Phase roughening transition in Josephson-junction ladders in random magnetic fields

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The dynamics of one-dimensional arrays of parallel coupled Josephson-junction ladder arrays in a random magnetic field is described. The fluctuation of superconducting phase difference becomes constant above the roughening transition current I_{C2}, and grows linearly in time for $I_{C1} < I < I_{C2}$, where I_{C1} is the usual critical current separating zero voltage and finite voltage regimes. The characteristic time $t^*(I)$ diverges as $(I_{C2}-I)^{-\gamma}$ when I approaches I_{C2} from below. The discontinuity of dV/dI at I_{C2} and the absence of hysteresis above I_{C2} in I-V curve are also discussed.

Josephson-junction arrays[1] (JJA's) have long been a subject of theoretical and experimental study for both interesting equilibrium properties as realizations of unfrustrated and frustrated XY models (under a magnetic field),[2,3] and dynamic properties including coherent mode locking,[4] hysteresis, and chaos.[5] Also inhomogeneous superconductors are often modeled by a network of Josephson junctions. In these systems where randomness can appear in coupling strengths between superconducting grains, grain positions, and sizes of grains, the disorder play an important role. Recently, Josephson-junction arrays were fabricated and studied in such a way that disorder could be deliberately introduced.[6–8]

Here, we consider the dynamics of one-dimensional (1D) arrays of parallel coupled Josephson ladder junctions under a random magnetic field as shown in Fig. 1, where parallel couplings are also Josephson couplings. A random magnetic field will generate random magnetic flux (hence, random frustration) through each plaquette. In experimental situations, an equivalent model could be realized by introducing random plaquette areas with a uniform external magnetic field.[7] Since there is a growing interest in the surface fluctuations of driven growth

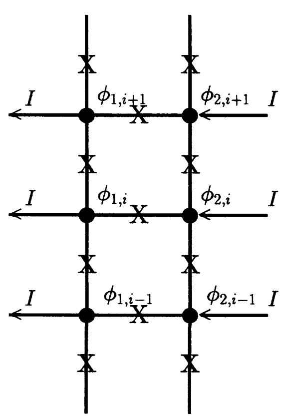

FIG. 1. Schematic picture of a ladder array of Josephson junctions. X represents a Josephson coupling.

models,[9] as an analogy, we concentrate on the fluctuation of θ_i which is the phase difference of the superconducting order parameter across the ladder junction. By measuring a standard deviation of θ_i, we show a roughening transition due to a random magnetic field. We also discuss the effect of the randomness on the I-V characteristic curve and the relation between JJA's and the sliding charge density wave.

The ladder array dynamics under a random magnetic field (Fig. 1) is modeled by the coupled resistively shunted junction (RSJ) model with random gauges assigned to horizontal junctions. External dc current I is uniformly injected at each node on one side and extracted on the other side. Due to the random magnetic field which gives a sort of random pinning potential, we can think of the whole array as a combination of strongly pinned regions and weakly pinned regions, i.e., regions with large local effective critical currents and regions with small effective critical currents. From an analogy between the dynamics of a single Josephson junction and that of a rigid rotor (pendulum) with an applied torque,[10] one can interpret a JJ ladder array as N rotors coupled in parallel. The randomness of the external magnetic field destroys the translational invariance along the direction of the ladder (y direction in Fig. 1), and hence produces a time-dependent fluctuation of θ_i. We find a phase roughening transition at the point I_{C2} in addition to the known transition at I_{C1} which separates the zero voltage and finite voltage regimes. Above the critical current I_{C2}, the randomness becomes irrelevant in the sense that all the rotors rotate with constant average speed ("flat phase"). Below I_{C2}, clusters of rotors move with different average speed where the fluctuations of θ_i increase linearly in time due to phase-slip processes ("rough phase"). This roughening transition is not a Kosterlitz-Thouless-type roughening transition,[11] but a kind of pinning-depinning transition.[12–14]

Figure 1 shows a ladderlike array of superconducting islands where nearest-neighboring islands are connected by Josephson junctions. Uniform dc external currents are injected through the right-hand-side nodes (islands) and extracted through left-hand-side nodes. For each junction, we use the RSJ model neglecting both capacitance and inductance. Finite inductance will produce an induced magnetic field. In our system, however, we are also assuming that the array is under a *random external* magnetic field. Therefore, the small induced time-dependent field will simply be added to the already present random magnetic field and generate another random field configuration. Hence, we expect that no qualitative change will occur to the dynamic charateristics of the array even if we include the inductance effect.

All horizontal junctions (we call these the "main" junctions) are assumed to have uniform values of critical current I_{cx}, junction resistance R_1, and also all vertical junctions (y direction) are assumed to have critical current I_{cy} and resistance R_2. I_{cy} controls the coupling between neighboring main juctions. The random external magnetic field is modeled by simply assigning a random gauge A_i uniformly distributed between $-\pi$ and π to each horizontal bond i.

Applying current conservation at each node, we get the following equations for superconducting phases $\phi_{1,i}$, $\phi_{2,i}$ ($i = 1, \ldots, N$; N is the number of parallel junctions or ladder size) where $\phi_{1,i}$, and $\phi_{2,i}$ refer to the phases of the ith island on the left-side and the right-side columns, respectively:

$$I = \frac{\hbar}{2eR_1}(\dot{\phi}_{2,i} - \dot{\phi}_{1,i}) + \frac{\hbar}{2eR_2}(\dot{\phi}_{1,i+1} - \dot{\phi}_{1,i}) + \frac{\hbar}{2eR_2}(\dot{\phi}_{1,i-1} - \dot{\phi}_{1,i}) + I_{cx}\sin(\phi_{2,i} - \phi_{1,i} - A_i) + I_{cy}\sin(\phi_{1,i+1} - \phi_{1,i}) + I_{cy}\sin(\phi_{1,i-1} - \phi_{1,i}) \quad (1)$$

and

$$I = \frac{\hbar}{2eR_1}(\dot{\phi}_{2,i} - \dot{\phi}_{1,i}) + \frac{\hbar}{2eR_2}(\dot{\phi}_{2,i} - \dot{\phi}_{2,i+1}) + \frac{\hbar}{2eR_2}(\dot{\phi}_{1,i} - \dot{\phi}_{1,i-1}) + I_{cx}\sin(\phi_{2,i} - \phi_{1,i} - A_i) + I_{cy}\sin(\phi_{2,i} - \phi_{2,i+1}) + I_{cy}\sin(\phi_{2,i} - \phi_{2,i-1}). \quad (2)$$

There are $2N$ equations for $2N$ phases $\phi_{1,i}$ and $\phi_{2,i}$ ($i = 1, \ldots, N$). Among them, one equation is redundant due to the overall $U(1)$ phase rotation symmetry. So one of the phases can be fixed arbitrarily and we can solve $2N - 1$ remaining equations. A periodic boundary condition is imposed along the y direction (perpendicular to the external current).

We have carried out a direct integration of Eqs. (1) and (2) in order to understand the effect of a random magnetic field in Josephson-junction (JJ) ladders. A fourth-order Runge-Kutta algorithm[15] was employed for this purpose with a time interval of $\delta t = 0.1 t_0$, where $t_0 \equiv 2eR_1/\hbar$. Most simulations were performed starting from an initially flat phase $\phi_{1,i}(0) = \phi_{2,i}(0) = 0$ for $i = 1, 2, \ldots, N$, $R_1 = R_2 = 1$, and $I_{cx} = I_{cy} = 1$. To facilitate the analogy with the surface roughening of driven growth models,[9] we have monitored the dynamic process by calculating a new quantity, the phase fluctuation of θ_i,

$$\Delta\theta(t) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\theta_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N}\theta_i\right)^2} , \quad (3)$$

where $\theta_i = \phi_{2,i} - \phi_{1,i}$. The discrete transform $\theta_i \to \theta_i + 2\pi$ does not change the dynamic equations (1) and (2). However, the quantity $\Delta\theta$ is a good parameter to investigate the phase roughening transition as described below.

In zero magnetic field, the θ_i are synchronized with $\Delta\theta$ being constant and the JJ ladder behaves as N identical single Josephson junctions. The I-V characteristics show a known result $V \sim (I - I_{C1})^{1/2}$ for $I > I_{C1}$ where I_{C1} is equal to the critical current I_{cx} for a single horizontal junction.

In a random magnetic field, the quenched vector potential A_i shifts the θ_i and effectively reduces I_{C1}. Starting with all $\theta_i = 0$, the fluctuation $\Delta\theta$ increases with time. Depending on the extent of the phase fluctuations, we can divide the dynamics into three different regimes.

(I) For $I < I_{C1}$, $\Delta\theta$ grows with time for the initial transient period and then becomes saturated without any oscillation where the voltage is zero.

(II) For $I_{C1} < I < I_{C2}$, $\Delta\theta$ grows linearly in time with additional small oscillations (Josephson oscillations). The external current I is big enough to produce nonzero voltage but $\langle\dot{\theta}_i\rangle$, the average time derivative of θ_i, depends on the position i.

(III) For $I > I_{C2}$, $\Delta\theta$ grows with time initially and then becomes saturated with Josephson oscillations. Even though the θ_i varies with position i, the average velocity $\langle\dot{\theta}_i\rangle$ is uniform. In this regime, there is no hysteresis in the I-V curves.

The critical currents I_{C1} and I_{C2} depend on the system size I_{cy} and the randomness of the magnetic fields. Here, we find that there is a phase roughening transition at I_{C2}. Above I_{C2}, θ_i and the instantaneous velocity of θ_i vary with position, but all the $\langle\dot{\theta}_i\rangle$ are the same where $\langle\ \rangle$ denotes the time average. We call this regime a "flat phase," because the $\langle\Delta\theta\rangle$ remains constant. For $I_{C2} > I > I_{C1}$, the fluctuation $\langle\Delta\theta\rangle$ increases linearly in time showing a " rough phase."

In regime (II), we propose a scaling form

$$\Delta\theta(I,t) \sim g(t/t^*) f(\omega(I)t) , \quad (4)$$

where $f(y)$ is an oscillating function with Josephson frequency $\omega(I)$ and $g(x)$ is an increasing function as shown in Fig. 2. There are two time scales: One is the Josephson frequency depending linearly on the average voltage

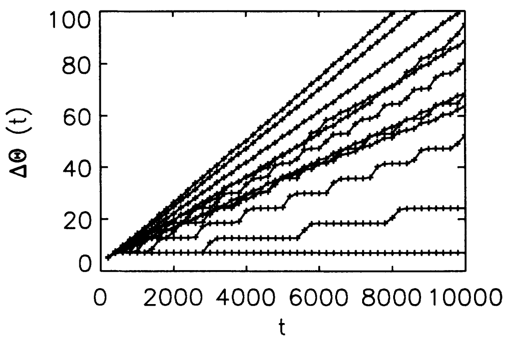

FIG. 2. The fluctuation $\Delta\theta$ as a function of time for regime (II). $I = 0.565 - 0.576$ from the top to the bottom with increment of current 0.001 for a given random configuration with $N = 40$ and $I_{C2} \approx 0.575\,526$.

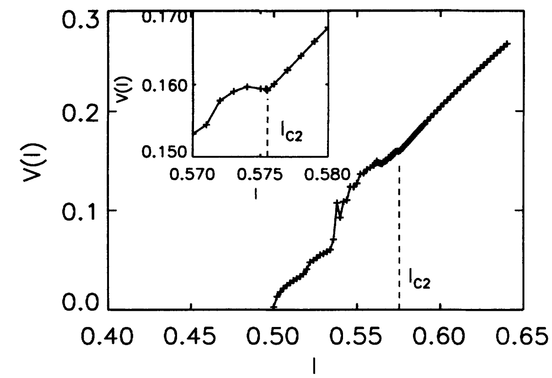

FIG. 4. I-V characteristics for $N = 40$ with the same random configuration of A_i as in Fig. 2. Inset: the I-V curve very near I_{C2}.

$[\omega(I) \approx (2e/\hbar)\langle V\rangle]$ and the other is the characteristic time $t^*(I)$ which diverges as the external current I approaches I_{C2} from below. In Fig. 2, since the $\Delta\theta$ is averaged over an interval larger than the Josephson periods, the $\langle\Delta\theta\rangle$ remains constant up to the time interval t^*. If we take an average of $\Delta\theta$ over the time interval larger than both t^* and the Josephson period, the phase fluctuation grows linearly in time,

$$\langle\Delta\theta(I,t)\rangle \sim S(I)t \,, \tag{5}$$

where $S(I) \sim 1/t^*(I)$ is the average slope of the curve in Fig. 2. We may define a domain as a cluster of junctions whose $\langle\dot\theta_i\rangle$ are the same. Then, N parallel junctions can be divided into domains. The reason that $\Delta\theta$ grows linearly in time is due to the different $\langle\dot\theta_i\rangle$ of the different domains.

Since t^* diverges at I_{C2}, one can assume

$$t^*(I) \sim (I_{C2} - I)^{-\gamma}. \tag{6}$$

We measure the slope S as a function of I. From the plot of $\ln(I_{C2} - I)$ versus $\ln t^*$ as shown in Fig. 3, a nice straight line is obtained with $\gamma = 0.65 \pm 0.15$ being consistent with the above scaling conjecture.

Figure 4 shows typical I-V characteristics for $N = 40$ with a random magnetic field. The nonsmooth I-V curve is due to the randomness of A_i. We find that a jump in

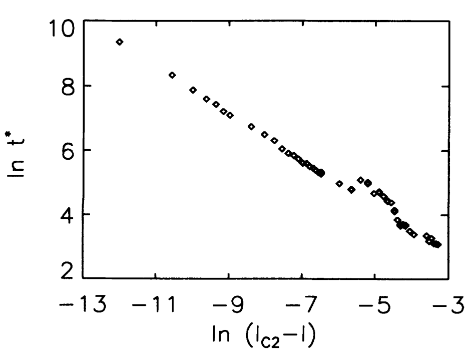

FIG. 3. $t^*(I)$ as a function of I_{C2}-I in a log-log plot near I_{C2} with the same random configuration A_i as in Fig. 2.

the I-V curve is closely related to a change of the slope S. Since $\Delta\theta$ is the fluctuation of θ_i, $S = d\langle\Delta\theta\rangle/dt$ has information on the $\dot\theta_i$ fluctuation, which is proportional to the voltage fluctuation in space. As shown in the inset of Fig. 4, the voltage decreases slightly as the external current I approaches I_{C2} very closely from below. However, just above I_{C2}, the I-V characteristics show good linear behavior $V(I) - V(I_{C2}) \sim I - I_{C2}$. Also there is a discontinuity of dV/dI at the transition current I_{C2} as shown in the inset of Fig. 4. One may find this roughening transition experimentally by measuring the I-V characteristics, dV/dI, and spatial fluctuations of time-averaged local voltages $V_i \sim \langle\theta_i\rangle$. There is a somewhat different experiment with positional shape disorder in superconducting wire networks and Josephson-junction arrays.[7] In the experiment, positional disorder was qualitatively introduced by displaying the centers of the islands, effectively changing the area of the cells. If we apply a constant external magnetic field, it will generate a random magnetic flux depending on the random area for the plaquette. Then we will expect smooth and linear I-V curves just above I_{C2}. A similar transition can be realized in the dynamics of a disordered flux line lattice as a transition between a plastic phase and a solid phase.[14]

Other evidence for the transition is that there is no hysteresis above I_{C2} where the voltage is independent of the initial condition $\theta_i(0)$. Above I_{C2} all $\langle\dot\theta_i\rangle$ are the same, making one big domain, and the phase fluctuation $\Delta\theta$ becomes constant as in Fig. 2. For constant magnetic fields, we could not find phase roughening regime. $\langle\Delta\theta\rangle$ remains constant above I_{C1}. This may be expected from the fact that a uniformly frustrated system does not generate a kind of random pinning potential as in the case of randomly frustrated arrays.

We have also measured the phase fluctuation with a free boundary condition. The change of the boundary condition is effectively equivalent to the change of gauge, resistance, and critical current for the boundary junction. Since we consider a random gauge, we expect that the boundary condition is irrelevant for the phase transition. We have found no qualitative changes for the free boundary condition numerically.

With a given random configuration of A_i, I_{C2} is sensitive to the y-directional critical current I_{cy}. It is not surprising since I_{cy} behaves like a diffusion constant in the y direction. From Eqs. (1) and (2), the condition of the zero net current along the vertical direction can be satisfied by assuming (this is a sufficient condition, but not a necessary one)

$$\phi_{1,i+1} - \phi_{1,i} = \phi_{2,i} - \phi_{2,i+1}. \tag{7}$$

Then we can reduce the equations of motion for $2N - 1$ phases in terms of only N phase differences $\theta_i \equiv \phi_{2,i} - \phi_{1,i}$. In order to simplify the equations further, we put $R_2 = \infty$. In dimensionless units (t in units of $\tau \equiv \hbar/2eR_1$ and the current in units of I_{cx}), we get a simpler set of equations,

$$\dot{\theta}_i + \sin(\theta_i - A_i) + I_{cy} \sin\left(\frac{\theta_i - \theta_{i+1}}{2}\right) + I_{cy} \sin\left(\frac{\theta_i - \theta_{i-1}}{2}\right) = I, \quad i = 1, \ldots, N. \tag{8}$$

The coupling between rotors (one can interpret a main junction as a rotor) becomes strong as the transverse critical current I_{cy} increases. Therefore, I_{C2} decreases as I_{cy} increases, which was seen exactly by our simulations. The phase roughening transition is controlled by both the randomness of the magnetic field and the y-directional critical current I_{cy}.

If we approximate the two last sinusoidal terms on the left-hand side of Eq. (8) by simple linear functions, it becomes

$$\dot{\theta}_i + \sin(\theta_i - A_i) - \frac{I_{cy}}{2} \nabla_y^2 \theta_i \approx I, \quad i = 1, \ldots, N. \tag{9}$$

One can see easily some similarities between our model and the sliding charge density wave, if we identify the superconducting phase difference in a ladder JJA with the phase of the sliding charge density wave (CDW).[12] However, there is an important difference between the two models. The usual coupling between neighboring phases in CDW models is Laplacian (diffusive). In the CDW model, neighboring phases are not allowed to have an arbitrarily large difference because of the elastic coupling that produces an arbitrarily large cost in energy to such a high-gradient configuration. On the other hand, in the ladder JJA, the y-directional coupling is another Josephson coupling, which can be approximated by a Laplacian only in the limit of small gradients in the phase-difference variable θ_i, while the full equation has an invariance under $\theta_i \to \theta_i + 2\pi$. So the θ variables in neighboring junctions can differ by an integer multiple of 2π without any extra cost in energy. This is the reason why $\Delta\theta$ grows linearly with time in regime (II). However, these 2π differences (winding numbers) are meaningful physically because they contain the history of time-dependent local voltage drops across individual junctions. Above the roughening transition point I_{C2}, all rotors have the same time-averaged winding numbers.

We also consider our model with random thermal noise current and find that the weak noise does not alter the nature of the transition. However, it is possible to destroy the flat phase [regime (III)] by a strong thermal noise. In the flat phase, since a vortex can sweep through from the left to the right of the sample, the phase fluctuation $\Delta\theta$ remains constant. The vortex movement can be measured experimentally to see the flat phase.

We have studied the fluctuation of θ_i in Josephson ladder arrays under a random magnetic field and found a rough to flat phase transition. Near the transition point, the characteristic time t^* diverges following a power law. Since the randomness can be realized by nonunifom sizes of plaquettes in JJA's even for constant external magnetic fields, we expect to see this kind of transition by measuring the I-V curves. The measurement of $\Delta\theta$ in a numerical simulation will provide more insight to our understanding of JJA's.

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