Time-dependent equations for phase differences and a collective mode in Josephson-coupled layered superconductors

L. N. Bulaevskii
Los Alamos National Laboratory, Los Alamos, New Mexico 87545

M. Zamora and D. Baeriswyl
Institut de Physique Théorique, Université de Fribourg, Pérolles, CH-1700 Fribourg, Switzerland

H. Beck
Institut de Physique, Université de Neuchâtel, Rue A.-L. Breguet 1, 2000 Neuchâtel, Switzerland

John R. Clem
Ames Laboratory and Department of Physics and Astronomy, Iowa State University, Ames, Iowa 50011
(Received 27 June 1994)

A time-dependent equation for the interlayer phase differences is derived for Josephson-coupled layered superconductors. It generalizes the sine-Gordon equation for the phase in a standard Josephson junction to the case of multilayer systems. With the help of this equation, the dispersion of a collective mode is found at low temperatures. In highly anisotropic systems the gap in the spectrum of this mode lies below the superconducting gap and is suppressed strongly by a magnetic field parallel to the layers. The effect of this mode on the dielectric function and specific heat is calculated.

For a Josephson junction formed by two bulk superconductors the time-dependent phase difference satisfies the well-known sine-Gordon equation,[1,2] which describes the full dynamics of the junction. The corresponding equation for multilayer systems of Josephson-coupled layers[3] of atomic thickness (high-T_c superconductors, organic layered superconductors, artificial superlattices of $YBa_2Cu_3O_{7-\delta}/PrBa_2Cu_3O_{7-\delta}$ type) has only been derived for the static limit.[4–6] As pointed out by Doniach and Inui,[7] the problem here is to account properly for the screened Coulomb repulsion which determines the kinetic energy of the phase variables.

In the following, using Maxwell's equations and the Josephson relation, we derive the equation for the time-dependent phase differences for Josephson-coupled multilayer systems. We then calculate the dispersion of the collective mode corresponding to coupled phase and charge variations at low temperatures T. This plasma mode lies well below the superconducting gap in highly anisotropic materials and affects low-energy properties of high-T_c superconductors.[8] We study the behavior of this mode in a magnetic field parallel to the layers and derive its contributions to the dielectric function and the specific heat at low T for the Meissner and the high-field regimes.

We consider a multilayer system within the Lawrence-Doniach model.[3] The z axis is chosen perpendicular to the layers (along the c axis). Let us denote by $\phi_n(\mathbf{r},t)$ the phase of the superconducting order parameter at position $\mathbf{r}=(x,y)$ in layer n and at time t. The magnetic field $\mathbf{B}(\mathbf{r},z,t)=\mathrm{curl}\mathbf{A}(\mathbf{r},z,t)$ is oriented along the layers, and the electric field is given by $\mathbf{E}(\mathbf{r},z,t)=-(1/c)\partial\mathbf{A}(\mathbf{r},z,t)/\partial t$, where we have chosen the gauge with zero scalar potential. The Maxwell equations read

$$\epsilon_0\nabla\cdot\mathbf{E}(\mathbf{r},z,t)=4\pi\rho(\mathbf{r},z,t), \tag{1}$$

$$\mathrm{curl}\mathbf{B}(\mathbf{r},z,t)=\frac{\epsilon_0}{c}\frac{\partial\mathbf{E}(\mathbf{r},z,t)}{\partial t}+\frac{4\pi}{c}\mathbf{j}(\mathbf{r},z,t), \tag{2}$$

where ϵ_0 is the high-frequency dielectric constant and $\rho(\mathbf{r},z)$ is the three-dimensional (3D) charge density. To proceed, we need constitutive equations relating the current density to the superconducting order parameter. We assume that the frequencies ω of phase variations are well below the superconducting gap $\Delta/\hbar$, so that the modulus of the order parameter is constant in space and time. Then the current density component parallel to the layers in layer n, averaged over the periodicity length s, is given by

$$\begin{aligned}\mathbf{J}_n(\mathbf{r})&=\frac{1}{s}\int_{s(n-1/2)}^{s(n+1/2)}dz\mathbf{j}(\mathbf{r},z)\\&=-\frac{c\Phi_0}{8\pi^2\lambda_{ab}^2}\left[\nabla\phi_n(\mathbf{r})+\frac{2\pi}{\Phi_0}\mathbf{A}_n(\mathbf{r})\right],\end{aligned} \tag{3}$$

where λ_{ab} is the penetration length for currents along layers, $\mathbf{A}_n(\mathbf{r})=\mathbf{A}(\mathbf{r},z=ns)$, and $\nabla=\partial/\partial\mathbf{r}$. Under the same condition, $\hbar\omega\ll\Delta$, the Josephson current density between layers n and $n+1$ is $J_0\sin[\varphi_{n,n+1}(\mathbf{r},t)]$, where

$$\varphi_{n,n+1}=\phi_n-\phi_{n+1}-\frac{2\pi}{\Phi_0}\int_{ns}^{(n+1)s}dzA_z \tag{4}$$

is the gauge-invariant phase difference between layers n and $n+1$. Here $J_0=c\Phi_0/8\pi^2s\lambda_{ab}^2\gamma^2$ is the Josephson critical current, and γ is the anisotropy parameter.[4] In the absence of pancake vortices we can choose the gauge $\phi_n = 0$. Then Eq. (4) provides the Josephson relation

$$E_{z;n,n+1}(\mathbf{r},t) = \frac{1}{s}\int_{ns}^{(n+1)s} dz E_z(\mathbf{r},z,t) = \frac{\hbar}{2es}\frac{\partial\varphi_{n,n+1}(\mathbf{r},t)}{\partial t}. \quad (5)$$

Here $E_{z;n,n+1}(\mathbf{r},t)$ is the average z component of electric field between layers n and $n+1$. In the context of the resistivity shunted junction model,[9] the total current density between layers n and $n+1$ is the sum of the Josephson and the quasiparticle currents:

$$J_{z;n,n+1} = J_0 \sin\varphi_{n,n+1} + \rho_c^{-1}E_{z;n,n+1}. \quad (6)$$

In general, ρ_c differs from the resistivity of the junction in the normal state. Since we assume the interlayer voltage to be much less than Δ/e, ρ_c arises due to thermally excited quasiparticles. Then $1/\rho_c \to 0$ at $T \to 0$.

Our goal is to find the equation for the time and space variations of $\varphi_{n,n+1}$. Following Ref. 10 we integrate $\mathbf{A}(\mathbf{r},z)$ along the contour C_x shown in Fig. 1 and a similar contour C_y parallel to the y axis with Δy instead of Δx. Using Eqs. (3) and (4), in the limit Δx, $\Delta y \to 0$ we obtain

$$\mathbf{z}_0 \times \mathbf{B}_{n,n+1} = \frac{4\pi\lambda_{ab}^2}{cs^s}(\mathbf{J}_{n+1} - \mathbf{J}_n) - \frac{\Phi_0}{2\pi s}\nabla\varphi_{n,n+1}, \quad (7)$$

where $\mathbf{B}_{n,n+1}$ is the averaged magnetic field between layers n and $n+1$ and $\mathbf{z}_0$ is the unit vector along the z axis. Inserting Eq. (7) into the Maxwell equation (2) projected onto the z axis we find

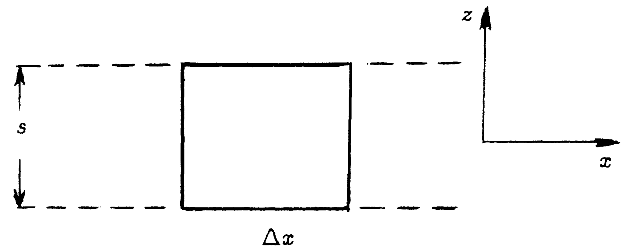

FIG. 1. The contour C_x is shown by solid lines. Dashed lines represent superconducting layers.

$$\frac{4\pi\lambda_{ab}^2}{cs^s}\nabla\cdot(\mathbf{J}_{n+1} - \mathbf{J}_n) - \frac{\Phi_0}{2\pi s}\nabla^2\varphi_{n,n+1} = -\frac{\epsilon_0}{c}\frac{\partial E_{z;n,n+1}}{\partial t} - \frac{4\pi}{c}J_{z;n,n+1}. \quad (8)$$

Now we express $J_{z;n,n+1}$ and $E_{z;n,n+1}$ in terms of $\varphi_{n,n+1}$ using Eqs. (6) and (5), respectively. To express $\nabla\cdot\mathbf{J}_n$ via phase differences we use the continuity equation

$$\frac{\partial\rho_n}{\partial t} + \nabla\cdot\mathbf{J}_n + \frac{J_{z;n,n+1} - J_{z;n-1,n}}{s} = 0, \quad (9)$$

where ρ_n is the averaged charge density in layer n,

$$\rho_n(\mathbf{r}) = \frac{1}{s}\int_{s(n-1/2)}^{s(n+1/2)} dz\rho(\mathbf{r},z). \quad (10)$$

We then use Eq. (1) to express ρ_n in terms the difference $E_{z;n,n+1}$ and $E_{z;n-1,n}$. The field components E_x and E_y can be neglected with accuracy $(\omega\lambda_{ab}/c)^2 \ll 1$. This argument has already been used in the case of a conventional Josephson junction.[11] Finally $E_{z;n,n+1}$ and $E_{z;n-1,n}$ are again expressed in terms of phase differences using Eq. (5), and we arrive at

$$\frac{1}{c_0^2}\left(\frac{\partial^2}{\partial t^2} + \frac{4\pi}{\rho_c\epsilon_0}\frac{\partial}{\partial t}\right)\left[\left(2 + \frac{s^2}{\lambda_{ab}^2}\right)\varphi_{n,n+1} - \varphi_{n+1,n+2} - \varphi_{n-1,n}\right] = \nabla^2\varphi_{n,n+1} + \frac{1}{\lambda_J^2}\left[\sin\varphi_{n+1,n+2} + \sin\varphi_{n-1,n} - \left(2 + \frac{s^2}{\lambda_{ab}^2}\right)\sin\varphi_{n,n+1}\right]. \quad (11)$$

Here $c_0 = \omega_{ab}s$ plays the role of the Swihart velocity, $\omega_{ab} = c/\lambda_{ab}\sqrt{\epsilon_0}$ is the plasma frequency for $\mathbf{E}$ parallel to the layers, and $\lambda_J = \gamma s$ is the Josephson length.

The terms on the left-hand side of Eq. (11) proportional to $1/\rho_c$ describe dissipation via quasiparticle tunneling in the cores of Josephson vortices. This dissipation is analogous to the Bardeen-Stephen mechanism for Abrikosov vortices. There is an additional dissipation due to phonons. According to Eqs. (1) and (5), the time variation of phase differences produces the charge variation

$$\rho_n(\mathbf{r}) = \frac{\hbar\epsilon_0}{8\pi es^2}\frac{\partial}{\partial t}\left[\varphi_{n,n+1}(\mathbf{r}) - \varphi_{n-1,n}(\mathbf{r})\right]. \quad (12)$$

These are coupled directly to phonons and thus phase variations in time and space (along z) can relax by phonon emission. In the following we include this mechanism within the parameter ρ_c.

Our main result, Eq. (11), generalizes the time-dependent sine-Gordon equation for a standard Josephson junction to the case of a multilayer system. We notice that, as a consequence of the peculiar nature of screening in quasi-2D systems, time derivatives and spatial differences are intertwined in the equation for phases $\varphi_{n,n+1}$. In the case of a two-layer system, $\varphi_{n,n+1} = \varphi$ for $n = 0$, and $\varphi_{n,n+1} = 0$ otherwise; Eq. (11) is thus reduced to the standard sine-Gordon equation because terms with s^2/λ_{ab}^2 are small and may be omitted.

At equilibrium, and low fields, $H_{c1,\parallel} \ll B \ll H_0$, Eq. (11) predicts the anisotropic triangular lattice of Josephson vortices similar to that of Abrikosov vortices;

see Ref. 12. Here $H_0 = \Phi_0/\gamma s^2$ and $H_{c1,\parallel}$ is the lower critical field parallel to the layers.[10] Josephson vortices form a triangular lattice with periods $a = (\gamma\Phi_0/B\sqrt{3})^{1/2}$ approximately in the ab direction and $l \approx a\sqrt{3}/\gamma$ approximately along z. A Josephson vortex does not have a normal core, unlike an Abrikosov vortex, but it has a central core with size of the order λ_J along the layers and s along the c axis, within which the nonlinear character of Eq. (11) is important. Outside this region Eq. (11) can be linearized, and here the current and field distributions are nearly identical to those of an Abrikosov vortex. As the field increases above H_0 [in $Bi_2Sr_2CaCu_2O_\delta$ (Bi-2:2:1:2), $H_0 \approx$ (1–3) T], the central cores overlap, and the vortex lattice transforms into the high-field configuration for $B \gg H_0$.[4,13] In this structure vortices are arranged periodically along layers with period $a = \Phi_0/sB \ll \lambda_J$. They still form a triangular lattice, and centers of vortices are shifted by $a/2$ in neighboring interlayer regions. For a given interlayer spacing this structure is similar to that in a long Josephson junction.

If a current is applied along the z axis, Josephson vortices start to move. Equation (11) describes this motion and provides the basis for studying fluctuations of vortex structure, current-voltage characteristics in the presence of a magnetic field parallel to the layers, and other nonstationary Josephson phenomena.[6]

On the basis of the equation of motion (11), we can derive the spectrum of the collective mode at low temperatures $T \ll T_c$ in the presence of a magnetic field parallel to the layers. Previously this mode in the absence of applied field was discussed in the framework of a microscopic theory, for isotropic superconductors by Anderson,[14] for highly anisotropic layered systems by Fertig and Das Sarma,[15] and recently by Artemenko and Kobel'kov.[16] Our treatment is more general, as it does not depend on microscopic details, and it is valid for any system with Josephson coupling of layers.

In isotropic superconductors the collective mode at low temperatures is the plasma oscillation mode with very high frequency, $\omega_p = c/\lambda_L(0)\sqrt{\epsilon_0}$.[14] In fact, the frequency of the plasma mode is not affected by the superconducting transition, as was shown by a sum rule.[17] In view of the very high energy of plasma oscillations as compared to the relevant energy scales for superconductivity, the collective mode can be neglected in the theory of isotropic and weakly anisotropic superconductors.

The situation is different in highly anisotropic layered superconductors. If the transfer of electrons between layers is strongly suppressed, we have to consider systems with 3D Coulomb interaction of electrons but almost 2D electron motion along the layers. Due to anisotropic screening, the plasmon spectrum in the normal phase is also anisotropic with low frequencies for wave vectors close to the c axis: charge oscillations of this type do not produce electric fields at large distances. Plasma oscillations are gapless for a strictly 2D band structure and have a very small gap in highly anisotropic layered systems.

In Josephson-coupled superconductors the frequency of the collective mode at low temperatures may be found using (11) and expanding in small variations of $\varphi_{n,n+1}$ around equilibrium values $\varphi^{(0)}_{n,n+1}$. We get the plasmon dispersion in the Meissner state [$\varphi^{(0)}_{n,n+1} = 0$]:

$$\omega_p(\mathbf{k}, q) = \omega_{ab}\left(\frac{1}{\gamma^2} + \frac{k^2}{Q^2 + \lambda_{ab}^{-2}}\right)^{1/2}. \tag{13}$$

Here $\mathbf{k}$ is the wave vector in the ab plane, $Q^2 = 2(1 - \cos q)/s^2$, and q is the dimensionless momentum along z, $-\pi \le q \le \pi$. At $q = 0$ the spectrum coincides with that of the Josephson junction. The relaxation rate of the collective mode is $2\pi/\rho_c\epsilon_0$. Equation (13) is in agreement with that obtained in Ref. 16. It differs from that of Ref. 15 where Meissner screening was omitted and thus Eq. (13) with $1/\lambda_{ab}^2 = 0$, was obtained [in calculation of the vortex matrix, Eq. (5) of Ref. 15, the only short range attractive potential was accounted for but the Coulomb repulsion was omitted].

The spectrum (13) for the collective mode corresponds to the longitudinal dielectric function

$$\epsilon_{zz}(\omega) = \epsilon_0 - \frac{\omega_p^2(0,q)\epsilon_0}{\omega(\omega + i\tau^{-1})}, \quad \frac{1}{\tau} = \frac{4\pi}{\rho_c\epsilon_0}. \tag{14}$$

The same result was obtained by Tachiki *et al.*[8] for the transverse dielectric function. They have used it to explain the infrared reflectivity spectra of $La_{1-x}Sr_xCuO_4$ for $\mathbf{E} \parallel \mathbf{c}$ obtained by Tamasaku *et al.*[18]

In highly anisotropic materials, the plasmon gap $\delta = \hbar\omega_{ab}/\gamma$, is very small compared to Δ. For Bi-2:2:1:2, $\omega_{ab} \approx$1 eV, $\lambda_{ab} \approx 2000$ Å, $s = 15.6$ Å, and $\gamma \approx$ 300–1000,[6] so that $\delta \approx$10–30 K, well below $\Delta \sim 300$ K. In $La_{1.84}Sr_{0.16}CuO_4$, the gap $\delta \approx 80$ K at $T = 8$ K was found.[18]

The dispersion of the collective mode can be changed by applying a magnetic field parallel to the layers. At $B > H_{c1,\parallel}$, in the vortex state, the collective mode becomes gapless due to the translational symmetry of the vortex lattice with respect to the underlying crystal lattice. As in the standard Josephson junction, the weight of oscillations with $\hbar\omega < \delta$ increases with B. We calculate now the dispersion of the collective mode at high fields $B \gg H_0$. With a field oriented along the y axis the equilibrium solution up to terms of order H_0^2/B^2 is $\varphi^{(0)}_{2n,2n+1} = 2\pi x/a$, $\varphi^{(0)}_{2n+1,2n+2} = 2\pi x/a + \pi$. The plasmon spectrum is obtained by solving the eigenvalue problem for small distortions $u_n = \varphi_{n,n+1} - \varphi^{(0)}_{n,n+1}$. The equation to be solved is the Mathieu equation

$$\begin{aligned}\frac{\omega^2}{c_0^2}&\left[u_{n+1} + u_{n-1} - \left(2 + \frac{s^2}{\lambda_{ab}^2}\right)u_n\right]\\ &= \nabla^2 u_n + \frac{1}{\lambda_J^2}\left[u_{n+1} + u_{n-1} + \left(2 + \frac{s^2}{\lambda_{ab}^2}\right)u_n\right]\\ &\quad\times\cos\frac{2\pi x}{a}.\end{aligned} \tag{15}$$

At given q the lowest frequency of collective mode is

$$\omega_p(0,q) = \frac{\omega_{ab}}{\gamma}\,\frac{H_0}{\pi B\sqrt{2}}\,\frac{\cos^2(q/2) + s^2/4\lambda_{ab}^2}{[4\sin^2(q/2) + s^2/\lambda_{ab}^2]^{1/2}}. \tag{16}$$

It reaches a minimum at $q = \pi$. Around $q = \pi$ and for $\omega \ll \Delta$ with accuracy H_0^2/B^2 the spectrum of the collective mode has 2D character because the Josephson coupling is suppressed by a strong parallel magnetic field:[19]

$$\omega_p(\mathbf{k}, q) = \frac{c_0 k}{2\sin(q/2)}. \tag{17}$$

Now Eq. (14) holds with $\omega_p(0, q)$ given by Eq. (16). Thus parallel magnetic fields of the order H_0 affect strongly the low-frequency collective mode and longitudinal dielectric function for large momentum along the c axis.

Next, we calculate the plasmon contribution to the specific heat. At $B = 0$ and $T \ll \delta$ we obtain

$$C_v = \frac{8k_BT}{s^3\hbar\omega_{ab}}\left(\frac{1}{3\gamma} + \frac{T}{\hbar\omega_{ab}}\right)\exp\left(-\frac{\delta}{T}\right). \tag{18}$$

At high fields $B \gg H_0$ the result is

$$C_v = \frac{16\zeta(3)k_BT^2}{\pi^2 s^3\hbar^2\omega_{ab}^2}. \tag{19}$$

Here we have neglected relaxation. Our approach is valid if $T \gg 4\pi\hbar/\rho_c$, otherwise the specific heat comes mainly from overdamped quantum oscillators, with $C_v \approx 8k_BT/\hbar\omega_{ab}^2 s^3\rho_c$.[20] Comparing Eqs. (18) and (19), we see that Josephson vortices enhance the specific heat by decreasing the plasmon frequencies.

Although the collective modes penetrate into the quasiparticle gap, they affect the thermodynamics of the superconductor weakly because of their strong dispersion (high Swihart velocity). In fact, the collective mode frequencies are below the superconducting gap for orientations of the 3D momentum close to the c axis, within an angle of $\Delta/\hbar\omega_{ab}$ (≈ 0.03 in Bi-2:2:1:2). Thus their contribution to the free energy is much smaller than that of phonons and quasiparticles near and even well below T_c. For Bi-2:2:1:2 C_v of Eq. (19) is $\sim 10^{-3}$ erg/K cm^3 at $T = 1$ K, while the phonon contribution is several orders of magnitude larger. The low-temperature specific heat due to perpendicular vortices in highly anisotropic layered superconductors was estimated to be much larger, ~ 100 erg/K cm^3 (see Ref. 20).

In summary, we have derived equations for the dynamics of Josephson-coupled layered superconductors when only Josephson vortices are present. We showed that in highly anisotropic systems the plasmon mode has a very small gap, which is further reduced when a magnetic field parallel to the layers is applied. At low temperatures the contribution of this mode to the dielectric function and to the specific heat depends rather sensitively on the field value.

This work was supported by the U.S. Department of Energy and by the Swiss National Science Foundation through Grant No. 21-29021.90. Ames Laboratory is operated for U.S. Department of Energy by Iowa State University under Contract No. W-7405-Eng-82. This work was supported by the Director for Energy Research, Office of Basic Energy Sciences. One of the authors (L.N.B.) thanks G. Grüner, V. Kogan, M. Maley, and V. L. Pokrovskii for useful discussions.

[1] I. O. Kulik and I. K. Yanson, *The Josephson Effect in Superconductive Tunneling Structures* (Israel Program for Scientific Translations, Jerusalem, 1972).
[2] A. Barone, *Physics and Applications of the Josephson Effect* (J. Wiley, New York, 1984).
[3] W. E. Lawrence and S. Doniach, in *Proceedings of the 12th International Conference on Low Temperature Physics, Kyoto, 1970,* edited by E. Kanda (Keigaku, Tokyo, 1970).
[4] L. N. Bulaevskii and J. R. Clem, Phys. Rev. B **44**, 10 234 (1991).
[5] S. N. Artemenko and A. N. Kruglov, Physica C **173**, 125 (1991).
[6] R. Kleiner and P. Müller, Phys. Rev. B **49**, 1327 (1994).
[7] S. Doniach and M. Inui, Phys. Rev. B **41**, 6668 (1990).
[8] M. Tachiki, T. Koyama, and S. Takahashi (unpublished).
[9] B. G. Orr, J. R. Clem, H. M. Jaeger, and A. M. Goldman, Phys. Rev. B **34**, 3491 (1986).
[10] J. R. Clem and M. W. Coffey, Phys. Rev. B **42**, 6209 (1990); M. W. Coffey and J. R. Clem, *ibid.* **44**, 6903 (1991).
[11] U. Eckern, G. Schön, and V. Ambegaokar, Phys. Rev. B **30**, 6419 (1984).
[12] L. J. Campbell, M. M. Doria, and V. G. Kogan, Phys. Rev. B **38**, 2439 (1988).
[13] A. F. Volkov, Physica C **192**, 306 (1992).
[14] P. W. Anderson, Phys. Rev. **112**, 1900 (1958).
[15] H. A. Fertig and S. Das Sarma, Phys. Rev. B **44**, 4480 (1991).
[16] S. N. Artemenko and A. G. Kobel'kov, Pis'ma Zh. Eksp. Teor. Fiz. **58**, 435 (1993) [JETP Lett. **58**, 445 (1993)].
[17] M. Y. K. Wong and S. Takada, Phys. Rev. B **37**, 5644 (1988).
[18] K. Tamasaku, Y. Nakamura, and S. Uchida, Phys. Rev. Lett. **69**, 1455 (1992).
[19] K. B. Efetov, Zh. Eksp. Teor. Fiz. **76**, 1781 (1979) [Sov. Phys. JETP **49**, 905 (1979)].
[20] L. N. Bulaevskii and M. P. Maley, Phys. Rev. Lett. **71**, 3541 (1993).