible explanation may be that the diamagnetism is due to the persistence of Cooper pairs (spinless bosons) well above $T_c$. We thank N. Kumar for this observation. See also A. S. Alexandrov and N. F. Mott, Phys. Rev. Lett. **71**, 1075 (1993); D. C. Bardos *et al.*, Phys. Rev. B **49**, 4082 (1994).

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