Stability and single-particle properties of bosonized Fermi liquids

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We study the stability and single-particle properties of Fermi liquids in spatial dimensions greater than one via bosonization. For smooth nonsingular Fermi-liquid interactions we obtain Shankar's renormalization-group flows to second order in the BCS coupling and reproduce well-known results for quasiparticle lifetimes. We demonstrate by explicit calculation that spin-charge separation does not occur when the Fermi-liquid interactions are regular. We also explore the relationship between quantized bosonic excitations and zero-sound modes and present a concise derivation of both the spin and the charge collective-mode equations. Finally we discuss some aspects of singular Fermi-liquid interactions.

I. INTRODUCTION

Landau's Fermi-liquid theory is an early example of what we would now call bosonization. The anticommuting operators which appear in the bare Hamiltonian describing the interactions among fermions disappear in Landau's effective theory. Instead only c-number quasiparticle occupancies appear in the semiclassical energy functional. That the low-energy semiclassical behavior of the Fermi liquid can be described in terms of these commuting variables suggests that a fully quantum bosonic description is obtainable.

Indeed, the Fermi-liquid state itself is an example of a zero-temperature quantum critical fixed point.[1] This fixed point is characterized by infinite U(1) symmetry which is *not* exhibited by the bare Hamiltonian. The infinite U(1) symmetry simply reflects the conservation of quasiparticle occupancy at each point on the Fermi surface.[2,3] Shankar[4] has used the functional renormalization-group (RG) approach to show that the Fermi-liquid state is a generic feature of interacting fermions, at least at weak coupling and in the absence of the usual superconducting and charge- and spin-density wave instabilities. By establishing rigorous bounds, other workers have studied the stability question at all orders in perturbation theory, but under more restrictive conditions such as a perfectly circular Fermi surface.[5]

Haldane has asserted recently that a fully quantum description of Fermi liquids in dimensions greater than 1 is obtainable via bosonization.[2] This viewpoint has been elaborated on by two of us.[3] In the present paper we continue to develop this theory, first by showing that Shankar's renormalization-group result is obtained easily in the bosonized picture. Next we investigate the bosonic excitations in more detail. We show that collective modes are obtained in a semiclassical limit; furthermore, the calculation of the single-particle boson Green's function yields information about the quasiparticle properties. In particular, by using the bosonization transformation we obtain the *exact* fermion quasiparticle propagator. We expand the solution to second order in f_0 to compare it with earlier work and recover the well-known result that the imaginary part of the fermion self-energy is proportional to $\omega^2 \ln|\omega|$ in two dimensions and just ω^2 in three dimensions. We emphasize that the bosonization method yields nonperturbative information, so a natural next step would be to use it to study the effects of singular interactions. We comment on the nature of several such singular interactions.

II. RENORMALIZATION-GROUP ANALYSIS IN THE BOSONIC BASIS

Now that we know how to bosonize Fermi liquids we may use this picture to investigate the stability of the zero-temperature Fermi-liquid fixed point to perturbations. First we reproduce the renormalization-group results of Shankar[4] in the bosonic basis. Three channels of fermion two-body interactions are marginal in the RG sense: forward scattering zero sound (ZS), exchange scattering (ZS$'$), and Cooper pairing (BCS). For simplicity we consider a system of spinless fermions in two dimensions and a circular Fermi surface. The second assumption eliminates the possibility of nesting instabilities in the zero sound channels which might produce charge- or spin-density waves. We also assume that the BCS coupling function $V_{\rm BCS}(S)$ is rotationally invariant. The BCS interaction pairs particles of equal but opposite momenta. For now we turn off the two zero-sound channels; later we show that these channels have no effect on the renormalization of the BCS interactions.

Fermi fields ψ may be expressed[3] in terms of the boson fields ϕ as

$$\psi(S;\mathbf{x}) = \frac{1}{\sqrt{V}} \sqrt{\frac{\Omega}{a}} e^{i\mathbf{k}_S\cdot\mathbf{x}} \exp\left\{ i\frac{\sqrt{4\pi}}{\Omega}\phi(S;\mathbf{x})\right\} O(S), \tag{1}$$

where the dependence on time t is included implicitly in the spatial coordinates $\mathbf{x}$. S runs from 0 to 2π and labels the patch on the Fermi surface with momentum $\mathbf{k}_S$. V is the volume of the system which equals L^2 in two dimensions; the factor of $V^{-1/2}$ is introduced to keep the fermion anticommutation relations canonical. Both the ψ and ϕ fields live inside a squat box centered on S with height λ in the radial (energy) direction and width Λ along the Fermi circle. These two scales must be small in the following sense: $k_F \gg \Lambda \gg \lambda$. We satisfy these limits by setting $\lambda \equiv k_F/N$ and $\Lambda \equiv k_F/N^\alpha$ where $0 < \alpha < 1$ and $N \to \infty$. The quantity a in the bosonization formula Eq. (1) is a real-space cutoff given by $a \equiv 1/\lambda$. Here $\Omega \equiv \Lambda(L/2\pi)^2$ equals the number of states in the squat box divided by λ. Finally, $O(S)$ is an ordering operator introduced[3,6] to maintain Fermi statistics in the angular direction along the Fermi surface. (Anticommuting statistics are obeyed automatically in the radial direction.)

With this connection between the fermion and boson fields we may check a number of relationships. For example, the fermion fields obey canonical equal-time anticommutation relations,

$$\{\psi(S;\mathbf{x}), \psi^\dagger(T;\mathbf{y})\} = \delta_{S,T}\delta^2(\mathbf{x}-\mathbf{y}), \tag{2}$$

because the boson fields in configuration space obey equal-time commutation relations,

$$[\phi(S;\mathbf{x}), \phi(T;\mathbf{y})] = \begin{cases} \frac{i}{4}\Omega^2\ \delta_{S,T}\epsilon(\hat{\mathbf{n}}_S\cdot[\mathbf{x}-\mathbf{y}]), & |x_\perp - y_\perp| \ll 1/\Lambda \\ 0, & |x_\perp - y_\perp| \gg 1/\Lambda. \end{cases} \tag{3}$$

Here $\perp$ denotes directions perpendicular to the surface normal $\hat{\mathbf{n}}_S$ at patch S, and $\epsilon(x) = 1$ for $x > 1$; otherwise it equals -1. Normal ordered charge currents are defined in configuration space in terms of both the Fermi and Bose fields as

$$\begin{aligned} J(S;\mathbf{x}) &= V : \psi^\dagger(S;\mathbf{x})\psi(S;\mathbf{x}) : \\ &\equiv V \lim_{\epsilon\to 0} \{\psi^\dagger(S;\mathbf{x}+\epsilon\hat{\mathbf{n}}_S)\psi(S;\mathbf{x}) \\ &\quad -\langle\psi^\dagger(S;\mathbf{x}+\epsilon\hat{\mathbf{n}}_S)\psi(S;\mathbf{x})\rangle\} \\ &= \sqrt{4\pi}\ \hat{\mathbf{n}}_S\cdot\nabla\phi(S;\mathbf{x}) \ . \end{aligned} \tag{4}$$

The momentum-space charge current is defined by

$$J(S;\mathbf{q}) \equiv \sum_{\mathbf{k}} \theta(S;\mathbf{k}+\mathbf{q})\theta(S;\mathbf{k})\{\psi^{\dagger\alpha}_{\mathbf{k}+\mathbf{q}}\psi_{\alpha\mathbf{k}} - \delta^3_{\mathbf{q},0}n_{\mathbf{k}}\} \tag{5}$$

where $\theta(S;\mathbf{k}) = 1$ if $\mathbf{k}$ lies inside the squat box of dimensions $\lambda \times \Lambda$ centered at S and equals zero otherwise. Given this definition, plus the fact that the Fermi fields in momentum and real space are related in the usual way to preserve the canonical anticommutation relations with conventional normalization,

$$\psi(S;\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \theta(S;\mathbf{k})\ e^{i\mathbf{k}\cdot\mathbf{x}}\ \psi(\mathbf{k}) \ , \tag{6}$$

the two currents are related by a Fourier transform,

$$J(S;\mathbf{x}) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} J(S;\mathbf{q}) \ . \tag{7}$$

Both currents Eq. (4) and Eq. (5) are dimensionless. The free Hamiltonian, written in terms of the Fermi fields, may also be bosonized and the result is quadratic in the ϕ fields,

$$\begin{aligned} H_0 &= v_F \sum_S \int d^2x\ \psi^\dagger(S;\mathbf{x}) \left\{\frac{\hat{\mathbf{n}}_S\cdot\nabla}{i} - k_F\right\} \psi(S;\mathbf{x}) \\ &= \frac{2\pi v_F}{\Omega\ V} \sum_S \int d^2x\ \{(\hat{\mathbf{n}}_S\cdot\nabla)\phi(S;\mathbf{x})\}^2 \ . \end{aligned} \tag{8}$$

The unusual prefactor of $\frac{2\pi}{\Omega}\frac{v_F}{V}$ appearing in the bosonic Hamiltonian compensates for the anomalous right hand side of the boson commutation relations, Eq. (3), and thereby reproduces the correct spectrum.

The next step is to bosonize the BCS interaction. To simplify the following algebra we set the Fermi velocity equal to 1 ($v_F = 1$). A fermion in patch S of the Fermi surface is paired with a fermion in patch $-S$ which is directly opposite patch S. (Note that in angular coordinates, patches S and $-S$ correspond to θ and $\theta+\pi$, *not* θ and $-\theta$.) The BCS action expressed in terms of the four Fermi fields is

$$\begin{aligned} S_{\text{int}}[\psi,\psi^\dagger] = \sum_{S,T} \int dt\ d^2x\ &\frac{V_{\text{BCS}}(S-T)}{k_F}\psi^\dagger(-S;\mathbf{x}) \\ &\times\psi^\dagger(S;\mathbf{x})\ \psi(T;\mathbf{x})\ \psi(-T;\mathbf{x}) \ . \end{aligned} \tag{9}$$

Here, S and T only range over half of the Fermi surface to avoid double counting the pair interactions. The dimensionless coupling function V_{BCS} must change sign under inversion because the fermions are spinless (Pauli exclusion principle) so $V_{\text{BCS}}(\theta) = -V_{\text{BCS}}(\theta+\pi)$; also the interaction must be Hermitian so $V_{\text{BCS}}(S-T) = V_{\text{BCS}}(T-S)$. To avoid sign mistakes it is important to keep track of the order of the fermion operators during the transformation to bosons. Formally the correct sign is set via the ordering operator $O(S)$ but in practice it is easier to determine the sign by direct inspection of the fermion operators. To bosonize the interaction, each Fermi field is replaced by the right hand side of Eq. (1) which converts the interaction into the exponential of four ϕ fields.

$$\begin{aligned} S_{\text{int}}[\phi] = \left(\frac{\Lambda}{2\pi}\right)^2 \sum_{S,T} \int dt\ d^2x\ &\frac{V_{\text{BCS}}(S-T)}{k_F a^2} \\ &\times \cos\left[\frac{\sqrt{4\pi}}{\Omega}[\phi_S(\mathbf{x}) - \phi_T(\mathbf{x})]\right] \ , \end{aligned} \tag{10}$$

where $\phi_S(\mathbf{x}) \equiv \phi(S;\mathbf{x}) + \phi(-S;\mathbf{x})$.

Before implementing the RG transformation, first we discuss scaling at the zero-loop level. Since we are concerned with scaling in the direction parallel to the surface normal $\hat{\mathbf{n}}_S$, the integral over $\mathbf{x}$ space should be factorized into separate integrals over directions perpendicular and parallel to the Fermi surface normal. So $d^2x = dx_\perp dx_\parallel$ with only $dx_\parallel$ changing under scale transformations. Thus $\Lambda \to \Lambda$, $\lambda \to \lambda/s$, and $a \to sa$ with $s > 1$. Clearly, $dx_\parallel \to s dx_\parallel$ and $dt \to sdt$. The boson field ϕ is invariant under the scale transformation as this leaves both the quadratic part of the action, S_0, and the BCS interaction invariant (marginal).

Now we perform the renormalization-group transformation on the BCS interaction to derive the flow equation. The fast parts of the $\phi(T;\mathbf{k})$ fields are integrated out of the functional integral. To be precise, modes with momenta $\lambda/2s < |\mathbf{k}\cdot\hat{\mathbf{n}}_T| < \lambda/2$ will be eliminated. In practice, since it is easier to carry out the calculation in real space, instead we integrate out fields over short *distance* scales $2a < \mathbf{x}\cdot\hat{\mathbf{n}}_T < 2sa$. Next, we rescale space and time: $\hat{\mathbf{n}}_T\cdot\mathbf{x} \to s\hat{\mathbf{n}}_T\cdot\mathbf{x}$ and $t \to st$. After performing these two operations we obtain the new BCS interaction coefficients $V_{\rm BCS}(S-T;s)$ and we may repeat the process.

The integration over the fast modes is accomplished via the usual functional integral

$$\exp(-S[\phi;s]) = \prod_T \int_{2a<|\mathbf{x}\cdot\hat{\mathbf{n}}_T|<2sa} \mathcal{D}\,\phi(T;\mathbf{x}) \times \exp(-S_0 - S_{\rm int}) \,. \tag{11}$$

Since only the fast modes are integrated out it is convenient to break the boson fields into two parts, the slow modes ϕ' and the fast modes $\not{\phi}$. Now we may express the right hand side of Eq. (11) as $\langle\exp(-S_{\rm int})\rangle$ where the contraction is performed only over the fast $\not{\phi}$ fields. The renormalized interaction is obtained by treating the interaction perturbatively and expanding $\langle\exp(-S_{\rm int})\rangle$ in powers of $V_{\rm BCS}$. The first nontrivial term arises at second order, $\frac{1}{2}\langle S_{\rm int}^2\rangle$, and since the interaction is diagonal in real space, it is readily evaluated with the use of the real-space boson correlation function[3]

$$\langle \not{\phi}(T;\mathbf{x})\,\not{\phi}(T;\mathbf{0}) - \not{\phi}^2(T;\mathbf{0})\rangle \simeq \frac{\Omega^2}{4\pi}\ln\left(\frac{ia}{\mathbf{x}\cdot\hat{\mathbf{n}}_T + i\tau}\right), \quad |x_\perp\Lambda| \ll 1$$
$$\to -\infty, \quad |x_\perp\Lambda| \gg 1. \tag{12}$$

Here and in Eq. (11) we have made a Wick rotation to imaginary time to avoid the poles along the real-time axis. The evaluation of the cosine-cosine correlation function that appears in $\frac{1}{2}\langle S_{\rm int}^2\rangle$ is carried out by decomposing terms of the form

$$\cos\left[\frac{\sqrt{4\pi}}{\Omega}[\phi_S(\mathbf{x}) - \phi_T(\mathbf{x})]\right] \tag{13}$$

into exponentials involving the slow and fast fields, and then using the identity

$$e^A e^B =: e^{A+B} : \exp\langle AB + \tfrac{1}{2}(A^2+B^2)\rangle \,. \tag{14}$$

Since $V_{\rm BCS}(S-T) = V_{\rm BCS}(T-S)$ we obtain

$$\frac{1}{2}\langle S_{\rm int}^2\rangle = \left(\frac{\Lambda}{2\pi}\right)^3 \frac{1}{(k_F a)^2}\int d\tau d^2x \sum_{R,S}\cos\left[\frac{\sqrt{4\pi}}{\Omega}\,[\phi'_R(\mathbf{x}) - \phi'_S(\mathbf{x})]\right]\sum_T V_{\rm BCS}(R-T)\,V_{\rm BCS}(T-S)$$
$$\times\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu \int_{2a}^{2sa} du_\parallel\,\frac{1}{{u_\parallel}^2+\nu^2} + \{\text{irrelevant operators}\}\,, \tag{15}$$

where $u_\parallel$ is a spatial variable in the direction parallel to the surface normal at patch T and ν is an imaginary time variable. The second step is to replace all the variables with the rescaled ones. This procedure does not change Eq. (15) since $V_{\rm BCS}(S-T)$ is marginal. Thus the β function is

$$\frac{dV_{\rm BCS}(R-S;s)}{d\ln(s)} = -\frac{\Omega}{Vk_F}\sum_T V_{\rm BCS}(R-T;s)V_{\rm BCS}(T-S;s)\,. \tag{16}$$

The equation may be diagonalized by a Fourier transform over the interval $[0,2\pi)$: $V_m \equiv \int \frac{d\theta}{2\pi}e^{im\theta}\,V_{\rm BCS}(\theta)$. Note that only odd modes appear due to the requirement $V_{\rm BCS}(\theta) = -V_{\rm BCS}(\theta+\pi)$. Then

$$\frac{dV_m}{d\ln(s)} = -\frac{1}{2}\,V_m^2\,, \tag{17}$$

which agrees with Shankar's result in the fermion basis at one-loop order.[4] Clearly, a BCS instability exists if any of the channels are attractive ($V_m < 0$). If all the channels are repulsive, the Fermi-liquid fixed point is locally stable.

We now turn on the other two marginal interactions, forward and exchange scattering, and ask whether the Fermi-liquid interactions $f_c(S-T)$ which involve the ZS and ZS′ channels alter the RG flows.

$$H_0 \to H_0 + \frac{1}{2} \sum_{S,T} \int d^2x \; f_c(S-T) \; \psi^\dagger(S;\mathbf{x}) \; \psi(S;\mathbf{x}) \; \psi^\dagger(T;\mathbf{x}) \; \psi(T;\mathbf{x})$$

$$= H_0 + \frac{2\pi}{V^2} \sum_{S,T} \int d^2x \; f_c(S-T) \; [\hat{\mathbf{n}}_S \cdot \nabla\phi(S;\mathbf{x})] \; [\hat{\mathbf{n}}_T \cdot \nabla\phi(T;\mathbf{x})] \; . \tag{18}$$

Unlike the BCS interaction, the forward and exchange interactions are quadratic in the boson fields and therefore parametrize different Gaussian fixed points, each with the infinite U(1) symmetry. This symmetry is reflected in the fact that the Hamiltonian Eq. (18) is invariant under changes in the phase of the fermions by different amounts in each patch: $\psi(S;\mathbf{x},t) \to e^{i\theta(S)}\psi(S;\mathbf{x},t)$. Here we see the advantage of the bosonic representation: the Fermi-liquid parameters are incorporated in a nonperturbative way into H_0. We carry out the calculation of the modified bosonic correlation function in Sec. IV; here we just note that these modifications are subleading corrections to scaling that do not influence the leading RG flows of the BCS interaction. For example, though bosons in different patches are now correlated, this correlation is only of order $\frac{\Lambda}{k_F}$ times that of correlations within the same patch. So the leading behavior exhibited in Eq. (15) is unchanged. Thus we have the remarkable result that the leading-order stability of the Fermi-liquid fixed point against Cooper pairing is unaffected by the existence of either small or large Fermi-liquid parameters.

Actually, there is a subleading order instability: the Kohn-Luttinger effect.[4] The bare V_m's due to, say, a short-range repulsive interaction are all positive but tend rapidly to zero at large m. The ZS and ZS$'$ channels, on the other hand, generate irrelevant contributions in the BCS channel (down by a positive power of Λ/k_F) which renormalize the bare BCS interaction and therefore can make some of the V_m's slightly negative (unstable) at sufficiently large m. Because of the small size of these coefficients, this effect is important only at extremely low temperatures and therefore we expect that the essential physics remains controlled by the Fermi-liquid fixed point.

III. INTERACTING BOSONS IN THE SEMICLASSICAL LIMIT

Now that we have seen that the Fermi-liquid fixed point is locally stable in the absence of attractive BCS interactions, we turn to the problem of diagonalizing the bosonic Hamiltonian that describes the fixed point. As we shall see, the problem is not as simple as it might seem at first because the current operators behave as both creation and annihilation operators. So we begin with an approximate semiclassical solution that bypasses this difficulty and yields the familiar collective-mode equation.

The particle-hole excitations of Fermi liquids have a bosonic character. Furthermore, the excitations are of either the charge or spin type: the bosonized Hamiltonian may be written as $H = H_c + H_s$ to exhibit this factorization into charge and spin sectors. The charge sector in D dimensions (the volume $V = L^D$ now) is described by a Hamiltonian that is bilinear in the current operators[3] $J(\mathbf{S};\mathbf{q})$,

$$H_c = \frac{1}{2} \sum_{\mathbf{S},\mathbf{T}} \sum_{\mathbf{q}} V_c(\mathbf{S},\mathbf{T};\mathbf{q}) \; J(\mathbf{S};-\mathbf{q}) \; J(\mathbf{T};\mathbf{q}) \; , \tag{19}$$

where $\Omega \equiv (\frac{\Lambda L}{2\pi})^{D-1}(\frac{L}{2\pi})$. The Fermi-liquid interactions are $f_c(\mathbf{S},\mathbf{T}) \equiv F_c(\mathbf{S},\mathbf{T})/N(0)$, where the density of states at the Fermi surface, summed over both spin species, is given by $N(0) = \frac{k_F}{\pi v_F}$ for the case of the two-dimensional Fermi gas. These interactions are incorporated into V_c as matrix elements that couple currents in different patches:

$$V_c(\mathbf{S},\mathbf{T};\mathbf{q}) = \frac{1}{2} \; \Omega^{-1} v_F \delta^{D-1}_{\mathbf{S},\mathbf{T}} + \frac{1}{V} f_c(\mathbf{S}-\mathbf{T}) \; . \tag{20}$$

Note that with this definition, and given the relationship between the currents and the ϕ fields, Eq. (4), the charge Hamiltonian H_c of Eq. (19) agrees (up to a factor of 2 due to spin) with the form we found in the previous section, Eq. (18). The charge currents obey the equal-time U(1) current algebra,

$$[J(\mathbf{S};\mathbf{q}), J(\mathbf{T};\mathbf{p})] = 2\delta^{D-1}_{\mathbf{S},\mathbf{T}} \delta^D_{\mathbf{q}+\mathbf{p},\mathbf{0}} \Omega \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} \; ; \tag{21}$$

this algebra can be derived either from Eq. (3) and Eq. (4) or directly from Eq. (5) with the use of the canonical anticommutation relations for fermions.[3] The quadratic form of this Hamiltonian implies immediately that it describes a fixed point invariant under the scale transformations $\lambda \to \lambda/s$. Similarly, the spin sector is described by

$$H_s = \frac{1}{2} \sum_{\mathbf{S},\mathbf{T}} \sum_{\mathbf{q}} V_s(\mathbf{S},\mathbf{T};\mathbf{q}) \mathbf{J}(\mathbf{S};-\mathbf{q}) \cdot \mathbf{J}(\mathbf{T};\mathbf{q}), \tag{22}$$

where the spin currents commute with the charge currents and obey the more complicated SU(2) current algebra[3]

$$[J^a(\mathbf{S};\mathbf{q}), J^b(\mathbf{T};\mathbf{p})] = \frac{1}{2} \delta^{D-1}_{\mathbf{S},\mathbf{T}} \delta^{ab} \Omega \mathbf{q} \cdot \hat{\mathbf{n}}_{\mathbf{S}} \delta^D_{\mathbf{q}+\mathbf{p},\mathbf{0}} + i\delta^{D-1}_{\mathbf{S},\mathbf{T}} \epsilon^{abc} J^c(\mathbf{S};\mathbf{q}+\mathbf{p}) \tag{23}$$

($\{a,b,c\} = \{x,y,z\}$ label the three components of the spin). Fermi-liquid spin-spin interactions f_s appear in the Hamiltonian as coefficients that couple spin currents in different patches,

$$V_s(\mathbf{S},\mathbf{T};\mathbf{q}) = \frac{2}{3} v_F(\mathbf{S}) \Omega^{-1} \delta^{D-1}_{\mathbf{S},\mathbf{T}} + \frac{1}{V} f_s(\mathbf{S},\mathbf{T}) \; . \tag{24}$$

Here we consider only the case of Fermi-liquid interactions which are independent of the wave vector $\mathbf{q}$. For regular interactions, $\mathbf{q}$ dependence only gives rise to additional irrelevant operators which do not change the behavior of the system at leading order. With singular interactions, on the other hand, divergences arise as $\mathbf{q} \to \mathbf{0}$ and these divergences may in some instances introduce relevant interactions that destroy Fermi-liquid behavior.

The equations of motion for the charge and spin currents yield the corresponding collective-mode equations in the semiclassical limit. Using the Heisenberg equations of motion and the U(1) current algebra we readily obtain

$$\begin{aligned} i\frac{\partial}{\partial t}J(\mathbf{S};\mathbf{q}) &= [J(\mathbf{S};\mathbf{q}), H_c] \\ &= v_F\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}J(\mathbf{S};\mathbf{q}) \\ &\quad +\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}\frac{2\Omega}{V}\sum_\mathbf{T} f_c(\mathbf{S}-\mathbf{T})J(\mathbf{T};\mathbf{q}) \end{aligned} \tag{25}$$

for the charge sector. The first term on the right hand side has its origin in the free dispersion relation for particle-hole pairs of momentum $\mathbf{q}$ at patch $\mathbf{S}$. The second term couples currents in different patches. Note that

$$\frac{\Omega}{V}\sum_\mathbf{T} = \int \frac{d^Dk}{(2\pi)^D}\,\delta(|\mathbf{k}|-k_F), \tag{26}$$

so the second term reduces to the usual integral over the Fermi surface in the $N \to \infty$ continuum limit. The equation of motion for the non-Abelian spin currents contains, in addition to these two terms, a third term which makes the spins precess, when the system is magnetically polarized, in the local internal magnetic field:[8]

$$\begin{aligned} i\frac{\partial}{\partial t}J^a(\mathbf{S};\mathbf{q}) &= [J^a(\mathbf{S};\mathbf{q}), H_s] \\ &= v_F\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}J^a(\mathbf{S};\mathbf{q}) + \frac{1}{2}\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}\frac{\Omega}{V}\sum_\mathbf{T} f_s(\mathbf{S}-\mathbf{T})J^a(\mathbf{T};\mathbf{q}) - \frac{i}{V}\epsilon^{abc}\sum_\mathbf{k} J^b(\mathbf{S};\mathbf{k})\sum_\mathbf{T} f_s(\mathbf{S}-\mathbf{T})J^c(\mathbf{T};\mathbf{q}-\mathbf{k})\,. \end{aligned} \tag{27}$$

Note that the factor of 2/3 multiplying the Fermi velocity v_F in the free part of Eq. (24) does not appear in Eq. (27). The origin of the 2/3 factor is easy to understand in the Sugawara construction of the free fermion Hamiltonian out of current bilinears:[9] it reflects the SU(2) invariance of the spin currents which permits the replacement $\mathbf{J}(\mathbf{S};\mathbf{q})\cdot\mathbf{J}(\mathbf{S};-\mathbf{q}) \to 3J^z(\mathbf{S};\mathbf{q})J^z(\mathbf{S};\mathbf{q})$ for the purpose of computing the spectrum. The derivation given here, on the other hand, does not rely on this argument as spin rotational invariance is respected explicitly. Rather, the factor of 2/3 in Eq. (24) cancels contributions to the free spectrum which arise from both the δ^{ab} and the $i\epsilon^{abc}$ terms in the non-Abelian anomaly.

When an external magnetic field is applied, and $|\mathbf{q}|$ is small, the third term in Eq. (27) dominates. In the opposite limit of zero applied magnetic field, the spin equation is identical in form to the charge equation. Evidently bosonization captures all of the physics of charge and spin collective modes. The derivation is straightforward and relies only on the existence of the quadratic Hamiltonian and the Abelian and non-Abelian current-current commutation relations. In fact this approach may be useful in the study of highly spin-polarized Fermi liquids, a problem which has been examined recently by Meyerovich and Musaelian via the Green's function approach.[10]

As it stands these operator equations are exact, at least in the $N \to \infty$ limit in which the current algebras Eq. (21) and Eq. (23) become exact. The difficulty in obtaining exact solutions to either of these equations originates in the fact that neither algebra is equivalent to the canonical commutation relations for harmonic oscillators. Even the U(1) charge current algebra, Eq. (21), is nontrivial because the right hand side, the "anomaly," has indeterminate sign as $\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}$ can be either positive or negative. Therefore, even to diagonalize Eq. (25) requires a generalized Bogoliubov-unitary transformation which appears not to be reducible into a product of separate Bogoliubov and unitary transformations. This difficulty is in contrast to that found in one spatial dimension where a simple 2×2 Bogoliubov transformation is sufficient to decouple the currents associated with the left and right Fermi points.[3] Of course the equation of motion for the spin currents, Eq. (27), is even more difficult to solve as it is nonlinear.

In the next section we will diagonalize the charge current equation of motion by resumming the perturbative expansion for the propagator. First we find the semiclassical solution by taking the expectation value of both sides of these equations and identifying

$$\langle J(\mathbf{S};\mathbf{q})\rangle \equiv u(\mathbf{S};\mathbf{q}) \tag{28}$$

and

$$\langle J^a(\mathbf{S};\mathbf{q})\rangle \equiv S^a(\mathbf{S};\mathbf{q}) \tag{29}$$

as the amplitudes for charge and spin collective modes. Note that the collective-mode amplitudes are real valued in $\mathbf{x}$ space because $J^\dagger(\mathbf{S};\mathbf{q}) = J(\mathbf{S};-\mathbf{q})$. The replacement of the current operators by the c numbers u and $\mathbf{S}$ may be accomplished formally by introducing a coherent state basis that spans the space of geometric distortions of the Fermi surface.[7] For example, the charge collective-mode coherent states are generated by exponentials of the charge current operator,

$$|\Psi[u]\rangle = \exp\left\{\sum_\mathbf{S}\sum_\mathbf{q}\,\theta(\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S})\,\frac{u(\mathbf{S};\mathbf{q})}{2\Omega\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}}J(\mathbf{S};-\mathbf{q})\right\}|0\rangle, \tag{30}$$

where $|0\rangle$ represents the quiescent Fermi liquid. A simple computation then shows that

$$\frac{\langle\Psi[u]|J(\mathbf{T};\mathbf{p})|\Psi[u]\rangle}{\langle\Psi[u]|\Psi[u]\rangle} = u(\mathbf{T};\mathbf{p}), \tag{31}$$

consistent with our definition Eq. (28). In the spin equation we also must decouple the expectation value of the product of two spin current operators [the third term on the right hand side of Eq. (27)] into the product of expectation values $\mathbf{S}(\mathbf{T};\mathbf{k}) \times \mathbf{S}(\mathbf{T};\mathbf{q}-\mathbf{k})$. This decoupling is exact in the semiclassical limit of a macroscopically occupied zero-sound spin mode.

To shed light on the relationship between the semiclassical limit and the quantum regime, we present an alternative derivation of the collective-mode equation based on the usual identification of the pole in the two-point Green's function, but now in the presence of a background collective-mode field $u(\mathbf{S};\mathbf{q})$. For simplicity we focus on the charge sector. First note that H_c is diagonal in $\mathbf{q}$ space: the Hilbert space breaks up into a direct product of subspaces with different $\mathbf{q}$ and $-\mathbf{q}$. (States with $-\mathbf{q}$ on the hemisphere of the Fermi surface with $\mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S} > 0$, which we may call the "left" hemisphere, are coupled to states of $+\mathbf{q}$ on the opposite "right" hemisphere due to the indeterminate sign for the quantum anomaly.) Thus we may treat each $(\mathbf{q},-\mathbf{q})$ sector separately. Now we wish to compute the retarded Green's function

$$G_{\rm ret}([u];\mathbf{S};\mathbf{q},t) \equiv \frac{\langle\Psi[u]|J(\mathbf{S};\mathbf{q},t)J(\mathbf{S};-\mathbf{q},0)|\Psi[u]\rangle}{\langle\Psi[u]|\Psi[u]\rangle}\theta(t), \tag{32}$$

where

$$J(\mathbf{S};\mathbf{q},t) = \exp[iH_ct]J(\mathbf{S};\mathbf{q})\exp[-iH_ct] \tag{33}$$

is the current operator in the Heisenberg picture. We may choose $\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q} > 0$ so that $J(\mathbf{S};-\mathbf{q},0)$ creates a particle-hole pair at time $t=0$ while $J(\mathbf{S};\mathbf{q},t)$ destroys a pair at a later time $t>0$. The crucial step in our calculation is to ignore operator ordering within the time evolution operators $\exp[\pm iH_ct]$. This approximation is exact so long as the zero mode has macroscopic occupation since in this case the errors introduced by ignoring operator ordering are small compared to the total energy. In other words, we should think of the operator $J(\mathbf{S};\mathbf{q})$ as removing just one quantum out of the large number of quanta that make up the macroscopic zero mode. Macroscopic occupation corresponds to $|u(\mathbf{S};\mathbf{q})|^2 \gg \Omega|\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q}|$ which in physical terms means that there are a large number of quanta at each point in momentum space on the Fermi surface. (Macroscopic occupancy is possible only in the ω limit of $\lambda \gg |\mathbf{q}|$; in the opposite $\mathbf{q}$ limit the Pauli exclusion principle keeps the occupancy small.) Assuming macroscopic occupancy we have

$$\exp\{iH_ct\}J(\mathbf{S};\mathbf{q})\exp\{-iH_ct\} = J(\mathbf{S};\mathbf{p})\ \exp\{iE[u(\mathbf{S};\mathbf{q})]t\}, \tag{34}$$

where $E[u]$ is the c-number energy given by

$$\begin{aligned} E[u(\mathbf{S};\mathbf{q})] &= \frac{\langle\Psi[u]|[J(\mathbf{S};\mathbf{q},t),H_c]J(\mathbf{S};-\mathbf{q},0)|\Psi[u]\rangle}{\langle\Psi[u]|J(\mathbf{S};\mathbf{q},t)J(\mathbf{S};-\mathbf{q},0)|\Psi[u]\rangle} \\ &= \mathbf{q}\cdot\hat{\mathbf{n}}_\mathbf{S}\Bigg[v_F + \frac{2\Omega}{V}\,\frac{1}{u(\mathbf{S};\mathbf{q})} \\ &\quad \times \sum_\mathbf{T} f_c(\mathbf{S}-\mathbf{T})u(\mathbf{T};\mathbf{q})\Bigg]\,. \end{aligned} \tag{35}$$

Thus the Green's function is

$$G_{\rm ret}([u];\mathbf{S};\mathbf{q},t) = |u(\mathbf{S};\mathbf{q})|^2\ \exp\{iE[u]t\}\theta(t)\ ; \tag{36}$$

its poles in frequency space at $\omega = E[u]$ clearly correspond to solutions of the collective-mode equation Eq. (25) in the semiclassical limit.

IV. QUANTIZED BOSONS

In this section we calculate the *exact* boson Green's function in the quiescent state, that is, in the absence of macroscopic excitations. In this case the semiclassical approximation is inapplicable and the problem must be treated quantum mechanically. To simplify the calculation we restrict our attention to the case of spherical (circular in two dimensions) Fermi surfaces and only a single Fermi-liquid parameter, the constant term F_0. Furthermore, we consider only spinless fermions and again set the Fermi velocity equal to 1. None of these simplifications is essential.

First we write the currents in terms of boson operators that satisfy canonical commutation relations. The choice

$$\begin{aligned} J(\mathbf{S};\mathbf{q}) = \sqrt{\Omega|\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q}|}\,[&a(\mathbf{S};\mathbf{q})\theta(\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q}) \\ &+a^\dagger(\mathbf{S};-\mathbf{q})\theta(-\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q})] \end{aligned} \tag{37}$$

with

$$[a(\mathbf{S};\mathbf{q}),a^\dagger(\mathbf{T};\mathbf{p})] = \delta^{D-1}_{\mathbf{S},\mathbf{T}}\delta^D_{\mathbf{q},\mathbf{p}}\ , \tag{38}$$

and $\theta(x)=1$ if $x>0$ and zero otherwise, satisfies the U(1) commutation relations Eq. (21) up to a factor of 2 which does not appear here since the fermions are spinless. The Hamiltonian Eqs. (19) and (20) can now be written as $H_c = H_0 + H_{\rm int}$, where

$$\begin{aligned} H_0 = \sum_\mathbf{q}\Bigg\{&\sum_\mathbf{S}\theta(\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q})(\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q})a^\dagger(\mathbf{S};\mathbf{q})a(\mathbf{S};\mathbf{q}) \\ &+\sum_\mathbf{S}\theta(-\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q})(-\hat{\mathbf{n}}_\mathbf{S}\cdot\mathbf{q})a^\dagger(\mathbf{S};-\mathbf{q})a(\mathbf{S};-\mathbf{q})\Bigg\} \end{aligned} \tag{39}$$

and

$$H_{\rm int} = \sum_{\mathbf{q}} \Big\{ g_R \sum_{\mathbf{S},\mathbf{T}} \theta(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})\theta(\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})\sqrt{(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})(\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})}\,a^\dagger(\mathbf{S};\mathbf{q})a(\mathbf{T};\mathbf{q})$$
$$+g_L \sum_{\mathbf{S},\mathbf{T}} \theta(-\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})\theta(-\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})\sqrt{(-\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})(-\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})}\,a^\dagger(\mathbf{S};-\mathbf{q})a(\mathbf{T};-\mathbf{q})$$
$$+g \sum_{\mathbf{S},\mathbf{T}} \theta(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})\theta(-\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})\sqrt{(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})(-\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})}\,a(\mathbf{S};\mathbf{q})a(\mathbf{T};-\mathbf{q})$$
$$+\overline{g} \sum_{\mathbf{S},\mathbf{T}} \theta(-\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})\theta(\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})\sqrt{(-\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})(\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})}\,a^\dagger(\mathbf{S};-\mathbf{q})a^\dagger(\mathbf{T};\mathbf{q}) \Big\}, \tag{40}$$

with couplings $g_R = g_L = g = \overline{g} \equiv f_0 \frac{\Lambda^{D-1}}{(2\pi)^D}$. It will be convenient to denote $a(\mathbf{S};\mathbf{q})$ and $a(\mathbf{S};-\mathbf{q})$ by $a_R(\mathbf{S};\mathbf{q})$ and $a_L(\mathbf{S};\mathbf{q})$, respectively the right and left moving fields.

The generating functional for the zero-temperature correlation functions is given by an integral over the coherent state eigenvalues $a_i(\mathbf{S};\mathbf{q},t)$ and $a_i^*(\mathbf{S};\mathbf{q},t)$:

$$Z = \int \mathcal{D}a^*\mathcal{D}a \, \exp\left\{ i \int_{-\infty}^{\infty} dt \big[i a_i^* \frac{\partial}{\partial t}\, a_i - H_c(a^*, a)\big] \right\}, \tag{41}$$

where there is an implicit sum over $i = L, R$ and the patch index $\mathbf{S}$ which has been suppressed. The momentum-frequency space propagator

$$iG_i(\mathbf{S};\mathbf{q},\omega) = \langle a_i(\mathbf{S};\mathbf{q},\omega) a_i^\dagger(\mathbf{S};\mathbf{q},\omega)\rangle \tag{42}$$

is related to the propagator of the ϕ fields by

$$\langle \phi_i(\mathbf{S};\mathbf{q},\omega)\phi_i(\mathbf{S};-\mathbf{q},-\omega)\rangle$$
$$= \frac{\Omega}{4\pi \hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q}} \langle a_i(\mathbf{S};\mathbf{q},\omega)a_i^\dagger(\mathbf{S};\mathbf{q},\omega)\rangle \ . \tag{43}$$

We now calculate the propagator perturbatively with the use of the bare right and left propagators:

$$iG_R^0(\mathbf{S};\mathbf{q},\omega) = \langle a_R(\mathbf{S};\mathbf{q},\omega)a_R^\dagger(\mathbf{S};\mathbf{q},\omega)\rangle_0$$
$$= \frac{i}{\omega - \hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q} + i\eta\,\mathrm{sgn}(\omega)} \tag{44}$$

and

$$iG_L^0(\mathbf{S};\mathbf{q},\omega) = \langle a_L(\mathbf{S};\mathbf{q},\omega)a_L^\dagger(\mathbf{S};\mathbf{q},\omega)\rangle_0$$
$$= \frac{i}{\omega + \hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q} + i\eta\,\mathrm{sgn}(\omega)} \ . \tag{45}$$

The bare propagators are depicted in Fig. 1(a).

At first order in $H_{\rm int}$ there is only one connected contribution to the right two-point function and it is given by

$$iG_R^{(1)} = (-i)(iG_R^0)g_R(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})(iG_R^0), \tag{46}$$

where we have suppressed the patch, momentum, and frequency labels of the Green's function. Amputating the external legs, we find that the first order contribution to the self-energy is just $\Sigma^{(1)}(\mathbf{S};\mathbf{q},\omega) = g_R(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})$. At higher orders it is easy to see that the anomalous couplings g and $\overline{g}$ occur in pairs. In particular at second order there are two contributions to the one-particle self-energy, and these are shown in Fig. 1(b). As we build up the complete set of contributions to the right moving propagator, we split each scattering process, for example those shown in Fig. 2, into a forward scattered contribution (which involves an intermediate state in the same patch $\mathbf{S}$) and a remainder in which the boson has been scattered into a virtual state in a different patch. We can then construct the Dyson equation, depicted schematically in Fig. 3, where the irreducible self-energy $\Sigma^I(\mathbf{S};\mathbf{q},\omega)$ comprises all amputated diagrams that cannot be split into two by cutting a single bare right moving propagator. At the second order the contribution to the irreducible self-energy is therefore

$$\Sigma^{(2)}(\mathbf{S};\mathbf{q},\omega) = g^2(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})\Big\{ \sum_{\mathbf{T}\neq\mathbf{S}} \theta(\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})(\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})G_R^0(\mathbf{T};\mathbf{q},\omega) + \sum_{\mathbf{T}} \theta(-\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})(-\hat{\mathbf{n}}_{\mathbf{T}}\cdot\mathbf{q})G_L^0(\mathbf{T};\mathbf{q},-\omega)\Big\}. \tag{47}$$

Now we specialize to the case of two spatial dimensions. The sums over patches can be converted to integrals in the $N \to \infty$ limit where $\Lambda \to 0$ as

$$\Lambda \sum_S = k_F \int_0^{2\pi} d\phi = 2\pi N(0) \int_0^{2\pi} d\phi, \tag{48}$$

where $N(0) = \frac{k_F}{2\pi}$ for spinless fermions in units where $v_F = 1$. The second order contribution to the self-energy can now be written more concisely as

$$\Sigma^{(2)}(S;\mathbf{q},\omega) = -f_0 \frac{f_0\Lambda}{(2\pi)^2}\,(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})\chi^0(\mathbf{q},\omega), \tag{49}$$

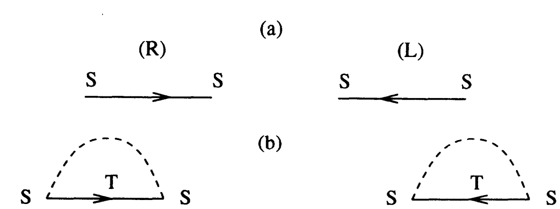

FIG. 1. Boson Green's functions. (a) Right and left moving bare boson propagators $G_R^0(\mathbf{S};\mathbf{q},\omega)$ and $G_L^0(\mathbf{S};\mathbf{q},\omega)$. (b) The two second order contributions to the self-energy which involve virtual states on the right and left sides of the Fermi surface.

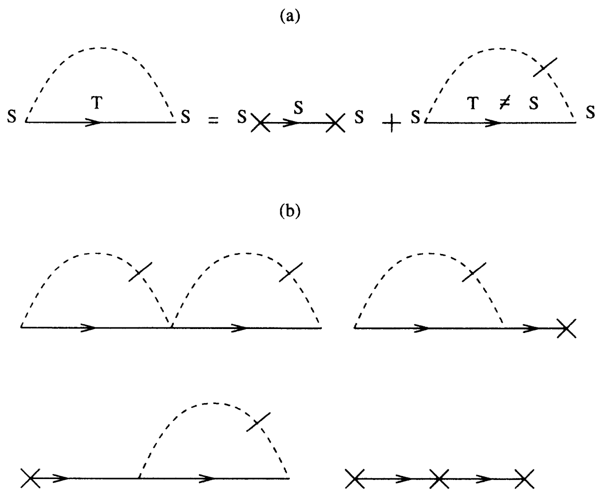

FIG. 2. Self-energy at second and third order for the boson propagator. (a) The second order contribution to the self-energy which involves virtual states on the right side of the Fermi surface. The first diagram on the right hand side of the equation (with two crosses) represents scattering into and out of the same patch S. The second diagram represents scattering into and out of a *different* patch $T \neq S$ (denoted by a dashed line with a slash). (b) Some of the third order contributions to the self-energy. Not shown are contributions which involve virtual states on the left side of the Fermi surface. Of the diagrams shown, only the first (with two dotted lines) contributes to the irreducible self-energy Σ^I; the remaining three diagrams break into two pieces when one of the bare propagators is cut.

where

$$\chi^0(x) = N(0) \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\cos(\phi)}{\cos(\phi) - x - i\eta \,\mathrm{sgn}(\omega)}$$
$$= N(0) \left\{ 1 - |x| \frac{\theta(x^2-1)}{\sqrt{x^2-1}} + i|x| \frac{\theta(1-x^2)}{\sqrt{1-x^2}} \right\}$$
$$\equiv N(0)\ \Omega_0(x) \tag{50}$$

is the two-dimensional Lindhard function and $x \equiv \frac{\omega}{|\mathbf{q}|}$. The exact solution of the Dyson equation Fig. 3 is then given by

$$\Sigma^I(S;\mathbf{q},\omega) = \frac{f_0 \Lambda}{(2\pi)^2}\ (\hat{\mathbf{n}}_S \cdot \mathbf{q})[1 - f_0 \chi(\mathbf{q},\omega)], \tag{51}$$

where

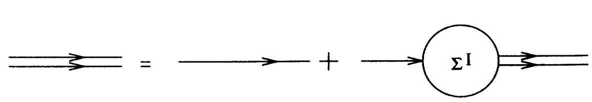

FIG. 3. The Dyson equation for the self-energy. The double line represents the exact one-particle boson propagator.

$$\chi(\mathbf{q},\omega) = \frac{\chi^0(\mathbf{q},\omega)}{1 + f_0 \chi^0(\mathbf{q},\omega)}\ . \tag{52}$$

Here we see that the equilibrium Fermi-liquid stability criterion $F_0 = f_0 N(0) > -1$ is necessary to keep the self-energy nonsingular in the $\mathbf{q}$ limit of $|x| \ll 1$. Also, the self-energy diverges at frequencies and momenta corresponding to the solutions of the collective-mode equation, as it should. A little algebra then shows that the exact right moving boson propagator can be written in the compact form

$$iG_R(S;\mathbf{q},\omega) = \frac{i}{\omega - \hat{\mathbf{n}}_S \cdot \mathbf{q} \left\{ 1 + \frac{f_0 \Lambda}{(2\pi)^2}\ [1 - f_0 \chi(\mathbf{q},\omega)] \right\}}. \tag{53}$$

Despite the fact that a perturbative expansion has been used as an intermediate step to obtain Eq. (51), all terms in the expansion have been summed to give the exact non-perturbative result valid for arbitrary dimension D. For example, in $D = 1$ the resummed expansion yields the well-known exact result for a Luttinger liquid.[3]

Quasiparticle damping occurs in the $\mathbf{q}$ limit when the Lindhard function has an imaginary part. In this regime we may write

$$\mathrm{Im}\ f_0 \chi(x) = \mathrm{Im} \left\{ \frac{F_0 \Omega_0(x)}{1 + F_0 \Omega_0(x)} \right\}$$
$$= \mathrm{Im} \left\{ \frac{A_0 \Omega_0(x)}{1 - A_0[1 - \Omega_0(x)]} \right\}$$
$$\approx A_0 |x|, \tag{54}$$

where $A_0 \equiv \frac{F_0}{1+F_0}$ and the boson Green's function then reads (for small A_0)

$$iG_R(S;\mathbf{q},\omega) = i \left\{ \omega - v_F' \hat{\mathbf{n}}_S \cdot \mathbf{q} + i \hat{\mathbf{n}}_S \cdot \mathbf{q}\ \frac{A_0^2 \Lambda |\omega|}{2\pi k_F |\mathbf{q}|} \right\}^{-1}, \tag{55}$$

where the velocity is slightly renormalized from its bare value of unity: $v_F' = 1 + F_0(1 - F_0)\frac{\Lambda}{2\pi k_F}$. The boson lifetime is now finite because of scattering into different patches. Note, however, that as the self-energy Eq. (51) scales to zero as $\Lambda \to 0$ it represents an irrelevant correction. In particular, the pole in the boson propagator remains unchanged as $N \to \infty$. We expect this to be true generally, regardless of the shape of the Fermi surface, the details of the Fermi-liquid parameters, or whether the fermions have spin or not. The renormalization-group calculation of the second section therefore holds, without alteration, when the Fermi-liquid interactions ZS and ZS′ are turned on.

V. FERMION QUASIPARTICLE PROPERTIES

In the previous section we saw that Fermi-liquid interactions modify the boson propagator. Though BCS processes were ignored, small-angle scattering processes made the boson lifetime finite. With these results we can use the bosonization formula Eq. (1) to infer the fermion quasiparticle lifetime. Since bosonization is carried out in $(\mathbf{x}, t)$ space we must carry out three operations. First we Fourier transform the boson propagator into real space. Next the exponential of the resulting expression yields the fermion propagator in real space. Finally an inverse Fourier transform of the fermion propagator back into momentum space allows us to extract the self-energy.

It is difficult technically to perform these steps in all generality. It will be sufficient for our purposes to first expand the boson propagator in powers of f_0, perform the three operations on each term, and then reassemble the pieces to find the fermion self-energy. Further, as we are interested only in the leading (second order) contribution to the imaginary part of the self-energy, we can avoid the first of the two Fourier transforms. The real-space and time boson Green's function may be written as

$$\begin{aligned} iG_\phi(S;\mathbf{x},t) &\equiv \langle \phi(S;\mathbf{x},t)\phi(S;\mathbf{0},0) - \phi^2(S;\mathbf{0},0)\rangle \\ &= \mathcal{F}\{iG_\phi\} \\ &= \mathcal{F}\{iG_\phi^{(0)} + iG_\phi^{(1)} + iG_\phi^{(2)} + \cdots\}, \end{aligned} \tag{56}$$

where $\mathcal{F}$ represents the Fourier transform operation that converts the variables $(\mathbf{q}, \omega)$ to $(\mathbf{x}, t)$. In the second line, iG_ϕ which is given by Eq. (43) has been expanded in powers of f_0. The Fourier transform of the leading term, $\mathcal{F}\ \{iG_\phi^{(0)}\}$, is given by Eq. (12). Rather than Fourier transforming the first and second order corrections, we exponentiate this expression to obtain the fermion propagator:

$$\begin{aligned} iG_\psi(S;\mathbf{x},t) &\equiv \langle \psi^\dagger(S;\mathbf{x},t)\ \psi(S;\mathbf{0},0)\rangle \\ &= \frac{\Omega}{Va} e^{i\mathbf{k}_S\cdot\mathbf{x}}\ \exp\left\{\frac{4\pi}{\Omega^2}\ iG_\phi(S;\mathbf{x},t)\right\} \\ &= \frac{i\Lambda}{(2\pi)^2}\ \frac{e^{ik_F x_\parallel}}{\mathbf{x}\cdot\hat{\mathbf{n}}_S - t + i\delta\ \mathrm{sgn}(t)}\ \exp\left\{\frac{4\pi}{\Omega^2}\ \mathcal{F}\big[iG_\phi^{(1)} + iG_\phi^{(2)} + \cdots\big]\right\} \\ &= \frac{i\Lambda}{(2\pi)^2}\ \frac{e^{ik_F x_\parallel}}{\mathbf{x}\cdot\hat{\mathbf{n}}_S - t + i\delta\ \mathrm{sgn}(t)}\ \{1 + i\mathcal{F}[\tilde{G}_\phi^{(1)}] - \tfrac{1}{2}(\mathcal{F}[\tilde{G}_\phi^{(1)}])^2 + i\mathcal{F}[\tilde{G}_\phi^{(2)}] + O(f_0^3)\}, \end{aligned} \tag{57}$$

where in the last two lines we have assumed $|x_\perp \Lambda| \ll 1$ and in the last line we have absorbed the factor of $4\pi/\Omega^2$ into $\tilde{G}_\phi^{(i)} \equiv (4\pi/\Omega^2)G_\phi^{(i)}$. We are interested primarily in the imaginary part of the fermion self-energy. The first order contribution to the boson self-energy contained in $iG_\phi^{(1)}$ is purely real and therefore does not contribute. The leading contribution to the imaginary part of the fermion self-energy comes from $iG_\phi^{(2)}$, which is given by

$$\begin{aligned} iG_\phi^{(2)}(S;\mathbf{q},\omega) = -i\frac{\Omega}{4\pi}\ \frac{f_0^2\Lambda}{(2\pi)^2}\ \chi^0(\mathbf{q},\omega) \\ \times[\omega - q_\parallel + i\eta\ \mathrm{sgn}(\omega)]^{-2}\ . \end{aligned} \tag{58}$$

Since $|x_\perp \Lambda| \ll 1$ this contribution to the boson propagator must be integrated over $q_\perp$, using Eq. (50) for χ^0, and we obtain

$$i\,\mathrm{Im} \int_{-\lambda/2}^{\lambda/2} dq_\perp \chi^0(S;q_\perp,q_\parallel,\omega) = -iN(0)|\omega|\ \ln\left|\frac{\omega^2 - q_\parallel^2}{\lambda^2}\right| . \tag{59}$$

The appearance of the logarithm in this equation is peculiar to two spatial dimensions. In three dimensions the integral is a two-dimensional one over the two coordinates perpendicular to the Fermi surface normal and the resulting imaginary part of the fermion self-energy is proportional simply to $\lambda\omega^2/k_F^2 + O(\omega^3/k_F^2)$. The appearance of the cutoff λ in this expression agrees with results from traditional Fermi-liquid theory.[8]

We now take the inverse Fourier transform $\mathcal{F}^{-1}$. Expanding the fermion propagator as $G_\psi \equiv G_\psi^0 + \delta G_\psi + \cdots$, the leading imaginary contribution to the fermion propagator in $(k_\parallel, \omega)$ space is given by

$$\begin{aligned} i\delta G_\psi(k_\parallel,\omega) = i\frac{f_0^2 N(0)}{(2\pi)^2} \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} dq_\parallel |\omega'| \ln\left|\frac{\omega'^2 - q_\parallel^2}{\lambda^2}\right| \\ \times\frac{1}{[\omega' - q_\parallel + i\eta\ \mathrm{sgn}(\omega')]^2\ [(\omega' - \omega) - (q_\parallel - k_\parallel) + i\eta\ \mathrm{sgn}(\omega' - \omega)]}\ . \end{aligned} \tag{60}$$

The integral may be performed by complex integration; no divergences occur since all the poles in the complex $q_\parallel$ plane lie on either one side of the real axis or the other unless ω' lies between 0 and ω. Thus, except for this limited range of frequencies, the contour in the $q_\parallel$ plane may be closed without enclosing any poles. The result is

$$i\delta G_{\psi}(k_{\|},\omega) = -\frac{1}{2}\,\frac{f_0^2 N(0)}{(2\pi)^2}\,\frac{1}{[\omega - k_{\|} + i\eta\ \mathrm{sgn}(\omega)]^2}\ \mathrm{sgn}(\omega)\,\Big\{[\omega^2 + (\omega - k_{\|})^2/4]\ \ln\frac{|\omega - k_{\|}|}{\lambda} + [\omega^2 - (\omega - k_{\|})^2/4]\ \ln\frac{|\omega + k_{\|}|}{\lambda} - \frac{\omega}{2}\,(2\omega - k_{\|})\Big\}, \tag{61}$$

and therefore the fermion self-energy at this order is given by

$$\mathrm{Im}\ \Sigma_f^{(2)}(k_{\|},\omega) = \frac{1}{2}\,\frac{f_0^2 N(0)}{(2\pi)^2}\mathrm{sgn}(\omega)\,\Big\{[\omega^2 + (\omega - k_{\|})^2/4]\ \ln\frac{|\omega - k_{\|}|}{\lambda} + [\omega^2 - (\omega - k_{\|})^2/4]\ \ln\frac{|\omega + k_{\|}|}{\lambda} - \frac{\omega}{2}\,(2\omega - k_{\|})\Big\}\ . \tag{62}$$

The imaginary part of the self-energy at the quasiparticle pole is the inverse of the quasiparticle lifetime. The location of the pole has been shifted from its bare value to $\omega = v_F' k_{\|}$ due to renormalization of the Fermi velocity, $v_F' = 1 + F_0(1 - F_0)\frac{\Lambda}{2\pi k_F}$. As a result the imaginary part of the self-energy at the pole is given by

$$\mathrm{Im}\ \Sigma_f^{(2)}(\omega)|_{\mathrm{pole}} = \frac{1}{2}\,\frac{f_0^2 N(0)}{(2\pi)^2}\mathrm{sgn}(\omega) \times\Big\{\omega^2\ \ln\frac{|F_0|\ \Lambda\ \omega^2}{\pi\lambda^2 k_F} - \frac{\omega^2}{2}\Big\}, \tag{63}$$

the form of which we immediately recognize from previous work on two-dimensional Fermi liquids.[11–13] The quantity in braces is always negative since $\Lambda\omega^2 \ll \lambda^2 k_F$. Despite the appearance of the logarithm, the weight of the quasiparticle pole, $Z_{\mathbf{k}}$, remains nonzero at the Fermi surface; the regular Fermi-liquid interaction F_0 does not destroy the Fermi-liquid fixed point. We expect that more general regular Fermi-liquid interactions, the inclusion of the spin sector, or extensions to nonspherical Fermi surfaces, will not change this result qualitatively.

VI. SINGULAR INTERACTIONS

Singular Fermi-liquid interactions in two-dimensions (2D) were proposed by Anderson[14] and studied perturbatively by Stamp.[15,13] The interaction studied couples opposite spins and diverges as $\mathbf{k} \to \mathbf{k}'$:

$$f^{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \frac{b}{N(0)}\delta_{\sigma,-\sigma'}\frac{(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}-\mathbf{k}'|^2} \times\theta(|\mathbf{k}| - k_F)\theta(k_F - |\mathbf{k}'|)\ . \tag{64}$$

Note that both $\mathbf{k}$ and $\mathbf{k}'$ lie off the Fermi surface (respectively above and below it). Thus this interaction is of a more general sort than the type Landau originally envisaged in the phenomenological theory. The interaction diverges like $\frac{1}{|\mathbf{k}-\mathbf{k}'|}$ as $\mathbf{k} \to \mathbf{k}'$. It is related to the *regular* interaction that arises at second order in a Taylor-series expansion of the Landau function. In three dimensions this regular interaction gives rise to a $T^3 \ln T$ contribution to the specific heat:[16,3]

$$f(\mathbf{k},\mathbf{k}') = a + b\ (\hat{\mathbf{p}}\cdot\hat{\mathbf{q}})^2 + \cdots = a + b\left\{\frac{(\mathbf{k}+\mathbf{k}')\cdot(\mathbf{k}-\mathbf{k}')}{|\mathbf{k}+\mathbf{k}'||\mathbf{k}-\mathbf{k}'|}\right\}^2 + \cdots \tag{65}$$

for $\mathbf{q} \equiv \mathbf{k} - \mathbf{k}'$ and $\mathbf{p} \equiv \mathbf{k} + \mathbf{k}'$. This interaction vanishes if $\mathbf{k}$ and $\mathbf{k}'$ lie on the Fermi surface and approach the same point. If, on the other hand, both momenta lie away from the surface then the interaction approaches a nonzero, but finite, limiting value as the two momenta converge.

The bosonized Hamiltonians Eq. (19) and Eq. (22) generalize Fermi-liquid theory in a different way: $\mathbf{S}$ and $\mathbf{T}$ lie on the Fermi surface but $\mathbf{q}$ need not be zero. Nevertheless, the interactions mentioned above have natural counterparts in the bosonized theory. The $T^3 \ln T$ contribution to the specific heat is recovered in this picture[3] by setting $\mathbf{k} = \mathbf{k_S} + \mathbf{q}$ and $\mathbf{k}' = \mathbf{k_T} - \mathbf{q}$ in Eq. (65), to obtain

$$V_c(\mathbf{S},\mathbf{T};\mathbf{q}) = \frac{1}{2}\ \Omega^{-1} v_F \delta^{D-1}_{\mathbf{S},\mathbf{T}} + \frac{1}{VN(0)}\left\{a + 4b\ \frac{(\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q})^2}{(\mathbf{k_S} - \mathbf{k_T})^2 + 4q^2} + \cdots\right\}. \tag{66}$$

Note that this contribution to the specific heat is a subleading correction which just reflects the fact noted above that any $\mathbf{q}$ dependence of regular Landau parameters is irrelevant to the leading-order behavior. That both generalizations yield the same nonanalytic thermodynamic behavior suggests that they are equivalent up to irrelevant terms. Therefore we proceed to make the same substitution in the singular interaction Eq. (64) of Anderson and Stamp to find in the spin sector

$$V_s(\mathbf{S},\mathbf{T};\mathbf{q}) = \frac{1}{2}\ \Omega^{-1} v_F \delta^{D-1}_{\mathbf{S},\mathbf{T}} + \frac{b}{VN(0)}\ \frac{k_F^2\ |2\hat{\mathbf{n}}_{\mathbf{S}}\cdot\mathbf{q}\ /k_F|^{\beta}}{(\mathbf{k_S} - \mathbf{k_T})^2 + 4q^2}\ , \tag{67}$$

where β interpolates between the Anderson-Stamp interaction ($\beta = 1$) and the regular interaction ($\beta = 2$) (now for the spin sector). We will assume that $S \neq T$ in the second term (otherwise the linear dispersion relation is destroyed at the outset) and consider the scaling of the largest part of the interaction as we take $N \to \infty$. Observing that the largest contribution comes from nearest-neighbor patches $\mathbf{S}$ and $\mathbf{T}$, and $|\mathbf{q}| \leq \lambda \ll \Lambda$, we see that the interaction scales as

$$\frac{b}{VN(0)}\, N^{2\alpha-\beta}\,, \tag{68}$$

where the exponent α was defined in Sec. II by the equation $\Lambda = k_F/N^{\alpha}$, and therefore the interaction is singular only if $\beta < 2\alpha < 2$. The strong dependence on the exponent γ may indicate that no fixed point exists for the Anderson-Stamp interaction.

The Anderson-Stamp interaction is singular only as $S \to T$. An example of an interaction which is singular for all S and T and which has more obvious physical significance is afforded by the Coulomb interaction. The bare interaction may be factorized into contributions to the three channels (ZS, ZS$'$, and BCS). We assume that the BCS channel renormalizes to zero since it is repulsive. Furthermore, for small $|\mathbf{q}|$ the ZS$'$ exchange channel is much smaller than the ZS direct channel. In this limit we find

$$V_c(\mathbf{S},\mathbf{T};\mathbf{q}) = \frac{1}{2}\,\Omega^{-1} v_F \delta^2_{\mathbf{S},\mathbf{T}} + \frac{1}{V}\,\frac{4\pi e^2}{|\mathbf{q}|^2} \tag{69}$$

in three spatial dimensions with V_s containing only regular interactions. It would be interesting to determine the effect of the bare interaction Eq. (69) on the single quasiparticle lifetime. If the technical problem of performing the Fourier transforms mentioned in the previous section can be overcome, the bosonization method would yield nonperturbative insight into the effect of singular interactions on the Fermi liquid.

VII. DISCUSSION

The Coulomb interaction Eq. (69) mentioned in the previous section illustrates the difference between collective modes and single-particle excitations. If we substitute the Coulomb interaction Eq. (69) into the charge collective mode Eq. (25), and compute the spectrum, we find in three dimensions a gap comparable to the plasma frequency.[17] Thus charged Fermi liquids do not support low-energy collective modes in the charge sector. This should not be confused with the single-particle spectrum, which remains gapless. Thus, the spectrum of single-particle bosonic excitations about the quiescent state, unlike the collective modes, must also remain gapless.

The fact that the bosonized Hamiltonian separates into a sum of charge and spin parts, $H = H_c + H_s$, raises the specter that, as in one dimension,[3] the quasiparticle propagator might also exhibit spin-charge separation, even in the case of regular Fermi-liquid interactions. Spin-charge separation in dimensions larger than 1 would, however, destroy the Green's function approach to Fermi-liquid theory as the key element in that approach, the pole of the single-particle Green's function with spectral weight $0 < Z < 1$, would be replaced by a branch cut and Z would equal zero. This does not happen because, as we saw at the end of Sec. IV, the location of the pole of the boson propagator is unchanged from its free value in the $\Lambda \to 0$ limit. Consequently the spin and charge velocities are equal and spin-charge separation does not occur.

Finally we note that our bosonic analysis of the renormalization-group flows near the fixed point does not rely on a particular form for the fermion propagator. For example, by hand we could set $v_c \neq v_s$ by introducing a third sort of singular interaction: one that couples the charge current in any given patch with itself. Because the renormalization flows separate into charge and spin sectors, they are not modified by spin-charge separation and we would conclude that a fixed point with different spin and charge velocities is locally stable provided the BCS interaction is repulsive. Thus RG in the boson basis is more general than RG in the Fermi basis,[4,5] which assumes that the one-particle propagator retains a Fermi-liquid form; consequently, non-Fermi-liquid fixed points are ruled out from the outset of the calculation. The bosonized theory contains nonperturbative information; only technical difficulties prevent us from evaluating directly the nonperturbative fermion propagator. It should be possible to surmount these difficulties.

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