

Dynamics of vortex liquids in Ginzburg-Landau theories with applications to superconductivity

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(Received 20 December 1993; revised manuscript received 8 March 1994)

This paper continues our study of vortices in Ginzburg-Landau theories with special attention to applications in superconductivity. In another paper, we derived asymptotic equations governing the dynamics of interacting vortices. Here, we study the hydrodynamic limit of these vortices. For vortices in the solutions of the nonlinear Schrödinger equation, the hydrodynamic equation is the incompressible Euler's equation in fluid mechanics. For vortices in the time-dependent Ginzburg-Landau equations, the hydrodynamic equations can be thought of as being the complement of the Euler equations. Preliminary results on the numerical studies of the hydrodynamic equations are presented. As applications of the hydrodynamic formalism, we study the pinning of vortex liquids by periodic potentials, and the propagation of magnetic fields into type-II superconductors. The hydrodynamic formalism suggests that to leading order, the vortex liquids are pinned even at small but positive temperature.

I. INTRODUCTION

This paper continues our study of vortices in time-dependent Ginzburg-Landau equations, with special attention to the applications in superconductivity. In a previous paper¹ (see also Refs. 2 and 3), we studied the dynamics of separated vortices and derived a reduced system of ordinary differential equations (ODE's) governing their evolution. In the present paper we will study the hydrodynamic limit of these interacting vortices and establish a continuum theory. The hydrodynamic equations resemble the equations in fluid mechanics and can be used as a very effective tool for analyzing many problems concerning the dynamics of vortices. As examples we will study the pinning of vortex liquids by a periodic potential, and the propagation of magnetic fields into type-II superconductors. We will focus on the intuitive, heuristic part of the argument, and leave the rigorous results to a separate paper. We concentrate on columnar vortices, i.e., axial variations are ignored. In other words, we will consider only the two-dimensional problem.

Our main motivation for studying the hydrodynamics of vortices comes from type-II superconductivity. In the classical mean-field picture, the phase diagram in the H - T plane consists of three parts. In the Meissner phase [$T < T_c$, $H < H_{c_1}(T)$], magnetic flux does not penetrate into the bulk of the sample. In the mixed state [$T < T_c$, $H_{c_1}(T) < H < H_{c_2}(T)$], magnetic fluxes penetrate the sample as quantized vortices and form the well-known Abrikosov flux lattice. The rest of the H - T plane is in the normal phase.⁴

It is generally accepted that, when effects due to fluctuations are taken into account, in a significant portion of the H - T phase diagram, generally close to the normal-mixed state transition curve, the Abrikosov vortex lattice obtained from mean-field theory can melt and form a liquid state. The vortices lose their positional order and the resulting "vortex liquid" has no shear rigidity at long wavelength. It is also expected that the flux flow of such

a vortex liquid will give rise to nonvanishing linear resistivity, even in the presence of weak pinning centers such as oxygen vacancies. This means that the material will cease to be truly superconducting.^{5,6}

Our primary interest is to study the intrinsic *nonlinear* effects in the dynamics of such a vortex liquid. The work presented in this paper and our ongoing work in this direction are strongly motivated by similar considerations in fluid mechanics. One main equation we will work with,

$$\begin{aligned}\rho_t + \nabla \cdot (\mathbf{v}\rho) &= 0, \\ \mathbf{v} &= -\nabla G, \\ -\Delta G &= \rho,\end{aligned}\tag{1.1}$$

which comes from the hydrodynamics of vortices in the Ginzburg-Landau theory, can be considered as the complement of the incompressible Euler equation in fluid mechanics,

$$\begin{aligned}\rho_t + \nabla \cdot (\mathbf{v}\rho) &= 0, \\ \mathbf{v} &= -\nabla^\perp G, \\ -\Delta G &= \rho,\end{aligned}\tag{1.2}$$

where $\nabla^\perp = (-\partial_y, \partial_x)$. While (1.2) describes a flow in the space of divergence-free vector fields, (1.1) describes a similar flow in the space of irrotational vector fields. Like its counterpart in fluid mechanics, (1.1) also admits a "velocity-pressure" formulation:

$$\begin{aligned}\mathbf{v}_t + \mathbf{v}(\nabla \cdot \mathbf{v}) + \nabla^\perp \varphi &= \mathbf{0}, \\ \nabla \times \mathbf{v} &= \mathbf{0}.\end{aligned}\tag{1.3}$$

These equations exhibit very interesting mathematical properties. These are discussed in a forthcoming paper.

We mention in passing that, by the same token, one obtains (1.2) as the hydrodynamic equations for vortices in the solutions of the nonlinear Schrödinger equation. The difference between this and the standard semiclassical

limit is explained in Sec. II.

Equation (1.1) arises when we consider variations on a scale much larger than the typical spacing between vortices. It should be of relevance for regimes when the external field is intermediate between the lower and upper critical fields. Close to the critical fields, either there are too few vortices and the hydrodynamic formalism is invalid, or the vortex cores overlap and the Ginzburg-Landau theory has to be used. The derivation of the hydrodynamic equations makes use of the assumption that the fields ρ, \mathbf{v}, G are smoothly varying on the scale being considered, the validity of which is very much in doubt when pinning disorders are present in the sample. However, under coarse graining, we expect that the effect of the weak and dense pinning centers (such as oxygen vacancies) can be taken into account by a renormalized friction coefficient in the second equation of (1.1). Strong and widely spaced pinning forces can be modeled by source terms in the equations (see Sec. IV).

We will also consider some preliminary applications of the hydrodynamic formalism. One example we will study is the pinning of vortex liquids in superconductors by periodic potentials whose scale is comparable to the penetration depth. Due to nonlinear effects, we obtain I - V curves that are very close to the ones measured in the experiments of Worthington *et al.* and Palstra *et al.*^{7,8} The discrepancy between our result and the conventional picture about the pinning of vortex liquids is explained in Sec. VI.

The hydrodynamic formalism also opens up new ways of studying dynamic instabilities in the mixed state. As an example we study the propagation of magnetic flux into type-II superconductors. Here we will take the preliminary step of looking for solutions which contain propagating fronts. The stability of these fronts will be studied in Ref. 9.

II. HYDRODYNAMICS OF VORTEX LIQUIDS IN GINZBURG-LANDAU THEORIES

A. Derivation of the hydrodynamic equations

In this section we consider a Ginzburg-Landau model in the absence of magnetic fields:

$$u_t = \Delta u + u(1 - |u|^2),$$

where u is a complex order parameter. This model may not be an accurate model of any physical systems, but it serves as a crude model for a wide variety of problems, ranging from magnetism to nematic liquid crystals. For us, it also serves the purpose of illustrating the technical points.

As was discussed in Ref. 1, under appropriate scaling

$$\frac{1}{\ln 1/\delta} u_t^\delta = \Delta u^\delta + \frac{1}{\delta^2} u^\delta (1 - |u^\delta|^2), \quad (2.1)$$

asymptotic analysis suggests that as $\delta \rightarrow 0$, $u^\delta(\mathbf{x}, t) \rightarrow e^{i\psi(\mathbf{x}, t)}$, except at the paths of the isolated vortices $\{\xi_1(t), \xi_2(t), \dots, \xi_N(t)\}$ whose dynamics is governed by a simple system of ODE's,

$$\dot{\xi}_j(t) = -\nabla_{\xi_j} \mathcal{H}(\xi_1, \dots, \xi_N), \quad (2.2)$$

where

$$\mathcal{H}(\xi_1, \dots, \xi_N) = -\sum_{i \neq j} n_i n_j \ln |\xi_i - \xi_j| \quad (2.3)$$

and n_j is the degree of the j th vortex. Obviously some requirements on the initial data $u_0^\delta(\mathbf{x})$ are necessary for the above to be true, but here we will not elaborate on that. For stability considerations, we will take $n_j = \pm 1$.

Equations (2.2) and (2.3) are the starting point of this paper. Let us consider a cloud of vortices $\{\xi_1^\epsilon, \xi_2^\epsilon, \dots, \xi_{N_\epsilon}^\epsilon\}$ with degrees $\{n_1^\epsilon, n_2^\epsilon, \dots, n_{N_\epsilon}^\epsilon\}$, respectively, evolving under the law (2.2) with

$$\mathcal{H}\{\xi_1^\epsilon, \dots, \xi_{N_\epsilon}^\epsilon\} = -\epsilon \sum n_i^\epsilon n_j^\epsilon \ln |\xi_j^\epsilon - \xi_i^\epsilon|. \quad (2.4)$$

A scaling factor ϵ is added to (2.4) to guarantee that the velocities of the vortices remain bounded as $\epsilon \rightarrow 0$ and $N_\epsilon \rightarrow +\infty$. One can think of ϵ as being the weight of each vortex. In the context of superconductivity, which we will turn into in the next section, ϵ is the flux quantum carried by each vortex. We will study the limit when the density of these vortices approaches some well-defined function.

Let

$$\begin{aligned} \rho_1^\epsilon(\mathbf{x}, t) &= \epsilon \sum_{n_j^\epsilon = -1} \delta(\mathbf{x} - \xi_j^\epsilon(t)), \\ \rho_2^\epsilon(\mathbf{x}, t) &= \epsilon \sum_{n_j^\epsilon = 1} \delta(\mathbf{x} - \xi_j^\epsilon(t)). \end{aligned} \quad (2.5)$$

In order to have a nontrivial limit for $\rho_1^\epsilon, \rho_2^\epsilon$ as $\epsilon \rightarrow 0$, we need the intervortex distance to be of order $O(\sqrt{\epsilon})$. Therefore the total number of vortices should satisfy $N_\epsilon = O(1/\epsilon)$. These can be specified as conditions for the initial configuration $\{\xi_1^\epsilon(0), \dots, \xi_{N_\epsilon}^\epsilon(0)\}$.

Assume that $\rho_i^\epsilon \rightarrow \rho_i$ ($i=1,2$) as $\epsilon \rightarrow 0$. Let us derive the equations satisfied by ρ_1, ρ_2 . Let $G^\epsilon(\mathbf{x}, t) = -\epsilon \sum_{j=1}^{N_\epsilon} n_j^\epsilon \ln |\mathbf{x} - \xi_j^\epsilon(t)|$. Then we have

$$\begin{aligned} G^\epsilon(\mathbf{x}, t) &\rightarrow G(\mathbf{x}, t) \\ &= -\int \ln |\mathbf{x} - \mathbf{y}| [\rho_2(\mathbf{y}, t) - \rho_1(\mathbf{y}, t)] d^2 y, \end{aligned} \quad (2.6)$$

i.e., G satisfies

$$-\Delta G = 2\pi(\rho_2 - \rho_1). \quad (2.7)$$

To derive the evolution equations, we make the assumption that the velocity of the vortex at ξ_j^ϵ is approximated by the value of a smooth vector field \mathbf{v}_1 at this point if the degree of the vortex is -1 , and is given approximately by \mathbf{v}_2 at this point if the degree is 1 . Let φ be a smooth function on R^2 with compact support. Let us compute

$$\begin{aligned}
\frac{d}{dt} \int \rho_1^\varepsilon(\mathbf{x}, t) \varphi(\mathbf{x}) d^2x &= \frac{d}{dt} \varepsilon \sum_{n_j^\varepsilon=-1} \varphi(\xi_j^\varepsilon(t)) \\
&= \varepsilon \sum_{n_j^\varepsilon=-1} \nabla \varphi(\xi_j^\varepsilon(t)) \dot{\xi}_j^\varepsilon(t) \\
&\approx \varepsilon \sum_{n_j^\varepsilon=-1} \nabla \varphi(\xi_j^\varepsilon(t)) \mathbf{v}_1(\xi_j^\varepsilon(t), t) \\
&\rightarrow \int (\nabla \varphi)(\mathbf{x}) \mathbf{v}_1(\mathbf{x}, t) \rho_1(\mathbf{x}, t) d^2x .
\end{aligned} \tag{2.8}$$

Similarly, we have

$$\frac{d}{dt} \int \rho_2^\varepsilon(\mathbf{x}, t) \varphi(\mathbf{x}) dx \rightarrow \int (\nabla \varphi)(\mathbf{x}) \mathbf{v}_2(\mathbf{x}, t) \rho_2(\mathbf{x}, t) d^2x . \tag{2.9}$$

Integration by parts gives

$$\rho_{it} + \nabla \cdot (\mathbf{v}_i \rho_i) = 0, \quad i = 1, 2 . \tag{2.10}$$

This is the basic continuity equation. Next we relate \mathbf{v}_1 and \mathbf{v}_2 to G . For vortices of degree 1, we have $\dot{\xi}_j^\varepsilon \approx -\nabla G(\xi_j^\varepsilon)$, and therefore we obtain

$$\varepsilon \sum_{n_j^\varepsilon=1} \mathbf{v}_2(\xi_j^\varepsilon, t) \varphi(\xi_j^\varepsilon) \approx -\varepsilon \sum_{n_j^\varepsilon=1} \nabla G(\xi_j^\varepsilon) \varphi(\xi_j^\varepsilon) . \tag{2.11}$$

Passing to the limit as $\varepsilon \rightarrow 0$, we obtain

$$\int \rho_2(\mathbf{x}, t) \mathbf{v}_2(\mathbf{x}, t) \varphi(\mathbf{x}) d^2x = - \int \rho_2(\mathbf{x}, t) \nabla G(\mathbf{x}, t) \varphi(\mathbf{x}) d^2x .$$

Therefore we get

$$\mathbf{v}_2 = -\nabla G . \tag{2.12}$$

Similarly, we have

$$\mathbf{v}_1 = \nabla G .$$

(2.7), (2.10), and (2.12) are the equations we need. We write them together as

$$\begin{aligned}
\rho_{it} + \nabla \cdot (\mathbf{v}_i \rho_i) &= 0, \quad i = 1, 2, \\
-\Delta G &= 2\pi(\rho_2 - \rho_1), \\
\mathbf{v}_1 &= \nabla G, \\
\mathbf{v}_2 &= -\nabla G .
\end{aligned} \tag{2.13}$$

In the special case when all the vortices have the same sign, say, $n_j^\varepsilon = 1$ for all j , then $\rho_2 = \rho$, $\mathbf{v}_2 = \mathbf{v}$, $\rho_1 = 0$, and we get a simpler system for (ρ, \mathbf{v}, G) :

$$\begin{aligned}
\rho_t + \nabla \cdot (\mathbf{v} \rho) &= 0, \\
-\Delta G &= 2\pi \rho, \\
\mathbf{v} &= -\nabla G .
\end{aligned} \tag{2.13'}$$

B. Incompressible Euler limit for the nonlinear Schrödinger equation

Consider

$$-iu_t^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon^2} u^\varepsilon (1 - |u^\varepsilon|^2) \tag{2.14}$$

where $u^\varepsilon: R^2 \rightarrow \mathbb{C}^1$. This equation arises as a model for the dynamics of quantum fluids.¹⁰ The vortex dynamics associated with (2.14) has been studied in Refs. 10 and 11. Instead of (2.2) and (2.3), we now have

$$n_j \dot{\xi}_j = -\nabla_{\xi_j}^\perp \mathcal{H}(\xi_1, \dots, \xi_N), \tag{2.15}$$

where \mathcal{H} is given by (2.3), and $\nabla^\perp = (-\partial_y, \partial_x)$. One difference between (2.14), (2.15) and (2.2), (2.3) is that for (2.14), (2.15) the logarithmic scaling is not required. We recognize immediately that (2.15) is the same as the equations describing the dynamics of point vortices in incompressible ideal fluids where the circulation carried by the j th vortex is $2\pi n_j$. It is well known¹² that the continuum limit of these interacting vortices is given by the incompressible Euler equation

$$\begin{aligned}
\rho_t + \nabla \cdot (\mathbf{v} \rho) &= 0, \\
\mathbf{v} &= -\nabla^\perp G, \\
-\Delta G &= 2\pi \rho,
\end{aligned} \tag{2.16}$$

where ρ is identified as the vorticity and G the stream function. The factor 2π can be scaled away.

This line of argument suggests the following. Consider

$$-i\varepsilon u_t^\varepsilon = \varepsilon^2 \Delta u^\varepsilon + u^\varepsilon (1 - |u^\varepsilon|^2). \tag{2.17}$$

Here we change to convective (instead of diffusive) scaling in order to have finite velocity for the vortices. This changes the Hamiltonian in (2.15) from the one given by (2.3) to the one given by (2.4). Otherwise the sum on the right-hand side of (2.15) is of order $O(1/\varepsilon)$. Let us consider initial conditions of the form $u_0^\varepsilon(\mathbf{x}) = e^{i\theta_0^\varepsilon(\mathbf{x})/\varepsilon}$ where

$$\theta_0^\varepsilon(\mathbf{x}) = \varepsilon \sum_{j=1}^{N_\varepsilon} \tan^{-1} \left[\frac{y - \eta_j^0}{x - \xi_j^0} \right]. \tag{2.18}$$

$\{\xi_j^0 = (\xi_j^0, \eta_j^0), j = 1, \dots, N_\varepsilon\}$ is distributed in such a way that

$$\varepsilon \sum_{j=1}^{N_\varepsilon} \delta(\mathbf{x} - \xi_j^0) \rightarrow \rho_0(\mathbf{x}). \tag{2.19}$$

Notice that the harmonic conjugate function of $\theta_0^\varepsilon(\mathbf{x})$ is given by

$$\psi_0^\varepsilon(\mathbf{x}) = \varepsilon \sum_{j=1}^{N_\varepsilon} \ln |\mathbf{x} - \xi_j^0|. \tag{2.20}$$

Then we expect that, for $t > 0$, $u^\varepsilon(\mathbf{x}, t) \approx e^{i\theta^\varepsilon(\mathbf{x}, t)/\varepsilon}$, where $\theta^\varepsilon(\mathbf{x}, t)$ and the conjugate function of $\theta^\varepsilon(\cdot, t)$, $\psi^\varepsilon(\cdot, t)$ satisfy

$$\nabla \theta^\varepsilon(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t), \quad \psi^\varepsilon(\mathbf{x}, t) \approx G(\mathbf{x}, t), \tag{2.21}$$

where (ρ, G, \mathbf{v}) is the solution of (2.16) with ρ_0 as initial

data. Moreover, we should have

$$\theta^\varepsilon(\mathbf{x}, t) \approx \varepsilon \sum_{j=1}^{N_\varepsilon} \tan^{-1} \left[\frac{y - \eta_j(t)}{x - \xi_j(t)} \right], \quad (2.22)$$

where, for fixed t , the distribution of $\{\xi_j(t), j=1, \dots, N_\varepsilon\}$ is given approximately by $\rho(\cdot, t)$. It would be very interesting to check the validity of these statements even on the level of formal asymptotics.

The difference between the argument suggested here and the standard semiclassical limit (see, for example, Refs. 13 and 14) lies in the choice of the phase function. In the standard semiclassical limit the phase function is regular and independent of ε . Consequently the semiclassical limiting velocity fields are always irrotational. Here the phase functions are multivalued and depend on ε . The multivaluedness of the phase functions gives rise to quantized vortices which in turn give rise to rotational velocity fields in the continuum limit. It should be noticed that, to the leading order, the conjugate function of the phase which approximates the stream function does not depend on ε ; neither does the gradient of the phase function.

C. Remarks on the hydrodynamic equations

(1) Comparing (2.13') with (2.16), we see that the only difference is in the direction of the velocity. For (2.1), the interaction between the vortices is either attractive or repulsive, and the force is in the direction of the positional vector $\xi_i - \xi_j$. In contrast, for (2.14), the interaction between the vortices is always neutral, as in ideal fluids, and the force is in the direction perpendicular to the positional vector. This results in the difference in the expressions for \mathbf{v} : the ∇ operator in (2.13') is changed to the ∇^\perp operator in (2.16). Consequently, while \mathbf{v} in (2.16) is always divergence-free, the \mathbf{v} in (2.13') is always irrotational.

(2) *Shock formation.* Let us look for solutions of (2.13') that are independent of y . It is easy to see that (2.13') then becomes

$$v_t + \left[\frac{v^2}{2} \right]_x = 0. \quad (2.23)$$

(2.23) is the celebrated inviscid Burgers equation. It is well known¹⁵ that for generic smooth initial data, shocks, namely jump discontinuities, form in the solutions of (2.23) after finite time. These shocks can only arise from compressive waves for which ρ is negative. They are of no relevance to the physical problems considered here since (2.13) is valid only when ρ is non-negative. However, when vortices of both signs are present, we do get such discontinuities, which represent interfaces between patches of vortices of different signs.

(3) In (2.13), let $\mathbf{v} = (\rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2) / (\rho_1 + \rho_2)$, $\rho = \rho_1 + \rho_2$; then we have

$$\rho_t + \nabla \cdot (\mathbf{v} \rho) = 0.$$

Consequently the total mass (the area integral of ρ) is always conserved in time. Obviously, we are not modeling

the effect of annihilation of vortices which should enter in the above equation as a sink term. Therefore we do not expect (2.13) to be of much use when vortices of opposite signs are mingled together. However, (2.13) is very useful when studying the dynamics of patches of vortices of different signs. In that case the annihilation of vortices can be modeled by interfacial boundary conditions.⁹

III. HYDRODYNAMICS OF VORTEX LIQUIDS IN SUPERCONDUCTORS

A. Derivation of the hydrodynamic equations

Our starting point is the phenomenological time-dependent Ginzburg-Landau (TDGL) equations. In Ref. 1 we studied the vortex dynamics associated with the TDGL equation

$$\begin{aligned} \varepsilon^2 \varphi_t + \gamma i V \varphi &= -(-i\varepsilon \nabla - \mathbf{A})^2 \varphi + (1 - |\varphi|^2) \varphi, \\ \frac{\varepsilon}{\gamma} \mathbf{A}_t + \nabla V &= -\nabla \times \nabla \times \mathbf{A} - \frac{i\varepsilon}{2} (\bar{\varphi} \nabla \varphi - \varphi \nabla \bar{\varphi}) - |\varphi|^2 \mathbf{A}, \end{aligned} \quad (3.1)$$

where $\varepsilon = 1/\kappa$ and κ is the Ginzburg-Landau parameter. We are interested in extremely type-II superconductors, i.e., $\kappa \gg 1$. (3.1) is a nondimensionalized form of the TDGL equation in which the only control parameter in the equations is κ . In particular, the basic length scale we are using is the penetration depth, i.e., $\lambda \approx 1$.

In the asymptotic limit as $\varepsilon \rightarrow 0$, a collection of vortices $\{\xi_1, \dots, \xi_N\}$ with degrees $\{n_1, \dots, n_N\}$ evolves according to

$$\dot{\xi}_j = -\nabla_{\xi_j} \mathcal{H}(\xi_1, \dots, \xi_N), \quad (3.2)$$

where

$$\mathcal{H}(\xi_1, \dots, \xi_N) = \sum n_i n_j K_0(|\xi_i - \xi_j|) \quad (3.3)$$

and K_0 is the modified Bessel function of zeroth order. In the same fashion as in Sec. II A, we can derive the following equations for the hydrodynamics of vortices interacting according to (3.2) and (3.3):

$$\begin{aligned} \rho_{it} + \nabla \cdot (\mathbf{v}_i \rho_i) &= 0, \quad i = 1, 2, \\ \mathbf{v}_1 &= \nabla G, \\ \mathbf{v}_2 &= -\nabla G, \\ -\Delta G + G &= \rho_2 - \rho_1. \end{aligned} \quad (3.4)$$

In the more general case when the relaxation time parameter γ is complex $\gamma = \Gamma_1 + i\Gamma_2$, (3.2) and (3.3) are changed to

$$n_j \Gamma_1 m_1 \dot{\xi}_j + \Gamma_2 m_2 \dot{\xi}_j = \nabla_{\xi_j} \sum_{i \neq j} n_i K_0(|\xi_j - \xi_i|), \quad (3.5)$$

where m_1 and m_2 are absolute constants and $m_1 < 0$. Here we used the notation $(v_1, v_2)^\perp = (-v_2, v_1)$. To get the hydrodynamic equations in this case, we must replace the second and third equations in (3.4) by

$$\begin{aligned} -\Gamma_1 m_1 \mathbf{v}_1 + \Gamma_2 m_2 \mathbf{v}_1^\perp &= \nabla G, \\ \Gamma_1 m_1 \mathbf{v}_2 + \Gamma_2 m_2 \mathbf{v}_2^\perp &= \nabla G. \end{aligned} \quad (3.6)$$

Some comments about Eqs. (3.4) and (3.6) are in order.

(1) The linearization of (3.6) at a uniform state $\rho = \rho_0$ reduces to the equations studied by Ambegaokar *et al.*,¹⁶ and more recently by Marchetti and Nelson.¹⁷ As we will see later, the nonlinear convection term is important in modeling some physical problems, such as the propagation of magnetic field into a type-II superconductor.

(2) A special form of (3.4) was obtained by Chapman and Rubinstein,¹⁸ who considered the case when all vortices have the same index (either +1 or -1). In this case (3.4) reduces to

$$\begin{aligned} \rho_t + \nabla \cdot (\mathbf{v}\rho) &= 0, \\ \mathbf{v} &= -\nabla G, \\ -\Delta G + G &= \rho. \end{aligned} \quad (3.7)$$

This is the analog of (1.1) obtained earlier by the author.

(3) *Comparison with Maxwell's equations.* In mks units, Maxwell's equations¹⁹ of electrodynamics are

$$\nabla \times \mathbf{H} = \mathbf{J}, \quad (3.8)$$

$$\mathbf{B}_t + \nabla \times \mathbf{E} = 0. \quad (3.9)$$

We also need the constitutive relations

$$\mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (3.10)$$

If the magnetic Reynolds number is large, $Rm = \sigma \mu \mathcal{U} \mathcal{L} \gg 1$, then the leading-order electromagnetic induction equation becomes

$$\mathbf{B}_t = \nabla \times (\mathbf{v} \times \mathbf{B}) \quad (3.11)$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}. \quad (3.12)$$

One additional equation is needed to relate \mathbf{v} to \mathbf{B} . In the flux-flow theory, this is simply given by the phenomenological equation

$$\eta \mathbf{v} = \mathbf{J} \times \mathbf{B} \quad (3.13)$$

balancing the Lorentz force with the frictional force, where η is a phenomenological mobility coefficient. Using (3.8) and (3.10), we get

$$\eta \mathbf{v} = \frac{1}{\mu} (\nabla \times \mathbf{B}) \times \mathbf{B}. \quad (3.14)$$

In the two-dimensional geometry considered here, $\mathbf{B} = (0, 0, B)$ and $\mathbf{v} = (v_1, v_2, 0)$, and hence we can write the previous equations as

$$B_t + \nabla \cdot (\mathbf{v}B) = 0,$$

$$\eta \mathbf{v} = -B \nabla B, \quad \bar{\eta} = \eta \mu,$$

or

$$B_t = \frac{1}{\bar{\eta}} \nabla \cdot (B^2 \nabla B). \quad (3.15)$$

We carried out a preliminary numerical study of the solutions of Eq. (3.7). For simplicity we adopted the periodic boundary conditions. We discretize the spatial

variables using the Fourier methods, and the temporal variable using the classical Runge-Kutta methods. Figure 1 is the contour plot of ρ at a later time with initial data

$$\rho(\mathbf{x}, 0) = \sin(x) \sin(y) + \sin(2x) \cos(2y).$$

The picture exhibits clearly the formation of cellular structures. This is generic when the initial data have a regular pattern.

B. The effect of thermal noise and pinning

Consider the dynamics of vortices interacting through (3.2) and (3.3), and also under the influence of thermal noises and pinning potential V :

$$d\xi_j = -[\nabla_{\xi_j} \mathcal{H}^\varepsilon(\xi_1, \dots, \xi_{N_\varepsilon}) + \nabla V(\xi_j)] dt + \sqrt{2T} d\beta_j \quad (3.16)$$

and

$$\mathcal{H}^\varepsilon(\xi_1, \dots, \xi_{N_\varepsilon}) = \varepsilon \sum_{i \neq j} K_0(|\xi_i - \xi_j|). \quad (3.17)$$

Here $\beta_1, \dots, \beta_{N_\varepsilon}$ are independent Brownian paths and T is the temperature.

In this case, the hydrodynamics has been studied extensively in the probability literature as the "propagation of chaos" (see Refs. 20–22, etc.). A similar problem, the convergence of the random vortex method, has also been studied in the numerical analysis literature.^{23,24} The hydrodynamics in this case is described by the mean-field limit. If initially at $t=0$, $\varepsilon \sum_{j=1}^{N_\varepsilon} \delta(\mathbf{x} - \xi_j^\varepsilon(0))$ converges in law to a nonrandom distribution $\rho_0(\mathbf{x})$, then for $t > 0$, $\varepsilon \sum_{j=1}^{N_\varepsilon} \delta(\mathbf{x} - \xi_j^\varepsilon(t))$ necessarily converges in law to a nonrandom distribution $\rho(\mathbf{x}, t)$, where ρ satisfies

$$\begin{aligned} \rho_t + \nabla \cdot (\mathbf{v}\rho) &= T \Delta \rho, \\ \mathbf{v} &= -\nabla G - \nabla V, \\ -\Delta G + G &= \rho, \end{aligned} \quad (3.18)$$

with initial data $\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x})$. These equations will serve as the starting point in the next section where we study the pinning of vortex liquids.

IV. PINNING OF VORTICES BY A PERIODIC POTENTIAL

We mentioned earlier that the motion of vortices causes the conversion of electromagnetic energy to kinetic energy, and hence resistance. Fortunately, real materials contain all sorts of impurities (or defects) such as oxygen vacancies, grain boundaries, and twin boundaries. These impurities or defects act as barriers or pinning centers to impede the motion of vortices. The effect of pinning on the overall dynamics is an extremely important problem which has attracted considerable attention. In this section, we study the pinning of vortices using the hydrodynamic formalism developed earlier. As we remarked in the Introduction, we model the weak pinning forces by a renormalized friction coefficient in (3.4)

and (3.6), and the strong pinning forces by a source term in these equations [see (3.18)]. We assume that these strong pinning centers exist on a scale much larger than the typical spacings between vortices, and much smaller than the sample size. Twin boundaries, grain boundaries, and columnar defects can all fit this description. We will examine the collective effects of these pinning centers on a cloud of interacting vortices. As a first step, we will model these effects by a periodic pinning potential. Currently, we are studying the effects of random potentials and extensions to higher-dimensional models.

Before studying the effect of periodic potentials on a cloud of vortices, let us consider a simpler example, namely, a single particle on the line traveling in a periodic (with period $[0,1]$) potential V under the influence of thermal noise. We will assume that the particle obeys overdamped dynamics, i.e., the inertia is negligible compared with friction. This is consistent with the TDGL models for the dynamics of vortices in superconductors.

Consider the Langevin equation satisfied by the trajectory of the particle, $x(t)$:

$$dx = [F - V'(x)]dt + \sqrt{2T}d\beta . \quad (4.1)$$

Here F is the mean force, β is the standard Brownian motion, and T is the temperature. This is the classical problem of a Brownian particle in a periodic potential. In the context of superconductivity, it has been used to model, among other things, the pinning of vortices near H_{c1} . We will use it here to illustrate the different phenomena caused by nonlinear interactions between vortices. We are interested in the relation between F and $\lambda(F)$ defined by

$$\lambda(F) = \lim_{t \rightarrow +\infty} \frac{\langle x(t) \rangle}{t} . \quad (4.2)$$

Here $\langle \dots \rangle$ means averaging with respect to the thermal

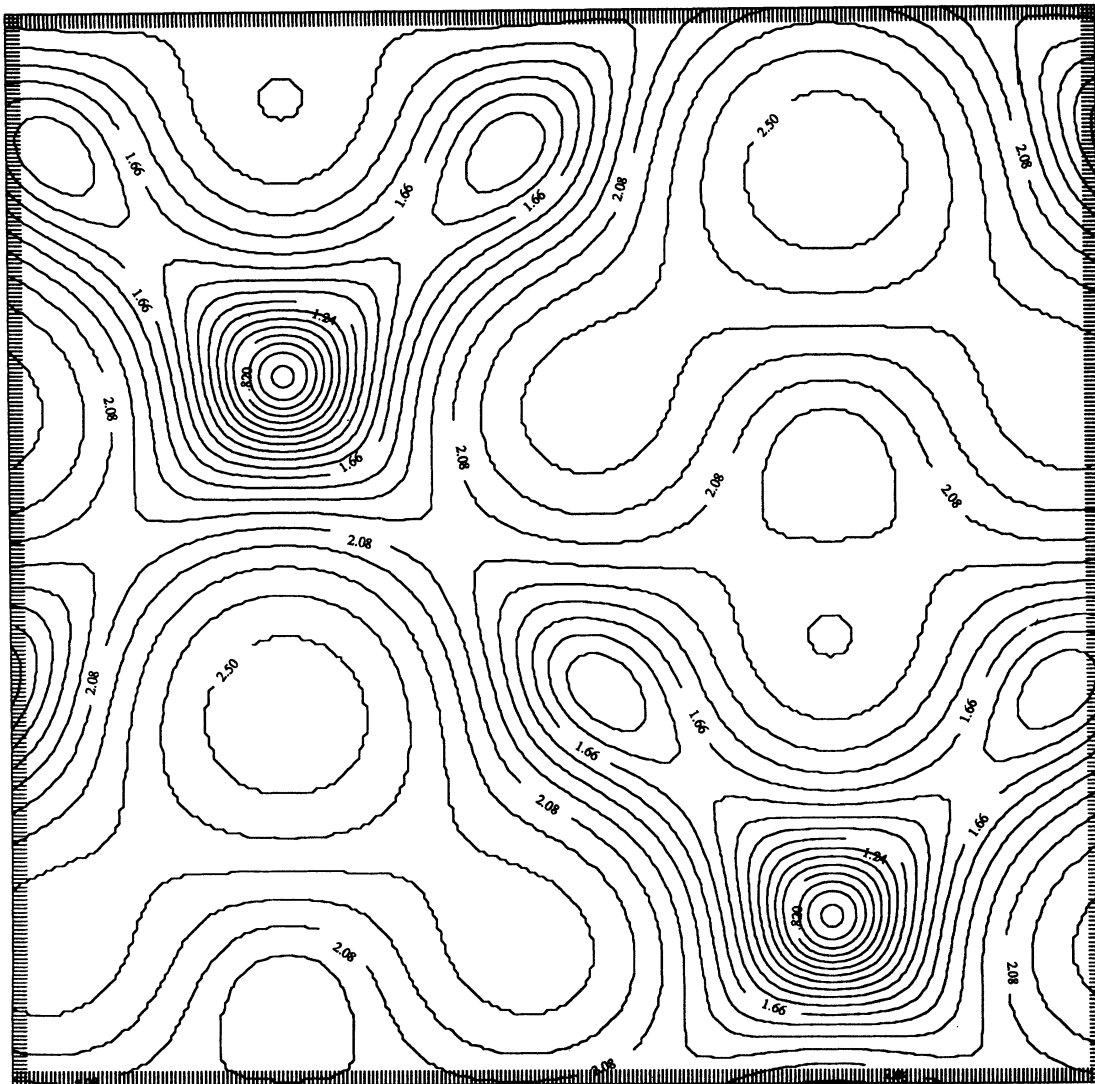


FIG. 1. Behavior of solutions of (3.7). Displayed is the contour of ρ with initial data $\rho(\mathbf{x},0) = \sin(x)\sin(y) + \sin(2x)\cos(2y)$. Regions of high density of vortices form a cellular pattern.

noise. $\lambda(F)$ is the average asymptotic velocity of the particle.

The F - λ relation corresponds roughly to the I - V curves often measured in experiments (where F , the applied force, is proportional to the applied current, and $\lambda(F)$ corresponds to the induced voltage). This relation gives the effective constitutive equation replacing the standard Ohm's law for normal conductors. Of special interest is the behavior of $\lambda(F)$ for small F . A truly superconducting state should have zero linear resistivity when the current density is infinitesimally small, i.e., $\lambda'(0)=0$, giving rise to a nonlinear response function. Such a nonlinear behavior is indeed observed in experiments on high- T_c materials (see, for example, Refs. 7 and 8). What is not yet completely understood is the origin of this nonlinear behavior. One attractive proposal is to attribute this to the existence of a vortex glass state in the presence of weak pinning centers (see Refs. 5 and 6).

Returning to (4.1) and (4.2), we can express $\lambda(F)$ as

$$\lambda(F) = \int_0^1 [F - V'(x)] m(x) dx, \tag{4.3}$$

where m is the invariant measure associated with (4.1):

$$[F - V'(x)] m_x = T m_{xx}. \tag{4.4}$$

We can evaluate (4.3) numerically and obtain the F - λ relation. This was done, for example, in Ref. 25. The important features are the following. If $T=0$, then

$$\lambda(F) = 0 \text{ for } F \leq F_c, \tag{4.5}$$

where F_c is the maximum value of V' . If $T > 0$, then we always have a linear relation between F and λ for small F , i.e., $\lambda'(0) \neq 0$, and $\lambda'(0)$ is proportional to T (= temperature) for small T .

Next we come to the main interest of this section: pinning of vortex liquids by a periodic potential. At this preliminary stage, we will restrict ourselves to the one-dimensional version of (1.1). This might be of relevance for studying pinning by twin boundaries when the applied current is in the direction of the twin and the Lorentz force is in the direction perpendicular to the twin. Moreover, we expect that, at least qualitatively, the overall picture described below should also hold for (3.10) with a two-dimensional periodic potential.

In the presence of thermal noise and the pinning potential, (1.1) changes to (in one space dimension)

$$\begin{aligned} \rho_t + (\rho v)_x &= T \rho_{xx}, \\ v &= -(G_x + V_x) + F, \\ -G_{xx} &= \rho, \end{aligned} \tag{4.6}$$

where V is a periodic potential with period $[0,1]$ and F is the mean applied force. We imagine that there are many twin boundaries inside the sample, so the period of the potential is small compared to the sample size, which is taken to be infinite here. We will study periodic solutions of (4.6).

(4.6) is equivalent to

$$v_t + \left[\frac{v^2}{2} \right]_x + vV'' - \int_0^1 v(y,t)V'''(y)dy = Tv_{xx} \tag{4.7}$$

with initial data $v(x,0) = v_0(x)$ satisfying

$$\int_0^1 v_0(x) dx = F. \tag{4.8}$$

To define $\lambda(F)$, let $X(t)$ be the solution of

$$\dot{X}(t) = v(X(t), t) \tag{4.9}$$

with $X(0) = x_0 \in [0, 1]$. Define

$$\lambda(F) = \lim_{t \rightarrow +\infty} \frac{X(t)}{t}. \tag{4.10}$$

It is easy to see that $\lambda(F)$ does not depend on x_0 . In the case when $T=0$, the solutions of (4.7) may contain shocks. If the particle $X(t)$ falls into the shock, then we specify that the particle travels with the shock.

The behavior of $\lambda(F)$ for the potential $V''(x) = 2 \sin(2\pi x)$ is displayed in Fig. 2, at different temperatures $T = 0, 0.01, 0.03, 0.04, 0.07, \text{ and } 0.1$. For $T=0$ we get qualitatively the same kind of behavior as for the simple model (4.1). However, the picture changes qualitatively for small but finite T . For the present model, there is a finite value T^* , and a function $F_c(T)$ defined for $T < T^*$, such that if $0 < T < T^*$ and $0 < F < F_c(T)$ then $\lambda(F) = 0$. This is depicted in the phase diagram in Fig. 3. Notice that Fig. 2 is qualitatively very close to the experimental results of Refs. 7 and 8.

The phenomena described above can be explained as follows. As $t \rightarrow +\infty$, $v(\cdot, t)$ converges to a steady-state solution of (4.7), $\bar{v}(\cdot)$, which satisfies

$$\left[\frac{\bar{v}^2}{2} \right]_x + \bar{v}V'' - \int_0^1 \bar{v}(y)V'''(y)dy = T \bar{v}_{xx} \tag{4.11}$$

and $\int_0^1 \bar{v}(x) dx = F$. When $T < T^*$ and $0 \leq F < F_c(T)$, \bar{v} has zeros in the interval $[0,1]$. A typical \bar{v} is displayed in Fig. 4. To compute $\lambda(F)$, we can replace (4.9) by

$$\dot{X}(t) = \bar{v}(X(t)). \tag{4.12}$$

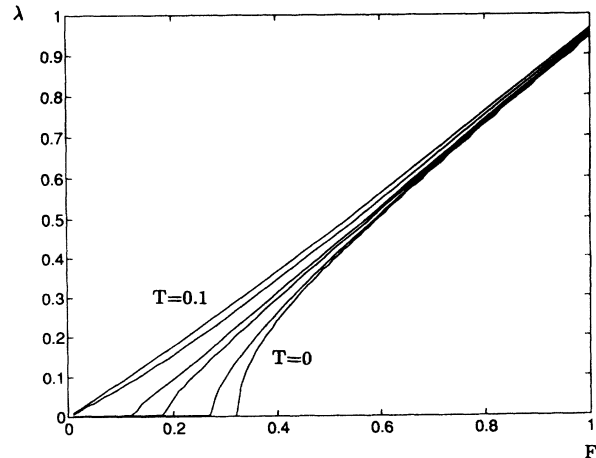


FIG. 2. F - λ curves for the one-dimensional vortex liquid model in a periodic potential. The vortex liquid is pinned even for small but positive temperature.

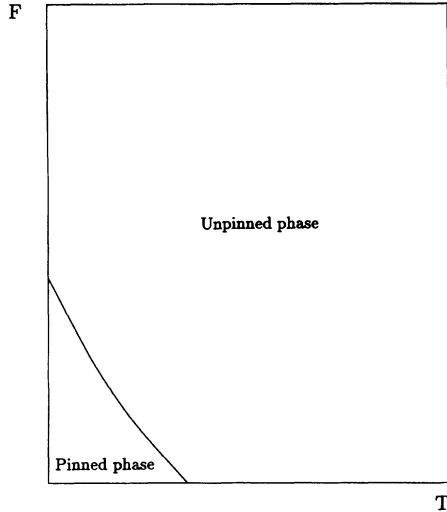


FIG. 3. Schematic phase diagram for the pinning of vortex liquids.

It is easy to see that when \bar{v} has zeros we get $\lambda(F)=0$.

We should emphasize that the above is based on the hydrodynamic formalism which in the present situation is given by the mean-field equations (4.6). Thermal fluctuations are of the order $(1/\kappa)^{1/2}$. Therefore, in the large- κ limit, they do not contribute to the leading order. If we do take into account the thermal noise, then the results should be modified by a term of order $(1/\kappa)^{1/2}$.

V. PROPAGATION OF MAGNETIC FIELDS INTO TYPE-II SUPERCONDUCTORS

As another example of the application of the hydrodynamic formalism, we study the propagation of magnetic fields into type-II superconductors. This is motivated mainly by the theoretical work of Vinokur, Feigel'man, and Geshkenbein²⁶ and the experimental work by Welp *et al.* To explain recent experimental results on the time

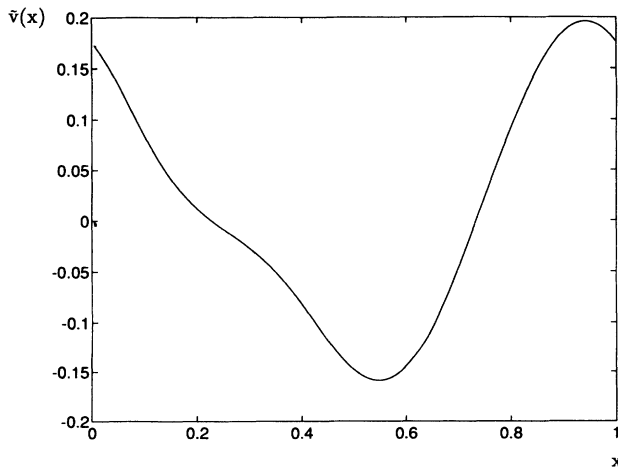


FIG. 4. Typical profiles of \bar{v} in the pinned phase. Notice that \bar{v} contains zeros. This is the reason why the vortex liquid is pinned.

evolution of magnetization curves, Vinokur, Feigel'man, and Geshkenbein studied the flux creep in the presence of impurities with a renormalized mobility coefficient which depends exponentially on the activation barrier, which in turn depends logarithmically on the current,

$$\eta(\mathbf{J}) = \eta_0 e^{-U(\mathbf{J})/T}, \quad U(\mathbf{J}) = U_0 \ln \left| \frac{\mathbf{J}_c}{\mathbf{J}} \right|. \quad (5.1)$$

This renormalized mobility coefficient accounts for the average effect of the impurities. Notice that it depends on the current density. With (5.1), (3.15) becomes (neglecting coefficients)

$$B_t = \nabla(|\nabla B|^m \nabla B), \quad m = \frac{U_0}{T} + 1. \quad (5.2)$$

Or, in terms of \mathbf{J} ,

$$\mathbf{J}_t = \Delta(|\mathbf{J}|^m \mathbf{J}). \quad (5.2')$$

This is a well-known equation which occurs in models of filtration of gases in porous media.²⁷ Unlike the standard diffusion equation, this equation supports front propagation with a finite speed. In particular, it has the following self-similar solutions for arbitrary $c_0 > 0$:

$$\begin{aligned} J_1(x, t) &= 0, \\ J_2(x, t) &= \frac{1}{t^{1/(m+2)}} \left[c_0 - \frac{m}{2(m+1)(m+2)} \frac{x^2}{t^{2/(m+2)}} \right]_+^{1/m}. \end{aligned} \quad (5.3)$$

Here we used the notation $x_+ = \max(x, 0)$. For these solutions, the fronts at time t are located at $x = \pm\gamma(t)$, with

$$\gamma(t) = c_1 t^{1/(m+2)}, \quad (5.4)$$

where c_1 is related to c_0 by a simple relation. It is also known that the propagating fronts are highly stable under the dynamics given by (5.2).

However, recent experiments by Welp *et al.* suggest that the flux fronts in type-II superconductors are extremely unstable. Since these instabilities cannot be studied within the flux creep theory of Vinokur, Feigel'man, and Geshkenbein,²⁶ we are motivated to study the possible dynamic instabilities caused by nonlinear interactions in the vortex liquid. Here we will only report our preliminary results on the existence of planar fronts. The dynamic instabilities of these fronts will be studied in Ref. 9.

The thickness of the flux front is on the order of the penetration depth λ . This parameter was set to be 1 in all previous discussions. In order to study front propagation, we must restore this parameter. This has the effect of changing $K_0(x)$ to $K_0(x/\lambda)$. Therefore we change (3.7) to

$$\begin{aligned} \rho_t + \nabla \cdot (\mathbf{v}\rho) &= 0, \\ \mathbf{v} &= -\nabla G, \\ -\lambda^2 \Delta G + G &= \rho. \end{aligned} \quad (5.5)$$

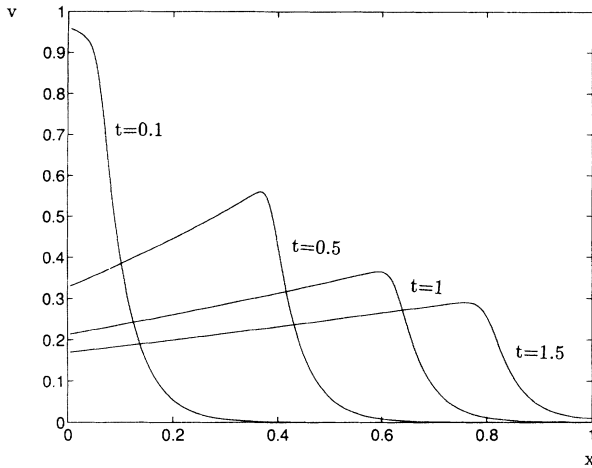


FIG. 5. Propagation of magnetic fields into the sample. v , which corresponds to the current in experimental situations, is plotted at different times $t=0.1, 0.5, 1$, and 1.5 . $\lambda=0.04$.

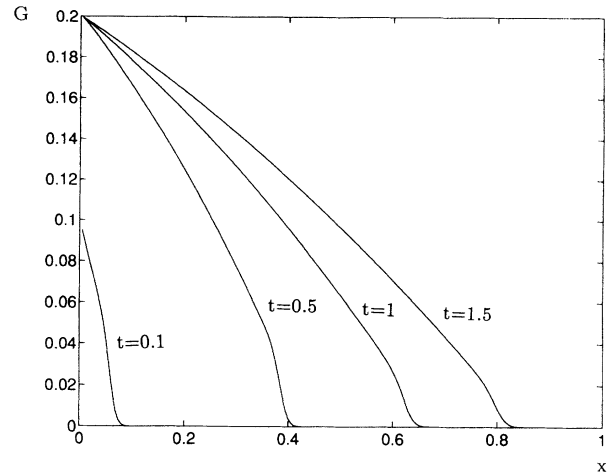


FIG. 6. Propagation of magnetic fields into the sample. The magnetic field G is plotted at different times $t=0.1, 0.5, 1$, and 1.5 . $\lambda=0.04$.

To leading order in λ , we have $G = \rho$, and

$$G_t = \frac{1}{2} \Delta G^2. \quad (5.6)$$

This equation has the same form as (5.2') with $m=1$. Hence it admits solutions which contain fronts propagating at finite speeds. This also vindicates our choice of time scale in (5.5).

(5.5) was integrated numerically in one space dimension. Typical profiles for v and G are presented in Figs. 5 and 6. Here ρ, G, v at $t=0$ were chosen to be zero. The boundary data at $x=0$ were chosen to be

$$\rho_0(t) = G_0(t) = \begin{cases} t & \text{for } t < 0.2, \\ 0.2 & \text{for } t > 0.2. \end{cases} \quad (5.7)$$

At $x=1$, we chose

$$\rho_1(t) = G_1(t) = 0. \quad (5.8)$$

Qualitatively, the profiles in Figs. 5 and 6 are very close to the ones presented in Ref. 26. In particular, we see a clear propagation of fronts into the bulk of the sample.

VI. CONCLUDING REMARKS

The main purpose of this paper was to develop a hydrodynamic formalism for vortices in extremely type-II superconductors in the regime of magnetic fields intermediate between the lower and upper critical fields. This hydrodynamic formalism is useful when studying variations on the scale much larger than the typical spacing between vortices. It enables us to treat such problems in the same way as problems of fluid mechanics. In particular, we expect interesting phenomena to happen because of the nonlinear effects. In high- T_c materials, effects of thermal noise, pinning forces, etc., all combine to render the vortex distribution highly nonuniform. In such cir-

cumstance the nonlinear terms should be of major importance.

Our strategy is to model the strong, widely spaced pinning centers by pinning potentials in the hydrodynamic equations. We illustrated this by studying the pinning of vortex liquids through a one-dimensional example. While the predictions of this one-dimensional model have to be validated by studies of more realistic models, it does suggest the importance of *collective effects*. We know that a single isolated vortex cannot be pinned by an array (or a distribution) of pinning barriers in the presence of thermal noise, and it is intuitively quite clear that a single pinning center can hardly pin a vortex liquid. Yet, collectively, an array of pinning barriers such as twin boundaries can effectively pin a vortex liquid (which in our model is a cloud of interacting vortices). This is the main suggestion of Sec. IV.

The next obvious step is to develop a parallel hydrodynamic stability theory. In Sec. V, we took one preliminary step of showing that the hydrodynamic equations do have solutions which represent propagating fronts. The dynamic instability of these fronts, studied experimentally by Welp *et al.*, is of great interest and should be amenable to a hydrodynamic treatment.

ACKNOWLEDGMENTS

Part of the work presented here was done during my visits to the Argonne National Lab under the support of the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy Contract No. W-31-109-Eng-38. I am very grateful to Hans Kaper, Man Kwong, Gary Leaf, and Valerii Vinokur for their hospitality and many stimulating discussions. I also want to thank Fong Liu for his helpful remarks. This work was also supported by the NEC Research Institute Inc. and the Sloan Foundation.

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