Comments

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Critical behavior and the Harris criterion for the random Potts model on hierarchical lattices

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The critical properties of the diluted Potts model on diamond-type hierarchical lattices of different fractal dimensions were studied recently by Wu *et al.* [Phys. Rev. B **48**, 3171 (1993)]. Although the aim of the paper is very interesting, we show in this comment that the fixed bimodal bond-probability distribution used in their renormalization-group work is not appropriate to deal with this disordered model. Such treatment violates the Harris criterion and may lead to spurious results. We point out that the correct way to deal with randomness is to rescale the full distribution and take its second moment as an additional parameter. We use this method and find that the critical properties are different from those which Wu *et al.* predicted.

An important question in the field of critical phenomena is how the properties of a system near its critical points and its critical exponents are affected by quenched bond disorder. A fundamental advance in the study of criticality under randomness was achieved with the physical argument due to Harris.[1] The Harris criterion predicts that when the specific-heat exponent α of the pure system is positive, the disorder will change the critical regime of the pure system; if α is negative the randomness does not affect the critical behavior. A simple way of expressing this criterion, in the language of the renormalization-group theory, is that when $\alpha > 0$ the pure fixed point becomes unstable, causing the flow along the critical boundary be in the direction of a new random fixed point. The new critical point will control the critical behavior of the system.

The use of the Migdal-Kadanoff renormalization-group method to study pure and disordered systems is not new.[2–4] It is a very simple method and contains several important features, which are common to all real-space renormalization-group (RSRG) approaches. For the pure systems it gives approximate results for Bravais lattices, but it was realized[5] that one can construct some scale-invariant lattices (called hierarchical lattices), on which the Migdal-Kadanoff method gives exact results. For this reason hierarchical lattices have received much attention in the field of phase transition and critical phenomena.[6]

The RSRG method is particulary suitable for critical properties of systems with quenched randomness, since the position space formulation can deal directly with local bond disorder. The simplest model of a disordered system is defined by a probability distribution of the nearest-neighbor interaction. With the Migdal-Kadanoff method, one can write the recursion relations, which the probability distributions of the Hamiltonian parameters evolve along the renormalization proccess. However, the space, of a probability distribution is an infinite-dimensional space and it is a very difficult mathematical problem to obtain a fixed (scale-invariant) distribution. This problem had important advances ten years ago by the work of Andelman and Berker,[7] who derived a true fixed-point distribution using numerical methods and by Derrida and Gardner,[8] who used a weak-disorder expansion to calculate analytically the random fixed points and their associated critical exponents. Both works were performed on the Potts model on diamond hierarchical (Berker) lattices with random exchange interaction. Later da Cruz and Stinchcombe[9] elucidated the violation of the Harris criterion[1] on the bond-diluted Potts model on the Berker lattice: they also argued that it is necessary to allow the bond probability distribution to evolve, under length scaling, even if one starts with a binary form.

Very recently, Wu, Xin, and Yang,[10] studied the phase diagram of the bond-diluted Potts model on a fractal family of diamond-type hierarchical lattices. The paper has a very interesting purpose, namely, whether the fractal dimension and other geometrical features of the different lattices and, in particular, for $D_f > 2$ and $D_f \leq 2$ has some effect on the disordered Potts model criticality. However, they used a two-delta function for the initial distribution (bimodal distribution), which are not allowed to evolve, and replace the renormalized distribution by another two-delta function, which has only the same first moment. As discussed in Refs. 8 and 9, it is not easy to estimate the accuracy of this approximation and, which is more serious, whether some effects are gain or lost.

TABLE I. The RG numerical results of the fixed points and critical exponents for the $M = 2$ and $L = 3$ ($D_f = 1.63$) lattice. The critical value $\tilde{u}^*$ is the position of the random fixed point.

q	u^*	λ_u	ν_u	α	$\tilde{u}^*$
10	0.622	2.40	1.24	-0.038	
11	0.613	2.44	1.22	-0.002	
11.5	0.608	2.46	1.21	0.013	0.03
15	0.581	2.57	1.16	0.105	0.17

In order to check on the predictions of Wu, Xin, and Yang,[10] we perform calculations on the same lattices. They defined diamond-type lattices with M branches of L bonds per branch. Thus we study the Berker lattice ($D_f = 2$), the $M = 2$, $L = 3$ lattice ($D_f = 1.63$), the $M = 3$, $L = 2$ lattice ($D_f = 2.58$), and the Wheatstone bridge lattice. The model is defined by the following Hamiltonian:

$$-\beta H = \sum_{\langle ij \rangle} K_{ij}\delta(\sigma_i, \sigma_j), \tag{1}$$

where K_{ij} is the exchange coupling between spins in nearest-neighbor sites and $\sigma = 1, 2, 3, \ldots, q$ are the possible spin values. It is common practice to introduce the following variable

$$u_j = \frac{e^{K_{ij}} - 1}{e^{K_{ij}} - 1 + q}, \tag{2}$$

which is defined in the range $0 \leq u_j \leq 1$.

On the other hand, for $p > p_c$, where p is the concentration of present bond and p_c the geometrical percolation critical value, starting the RSRG iteration with a bimodal distribution, the weight of the delta function of the missing bonds ($K_{ij} = 0$ or $u_j = 0$) will always decrease and eventually go to zero. Therefore, the properties of the diluted system for $p > p_c$ cannot be appropriately described by a diluted fixed point as in Ref. 10. Since the random fixed point is located in the $p = 1$ (no missing bonds) region, it is sufficient to use any continuous distribution with zero weight in $K_{ij} = 0$.[9,11] We choose to start with a Gaussian probability distribution

$$P(u_j) = \frac{1}{\sqrt{2\pi\tilde{u}^2}} \exp\left[-\frac{(u_j - u)^2}{2\tilde{u}^2}\right], \tag{3}$$

where $u = \langle u_j \rangle$ is the mean value and $\tilde{u} = \sqrt{\langle u_j^2 \rangle - \langle u_j \rangle^2}$ is the width of the probability distribution.

In order to test the method, we made a calculation on the Berker lattice ($D_f = 2$). In agreement with Refs. 8 and 9, we found that the critical exponent α vanishes at $q_0 = 6.8$. For $q > q_0$, α is positive and the system exhibits crossover to a new critical region dominated by the random fixed point. We also performed calculations on the Wheatstone bridge lattice ($D_f = 2.32$) and obtained $q_0 \simeq 5.8$. To make comparison with the paper of Wu, Xin, and Yang,[10] we summarize our results for the $M = 2$ and $L = 3$ lattice ($D_f = 1.63$), and $M = 3$ and $L = 2$ lattice ($D_f = 2.58$) in Table I and Table II, respectively. We present results for just a few values of q around $q_0 \simeq 11.1$, where the critical exponent α changes sign. The critical value u^* is the pure fixed point, and λ_u and ν_u are the eigenvalue and correlation length exponent relevant to the variable u. We also show the position of the random fixed point $\tilde{u}^*$, which is associated with the second moment (root mean square) of the bond probability distribution.

In the $D_f = 1.63$ lattice, the system criticality is dominated by the random fixed point, only for $q > 11.1$, in accord with the Harris criterion. For lower values of q our results for u^* and for the critical indices λ_u and ν_u agree with the calculations of Wu, Xin, and Yang. The main difference is that they claim that the system must cross over to a region dominated by a diluted fixed point for $q > 2.38$ (for $M = 2$ and $L = 3$). However, we can use the hyperscale relation $\alpha = 2 - \nu D_f$ to demonstrate that their results violate the fundamental Harris criterion.

For the $M = 3$ and $L = 2$ lattice, they found also a diluted fixed point at $q = 6.34$. As for the previous lattice ($D_f = 1.63$), this change of critical behavior also violates the Harris criterion. Furthermore, they found a tricritical point with mixed behavior, which they claim to be a particular property of the model on this type of fractal. In our calculations we did not find any indication of such different critical behavior. We found again that for $q < 11.1$ the system is dominated by the pure fixed point and for $q > 11.1$ by the random exchange fixed point. These results are summarized in Table II, which also shows that the random Potts model on both lattices have the same critical behavior.

In summary, we demonstrated by specific comparison, that the bimodal distribution commonly used to deal with diluted systems must also (as an additional pa-

TABLE II. The numerical results for the $M = 3$ and $L = 2$ ($D_f = 2.58$) lattice.

q	u^*	λ_u	ν_u	α	$\tilde{u}^*$
10	0.238	2.40	0.788	-0.038	
11	0.232	2.44	0.774	-0.002	
11.5	0.230	2.46	0.768	0.013	0.15
15	0.213	2.57	0.710	0.105	0.23

rameter) evolve under renormalization-group iterations. When one forces the probability distribution to maintain its bimodal form (as an additional approximation), as in the paper of Wu, Xin, and Yang,[10] some spurious critical behavior may be introduced, and the basic Harris criterion is strongly violated. In this comment we show the corrected way to deal with dilution and bond randomness in a general manner following along the lines of Refs. 7, 8, and 9. We also performed specific calculations and found that the two lattices studied by Wu, Xing, and Yang have the same critical properties if the disorder is treated exactly as we do here and that in this case the Harris criterion is not violated.

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