Phase locking of Fiske modes in sine-Gordon systems

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We investigate the dynamics of the sine-Gordon system when the length of the spatial interval is relatively small. In our analysis ac and dc forces are applied through the boundaries and a dc force is uniformly distributed over the entire spatial interval. Phase locking of Fiske modes to the external ac boundary signal is observed and analytical perturbation results are in excellent agreement with numerical experiments.

Phase locking of soliton oscillations in ac-driven sine-Gordon systems has been studied extensively over the past decade.[1–4] Both numerically and analytically the mechanisms for and the ranges of phase locking are reasonably well understood in the framework of the adiabatic perturbation technique. These studies have strong impact on possible future applications of Josephson junctions as local oscillators in superconducting electronics, since the sine-Gordon equation is a well established model for long Josephson junctions. For long Josephson junctions, oscillations of magnetic flux quanta [which manifest themselves as zero-field steps (ZFS) in the current-voltage (IV) characteristics] correspond to soliton oscillations in the sine-Gordon system. However, long Josephson junctions exhibit other resonant modes, which are just as important for technical applications as the ZFS's. Particularly, the Fiske modes in long Josephson junctions[5] have proven experimentally to be more stable than the ZFS's and therefore potentially very interesting for technical applications. This mode appears when the junction is embedded in an external dc magnetic field. Theoretically, the Fiske mode proves difficult to handle. Unlike the soliton oscillation mode, Fiske modes can only exist as a balance between external forces and dissipation in the system, and we do not have exact solutions for this particular mode. Two models have been proposed to explain Fiske steps in the IV characteristics of Josephson junctions. One model is the "Kulik theory"[6] which is valid for relatively short systems, and the other is a combined soliton and plasmon model valid for longer junctions.[7–10]

In this paper we investigate the Fiske modes when an external ac force is applied through the boundaries of the system. The parameter ranges of phase locking are found analytically and confirmed numerically. Experimentally, this corresponds to a situation where the junction is embedded in an external magnetic field, giving rise to the Fiske mode, and with a small ac component to which the mode can phase lock.

The model under investigation is

$$\phi_{xx} - \phi_{tt} - \sin\phi = \alpha\phi_t - \eta , \tag{1}$$

where ϕ describes the phase difference between the quantum mechanical wave functions of the two superconductors defining the Josephson junction. Space x is normalized to the characteristic Josephson length λ_J, and time t is normalized to the inverse of the plasma frequency, ω_J^{-1}. Tunneling of quasiparticles through the junction is represented by the dissipative parameter α, and η denotes a uniformly distributed dc bias current normalized to the maximum Josephson current of the junction. The boundary conditions for the finite size system are given by

$$\phi_x(0) = \phi_x(L) = \Gamma + \epsilon \sin\Omega t , \tag{2}$$

where Γ is a normalized magnetic dc field embedding the junction, and ϵ is the normalized amplitude of an ac magnetic field, Ω being its normalized frequency. The normalized length of the system is denoted L.

For $\epsilon = 0$ we can follow Ref. 6, writing the field in the form

$$\phi = \omega t + \Gamma x + \psi(x,t) , \tag{3}$$

where ψ is represented by the fundamental frequency only,

$$\begin{aligned}\psi &= \sum_{n=0}^{\infty} [A_n \cos\omega t + B_n \sin\omega t] \cos k_n x \\ &= \sum_{n=0}^{\infty} \sqrt{A_n^2 + B_n^2} \sin(\omega t + \theta_n) \cos k_n x , \end{aligned} \tag{4}$$

where

$$k_n = \frac{n\pi}{L} , \quad \tan\theta_n = A_n/B_n .$$

This function satisfies the boundary condition Eq. (2) for $\epsilon = 0$. Using this expression for ψ, inserting Eq. (3) into Eq. (1), and assuming that ψ is a small amplitude function we find the constants A_n and B_n to be

$$A_n = \frac{2\Gamma L}{1+\delta_{n,0}} \times \frac{[1-(-1)^n \cos \Gamma L]\left(\omega^2 - k_n^2\right) + \alpha\omega(-1)^n \sin \Gamma L}{[(\Gamma L)^2 - (n\pi)^2][(\omega^2 - k_n^2)^2 + (\alpha\omega)^2]}, \tag{5}$$

$$B_n = \frac{2\Gamma L}{1+\delta_{n,0}} \times \frac{(-1)^n\left(\omega^2 - k_n^2\right)\sin \Gamma L - \alpha\omega[1-(-1)^n \cos \Gamma L]}{[(\Gamma L)^2 - (n\pi)^2][(\omega^2 - k_n^2)^2 + (\alpha\omega)^2]}, \tag{6}$$

where $\delta_{n,0} = 1$ for $n = 0$ and zero otherwise. Clearly the system exhibits resonances for $\omega \simeq k_n$ and the dc component of Eq. (1) yields the following expression:

$$\eta = \alpha\omega + J(\omega), \tag{7}$$

where

$$J(\omega) = \sum_{n=0}^{\infty} \frac{2\alpha\omega(\Gamma L)^2[1-(-1)^n \cos \Gamma L]}{[(\Gamma L)^2 - (n\pi)^2]^2[(\omega^2 - k_n^2)^2 + (\alpha\omega)^2](1+\delta_{n,0})}. \tag{8}$$

The normalized energy of the system is defined by

$$H = \int_0^L [\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_t^2 + 1 - \cos\phi]dx. \tag{9}$$

Taking the time derivative of this expression and inserting Eqs. (1) and (2) we obtain

$$\dot{H} = \int_0^L \left[\eta\phi_t - \alpha\phi_t^2\right]dx + (\Gamma + \epsilon\sin\Omega t)\phi_t|_{x=0}^{x=L}. \tag{10}$$

Inserting the solution of Eqs. (3)–(6) into Eq. (10), assuming that $\omega = \Omega$, the following energy balance is obtained for the ac-driven phase-locked Fiske mode:

$$\int_0^{2\pi/\Omega}\int_0^L \dot{H}dx = 0 \tag{11}$$

$$\Rightarrow \frac{2\pi}{\Omega}L\eta\Omega = -\alpha\int_0^{2\pi/\Omega}\int_0^L \phi_t^2 dxdt + \pi\epsilon\sum_{n=0}^{\infty}\cos(\theta_{2n+1} - \theta) \times\sqrt{A_{2n+1}^2 + B_{2n+1}^2}, \tag{12}$$

where θ is a phase between the ac field and the Fiske mode. Note that only the odd spatial harmonics contribute to phase locking. From this expression it is now clear that a variation in $\eta = \eta_0 + \Delta\eta$, $\Delta\eta$ being small, can be compensated by an adjustment in θ. If we are considering phase locking at the mth Fiske step, i.e., when $\Omega \lesssim k_m$, we find that only the term originating from A_m and B_m contributes significantly; for $k_{m-1} < \Omega < k_m$, we can therefore write

$$2|\Delta\eta_m| < \delta\eta_m \equiv 2\frac{\epsilon}{L}\sqrt{A_m^2 + B_m^2} = \frac{8\epsilon\Gamma|\cos\frac{\Gamma L}{2}|}{\sqrt{[\Omega^2 - k_m^2]^2 + (\alpha\Omega)^2}[(\Gamma L)^2 - \pi^2 m^2]}, \tag{13}$$

for odd m, and

$$2|\Delta\eta_m| < \delta\eta_m \approx 0, \tag{14}$$

for even m.

The total size of the locking range in dc bias for the mth Fiske step is denoted $\delta\eta_m$. From this expression we see that phase locking appears as first order steps in the IV characteristics, symmetrically positioned around the IV curve of the Fiske step with no ac drive applied.

It is important here to discuss the limitations of the validity of this expression. The coefficients A_n and B_n are derived under the assumption that ψ is a small amplitude function, oscillating with the fundamental frequency only. This means that as long as these coefficients are relatively small we can be confident that the range of phase locking given above represents a good approximation. The range of phase locking is therefore a much better prediction than the predicted height [Eq. (8)] of the Fiske steps, since Eq. (8) is given for the largest possible amplitude of ψ.

The fact that all even Fiske steps show vanishing locking ranges is not surprising, since the even Fiske modes contain the symmetric spatial modes as their dominant contribution. The external ac magnetic field contributes with an asymmetric spatial oscillation and the power exchange between the external ac field and the even Fiske modes is therefore small. The odd Fiske modes are mainly generated by asymmetric spatial components and the coupling to the external asymmetric field is therefore possible.

In the limit of relatively small Γ the above theory fails due to the presence of higher harmonics in the dynamics. In this limit we can use the soliton picture[7,10] and assume that a Fiske mode is a soliton generated at one boundary, traveling through the system, and then annihilated at the other boundary. Phase locking of this mode can be described analogously to the phase-locked, shuttling soliton mode described in Ref. 4. The resulting size of phase-locking range is given by[4]

$$\delta\eta_1 = \frac{\epsilon}{L}\frac{\cosh\left\{\frac{\pi}{2L}\sqrt{1-(L\Omega/\pi)^2}\cos^{-1}[2(L\Omega/\pi)^2 - 1]\right\}}{\cosh\left[\frac{\pi^2}{2L}\sqrt{1-(L\Omega/\pi)^2}\right]}, \tag{15}$$

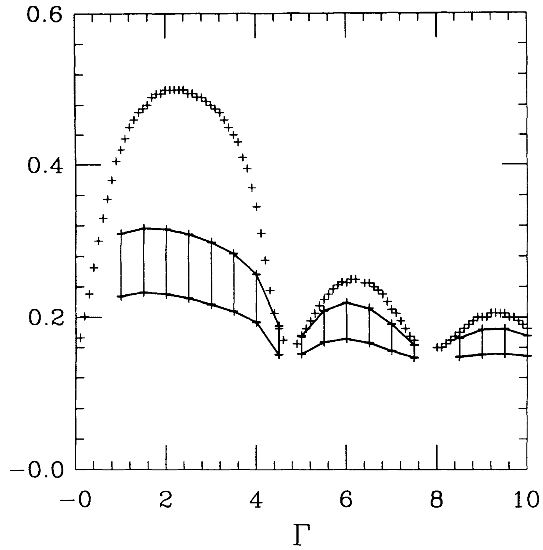

FIG. 1. Maximum dc bias η (+) for the first Fiske step as a function of dc magnetic field Γ. Parameters for the numerical simulations are $L = 2$, $\alpha = 0.1$, $\epsilon = 0$. For $\epsilon = 0.05$ and $\Omega = 1.5$ the range of phase locking is shown as markers connected by solid lines.

for $\Gamma L < 2\pi$. Note that this expression is valid only for the first Fiske step.

We have performed numerical simulations and compared the results to the above predictions. Due to the large parameter space we have limited simulations to the first Fiske step ($m = 1$), normalized length $L = 2$, and damping parameter $\alpha = 0.1$. In Fig. 1 we show the maximum dc bias value for the first Fiske step as markers (+) as a function of the external dc magnetic field Γ ($\epsilon = 0$). The height of the Fiske step has its minima for $\Gamma L = 3\pi/2+2\pi l$ (l integer) as given by Eq. (8). The error margin for the numerical data is $\sim 10^{-3}$, and therefore much smaller than the size of the markers. When $\epsilon \neq 0$ and $\Omega \lesssim \pi/L$, we may phase lock the normalized voltage $V \equiv \omega$ to the external frequency Ω. For a particular choice, $\epsilon = 0.05$ and $\Omega = 1.5 \lesssim \pi/L$, we show the resulting range in bias current for which the system phase locks. This range is displayed as vertical lines in Fig. 1. Clearly the size of the locking range follows the interference pattern of the size of the Fiske step, as predicted in Eq. (13). The linear dependence of the step size upon the ac amplitude is demonstrated in Fig. 2. Here we show

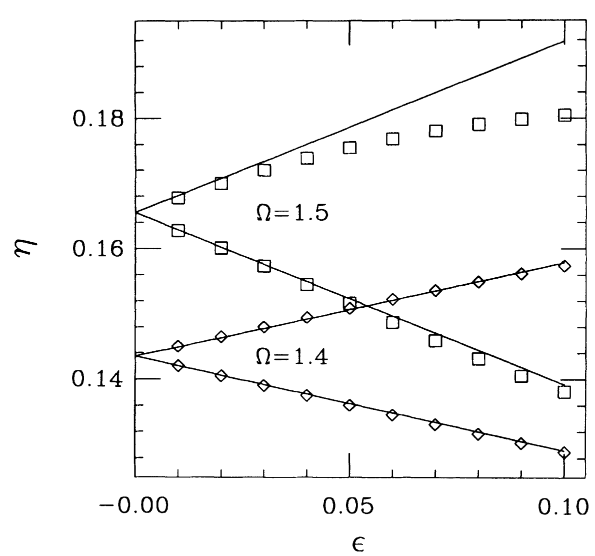

FIG. 2. Extrema of locking ranges as a function of the ac amplitude ϵ. Parameters are $L = 2$, $\alpha = 0.1$, $\Gamma = 5$. Markers represent results of numerical simulations, $\Omega = 1.4$ (◇) and $\Omega = 1.5$ (□), and solid lines represent the analytical result Eq. (13).

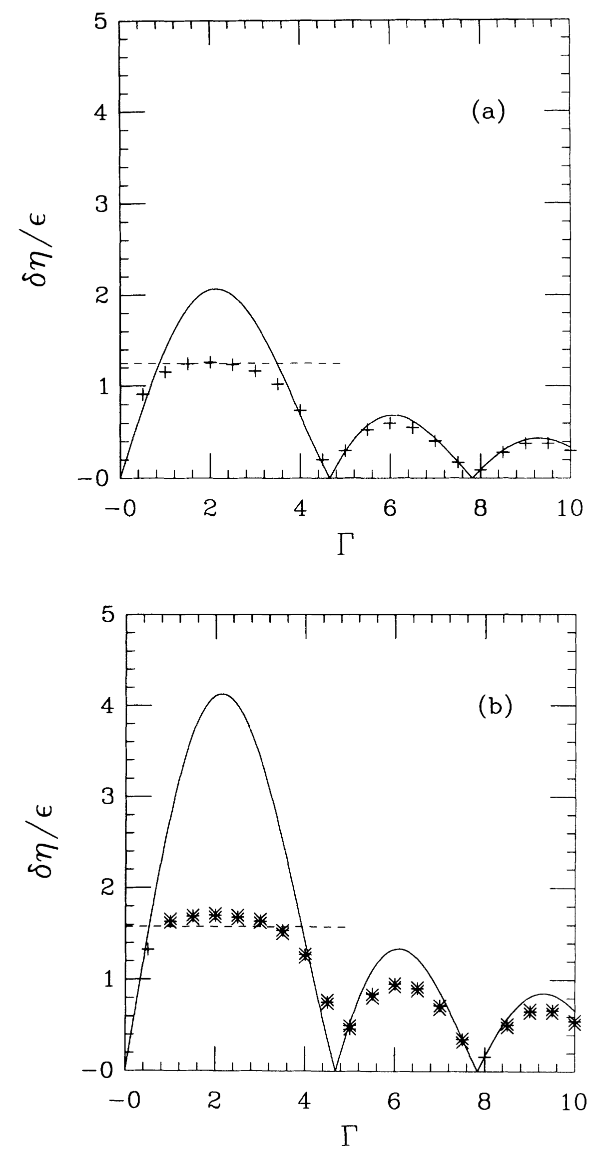

FIG. 3. Total size of the locking range in dc bias as a function of normalized dc magnetic field Γ. Parameters are $L = 2$, $\alpha = 0.1$, $\epsilon = 0.01$ (+), and $\epsilon = 0.05$ (×). Markers represent results of numerical simulations, $\Omega = 1.4$ (a) and $\Omega = 1.5$ (b), and solid lines represent the analytical result Eq. (13). Horizontal dashed line represents the soliton result Eq. (15).

the extrema of the phase-locked steps for $\Gamma = 5$ as a function of the ac amplitude for two different driving frequencies, $\Omega = 1.4$ ($\diamond$) and $\Omega = 1.5$ ($\square$). These markers represent the results of numerical simulations. It is worth noting that the linear dependence of the current amplitude of phase-locked steps (on Fiske singularities) on the external rf-drive amplitude was found in measurements on long Josephson junctions.[11]

The results presented in Figs. 2 and 3 are obtained by initiating the system in a phase-locked state and allowing a transient time of 4000 normalized time units before measuring the average frequency over 1000 normalized units. Then the normalized bias was changed by 2×10^{-5} and the procedure was repeated until the edge of the step was detected. The uncertainty of the displayed markers is therefore of the order of 4×10^{-5} in dc bias. For small ac amplitudes we find a linear and symmetric phase-locked step for both frequencies. The solid lines in Fig. 2 represent the predicted edges of the phase-locked step as given by Eq. (13). The agreement between the simulations and the analytical results is excellent. For increasing ac amplitudes the $\Omega = 1.5$ data deviate from the predicted curve. This is quite understandable since the maximum bias for the Fiske step for $\Gamma = 5$ is about $\eta = 0.18$. This is approximately the bias value at which the upper bound of the locking range levels for larger ϵ values for $\Omega = 1.5$. The fact that the lower bound of the step is found to follow the predicted slope almost perfectly also indicates that the deviation of the upper bound is caused by the upper limit of the Fiske step.

Figure 3 shows comparisons between numerical simulations (markers) and analytical predictions, Eq. (13) (solid curves) of the total size of the locking range as a function of the normalized dc magnetic field Γ. Simulations were carried out for $\epsilon = 0.01$ (+) and $\epsilon = 0.05$ ($\times$) [Fig. 3(b) only]. For high Γ the agreement is found to be very good for both $\Omega = 1.4$ [Fig. 3(a)] and $\Omega = 1.5$ [Fig. 3(b)]. However, we find large deviations for relatively small values of Γ. The reason for this is that the linear mode analysis of Eqs. (3)–(6) only holds for large Γ. When nonlinearity enters the spatial variation of ϕ we may use the soliton picture of Refs. 7 and 10 for $\Gamma L \lesssim 2\pi$, as stated above. We have shown the analytical prediction from the soliton model Eq. (15) as horizontal dashed lines and we find very close agreement with the numerical simulation in the expected region of small Γ.

We have demonstrated that Fiske modes can be phase locked to external ac signals and that the locking range in dc bias gives rise to a symmetric step, whose amplitude increases linearly with the ac field. Combining a linear mode description of the Fiske mode with an energy balance approach has enabled us to predict the locking ranges for relatively large values of the normalized dc magnetic field Γ. In the limit of small Γ, where the linear mode theory for the Fiske mode is known to be invalid, we have applied a soliton description of the Fiske mode and used a known expression for phase locking to external ac signals. Also in this parameter range we have found good agreement between numerical simulations and the analytical predictions.

In conclusion, we note that it is relatively easy to design and fabricate junctions presenting low-order Fiske modes in frequency ranges (X band, for example) where room temperature microwave equipment is available and it should, therefore, not be difficult to validate the model presented in this paper.

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