# Charged spin-texture excitations and the Hartree-Fock approximation in the quantum Hall effect

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We develop a Hartree-Fock approach to the charged spin-texture excitations (CSTE's) of the ferromagnetic incompressible ground state, which occurs in the quantum Hall effect at Landau-level filling factor $\nu = 1$. The CSTE's are the appropriate generalization of skyrmions to the situation when there is a nonzero Zeeman coupling. We find for Coulomb interactions that the charged spin-texture excitation energies are always smaller than the excitation energies of localized spin $\frac{1}{2}$ quasiparticles and quasiholes. However, the amount by which the energy is lowered is quite small for typical experimental situations. The net spin of the CSTE's is always much larger than $\frac{1}{2}$, suggesting that adding or removing charge from a filled Landau level rapidly degrades its spin polarization.

The properties of two-dimensional electron systems in strong magnetic fields exhibit a great richness when discrete degrees of freedom are possible beyond the orbital degrees of freedom of the electrons. For example, systems in which the Zeeman coupling is not so strong that the spin degrees of freedom are completely frozen out exhibit the fractional quantum Hall effect (FQHE) with unusual, two-component ground states.[1] Double layered electron systems are similarly interesting, since these may be exactly mapped to the spin system by identifying the layer index as a pseudospin. However, there is a further richness in these systems because they allow one to vary the relative interaction strengths between electrons with different pseudospins, leading to interesting phase transitions, even-denominator FQHE's, and a possible Josephson effect.[2]

Beyond these interesting ground state properties, it has also been observed that both the spin and layer systems should exhibit, at appropriate filling fractions, unusual charged excitations. In particular for the spin system the Hamiltonian is spin-rotationally invariant and at $\nu = 1$ the ground state is ferromagnetic. (Here $\nu \equiv 2\pi l_0^2 \rho = 1$, where $l_0^2 = \hbar c/eB$, $B$ is the magnetic field, and $\rho$ is the two-dimensional electron density.) In the absence of Zeeman coupling the lowest energy charged excitations at $\nu = 1$ have been shown to be *skyrmions*.[3–5] Skyrmions are the lowest gradient energy $O(3)$ spin textures with a unit winding number in two dimensions, and they play an important role in field-theoretic descriptions of two-dimensional ferromagnetic systems.[6] Most theoretical studies of skyrmions in the quantum Hall effect have relied on the ability to map the spin-polarized quantum Hall system to appropriate field theories.[3–5] These mappings, however, do not permit an accurate description of the excitations in the presence of a Zeeman coupling,[3] which we will see is important in experimentally relevant magnetic fields.

In this paper, we derive a Hartree-Fock (HF) description of the charged spin-texture excitations (CSTE's) which are the appropriate generalization of skyrmions for nonzero Zeeman coupling. These excitations will have the same topological winding number and charge as skyrmions, but the precise spin textures will depend on the strength of the Zeeman coupling. Our approach allows us to compute both the spin textures and the energies for experimentally accessible magnetic fields. Since we are essentially dealing with a strong field phenomenon, and we are interested in states for which the Zeeman splitting is much smaller than the cyclotron gap $\hbar\omega_c$, where $\omega_c = eB/mc$, and $m$ is the effective electron mass in the host crystal, we will only consider states contained completely in the lowest Landau level (LLL). In the presence of a Zeeman field the spins in the ground state are uniformly aligned. It is convenient to consider charged spin textures centered on the origin and to use a symmetric gauge for which the single particle states are given by $\phi_m(\vec{r}) = \frac{1}{\sqrt{\pi 2^m m!}} z^m e^{-r^2/4}$, with $l_0 \equiv 1$ as our unit of length. The key element in the HF description of the excitations is to consider states of the form

 

$$|\psi_-\rangle = \prod_{m=0}^{\infty}(u_m a_m^\dagger + v_m b_{m+1}^\dagger)|0\rangle,$$

$$|\psi_+\rangle = \prod_{m=0}^{\infty}(-u_m a_{m+1}^\dagger + v_m b_m^\dagger)a_0^\dagger|0\rangle, \qquad (1)$$

where $|u_m|^2 + |v_m|^2 = 1$, $a_m^\dagger$ creates a spin up electron in the $m$th angular momentum state in the LLL, and $b_m^\dagger$ creates a spin down electron. (We take the fully polarized ground state of the system to be $|\psi_0\rangle = \prod_m b_m^\dagger|0\rangle$.) We seek the lowest energy single Slater determinants of this form for which $u_m$ is nonzero at $m = 0$, and decays to zero as $m$ increases. The spin polarization then is directed upward at the origin, downward at infinity, and the projection of the spin-polarization onto the $\hat{x}-\hat{y}$ plane rotates by $2\pi$ along any path winding once around the origin; i.e., the spin-texture has a unit winding number. Note that the single-particle spinors in the single-Slater determinant state have spin up and spin down orbitals with angular momentum differing by a single unit. One can see explicitly that the transfer of weight from the $m$th spin up state to the $(m \pm 1)$th spin down state as $m \to \infty$ guarantees that there will be a total particle number difference from the ground state of $\pm 1$, which will be localized around the origin. It is interesting to note that the standard HF spin $\frac{1}{2}$ quasihole excitations[7] can be written in the form of Eq. (1), by setting $u_m = 0$ for all $m$. This state may be regarded as the zero-size limit for the spin texture.

Our main results for the physically realistic Coulomb interaction between the electrons are summarized in Figs. 1 and 2. Figure 1 illustrates the energy of the CSTE's (in units of $e^2/\kappa l_0$, where $\kappa$ is the dielectric constant of the host material for the two-dimensional electron gas) as a function of the (unitless) Zeeman energy, $\tilde{g} = g\mu_B B/(e^2/\kappa l_0)$. Here $\mu_B$ is the Bohr magneton, and $g$ is half the Landé $g$ factor. For comparison, the energies of the spin $\frac{1}{2}$ quasiparticles are illustrated as well. The size of the stable spin texture decreases as the Zeeman coupling is increased, since the energy cost of having many reversed spins becomes prohibitive. For $\tilde{g} = 0$ the system possesses large spin-texture excitations which are exactly skyrmions.[3] At general values of $\tilde{g}$ the spin-textures may be regarded as distorted skyrmions whose size is determined by a competition between the Coulomb interaction and the Zeeman coupling, the former trying to minimize the spin by having a small size, the latter favoring a uniform charge density and a large size.[3] The solutions to the HF approximation compromise between these competing effects. Several conclusions immediately follow from our results.

(1) The energy difference between spin texture and spin $\frac{1}{2}$ charged particles is small once Zeeman coupling is included. Indeed, even for a magnetic field of 0.5 T, and a Landé $g$ factor of 0.5 (appropriate for GaAs), we find that spin-texture particle-hole pair energies are only $\sim$ 15% smaller than the spin 1/2 particle-hole pair en-

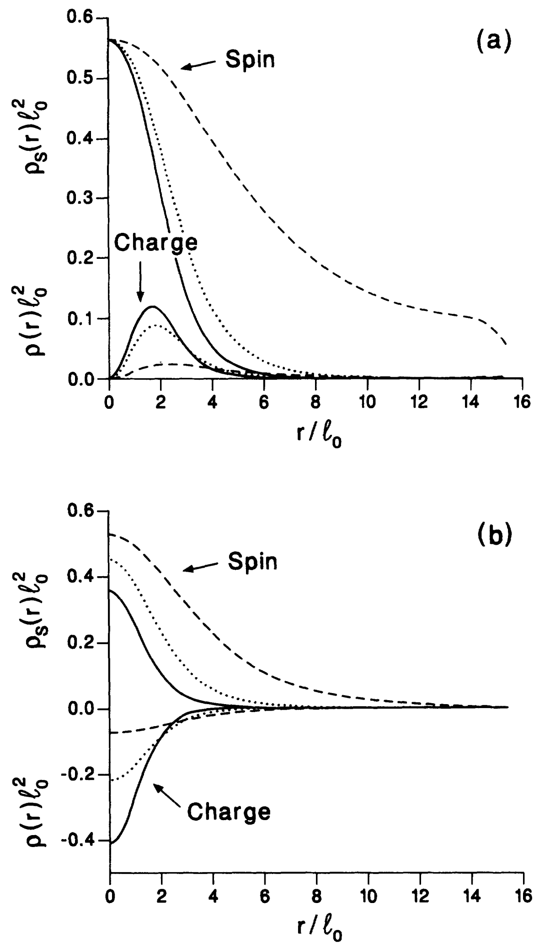


FIG. 2. (a) Spin and charge densities as a function of $r$ for charge +1 CSTE: $\tilde{g} = 0.001$ (solid lines), $\tilde{g} = 0.005$ (dotted lines), $\tilde{g} = 0.01$ (dashed lines). (b) Same, for charge $-1$ CSTE.

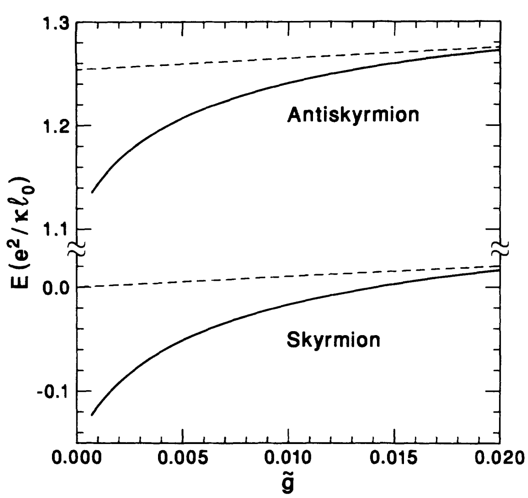


FIG. 1. Solid lines: energy of CSTE's as a function of reduced Zeeman coupling, $\tilde{g} = g\mu_B B/(e^2/\kappa l_0)$. Limiting values of the energies of $\tilde{g} \to 0$ are $-\frac{1}{4}\sqrt{\frac{\pi}{2}}$ for the skyrmion and $\frac{3}{4}\sqrt{\frac{\pi}{2}}$ for the antiskyrmion. Dotted lines: energy of introducing a spin 1/2 quasielectron or quasihole.

ergies. This contrasts with the limit $g = 0$ where the spin texture is a skyrmion whose energy can be evaluated exactly,[3] and the particle-hole creation energy is reduced by 50%. We find that one has to reduce the magnetic field to values smaller than those for which the QHE exists in real systems in order to obtain substantial reductions in the particle-hole pair energies. Experimentally, this means that transport measurements of the activation energy[9] — which essentially measure the energy to create free charged particles in the system — will show little effect due to the spin texture of the quasiparticles.

(2) The dependence of the energies on the parameter $\tilde{g}$ are identical, up to an overall energy shift. This is a result of particle-hole symmetry: under the transformation $a_m^\dagger \to b_m, b_m \to a_m^\dagger$, the Hamiltonian is invariant, except for an overall constant. This means that Eqs. (1) are particle-hole conjugates, the values of $u_m$ and $v_m$ are the same in both equations, and their energies are the same, up to an overall ($\tilde{g}$-independent) constant.

(3) While the energy difference between spin $\frac{1}{2}$ quasiparticles and CSTE's is small for experimentally relevant magnetic fields, it is always energetically favorable to form the latter over the former. In particular, this means that the total spin of the quasiparticles in this system will be much larger than $\frac{1}{2}$. Figure 2 illustrates both typical spin textures and charge densities for different values of the Zeeman coupling. From direct integration, we have found that the total spin of the CSTE's is approximately 9.0 for $\tilde{g} = 0.0044$ ($B = 2$ T if Landé $g = 0.5$), and is 3.5 for $\tilde{g} = 0.014$ ($B = 20$ T if Landé $g = 0.5$). The very large spin associated with these quasiparticles implies that the spin polarization of the $\nu = 1$ state rapidly degrades as charge is added or removed from the system.

The fact that CSTE's are always lower in energy than the corresponding quasiparticles, even for large values of $\tilde{g}$, may be understood as follows. The chemical potential to add or remove charge from the system $\mu_{\rm ST}^{\pm}$ may be written as[8,5] $\mu_{\rm ST}^{\pm} = 2\epsilon_H + \epsilon_X \pm E_{\rm ST}$, where $\epsilon_H$ and $\epsilon_X$ are the Hartree and exchange energies per electron in the single-Slater determinant ground state, and $E_{\rm ST}$ is the neutral spin-texture energy where the number of electrons is held fixed and the charge of the texture is absorbed by expanding or contracting the area occupied by the system.[11] These should be compared with the chemical potentials for spin $\frac{1}{2}$ quasiparticles at $\nu = 1$: $\mu_{\rm QP}^{+} = 2\epsilon_H + g\mu_B B/2$ and $\mu_{\rm QP}^{-} = 2\epsilon_H + 2\epsilon_H - g\mu_B B/2$. It follows that both the positively and negatively charged spin-texture charged excitations are thermodynamically stable when $\epsilon_X + E_{\rm ST} - g\mu_B B/2 < 0$.

To motivate our use of Eqs. (1) as appropriate CSTE wave functions, we first observe that the single-particle states going into the HF wave function should be eigenstates of a unitless Hamiltonian of the form

$$H_0 = -\vec{B}(\vec{r}) \cdot \vec{S}(\vec{r}), \tag{2}$$

where $\vec{S}(\vec{r})$ is the spin density, appropriately projected into the LLL, and $\vec{B}(\vec{r})$ must be determined self-consistently. $\vec{B}(\vec{r})$ represents an effective magnetic field proportional to the average spin density of the electron state, and we note that $H_0$ favors the local spin $\vec{S}(\vec{r})$ of an electron to point parallel to the local field; this arises due to the exchange interaction between electrons. To see what kind of single-particle states are relevant to the skyrmion, consider the single-particle problem in which $\vec{B}(\vec{r})$ is proportional to the spin texture of a skyrmion, as derived from the nonlinear $O(3)$ model of a ferromagnet.[3,6] In unitless form, this is given by

$$\vec{B}(\vec{r}) = \frac{(4\lambda x, \pm 4\lambda y, r^2 - 4\lambda^2)}{r^2 + 4\lambda^2}. \tag{3}$$

The parameter $\lambda$ sets the size scale of the skyrmion, and is a free parameter in the $O(3)$ model. Writing the spin raising and lowering operators $S^{\pm} = S_x \pm iS_y$, and $z = x + iy$, we have

$$H_0 = \frac{2\lambda z}{r^2 + 4\lambda^2} S^{\pm} + \frac{2\lambda z^*}{r^2 + 4\lambda^2} S^{\mp} + \frac{r^2 - 4\lambda^2}{r^2 + 4\lambda^2} S_z.$$

Note that $z$ acts as an angular momentum raising operator, and, after projecting into the LLL, $z^*$ acts as a lowering operator. The single-particle eigenstates of $H_0$ thus have the form

$$(u_m a_m^\dagger + v_m b_{m\mp 1}^\dagger)|0\rangle, \tag{4}$$

which is precisely the form entering our Eq. (1).

We now proceed to describe in detail our HF treatment of the full Hamiltonian. Working in the LLL, this is given by

$$\begin{aligned} H = & -g\mu_B B \sum_m [b_m^\dagger b_m - a_m^\dagger a_m] \\ & + \frac{1}{2} \sum_{m_1, m_2, m_3, m_4} V_{m_1 m_2 m_3 m_4} \\ & \times :[a_{m_1}^\dagger a_{m_2} + b_{m_1}^\dagger b_{m_2} - \delta m_1 m_2][a_{m_3}^\dagger a_{m_4} + b_{m_3}^\dagger b_{m_4} - \delta m_1 m_2]: \end{aligned}$$

where the colons represent the normal ordering of the operators, and

$$\begin{aligned} V_{m_1 m_2 m_3 m_4} = & \int d^2 r_1 d^2 r_2 \phi_{m_1}^*(\vec{r}_1) \phi_{m_2}(\vec{r}_1) \frac{e^2}{\kappa |\vec{r}_1 - \vec{r}_2|} \\ & \times \phi_{m_3}^*(\vec{r}_2) \phi_{m_4}(\vec{r}_2). \end{aligned}$$

(A uniform neutralizing background is also explicitly included in $H$ in the case of Coulomb electron-electron interactions.) We employ a HF decomposition of $H$, allowing the expectation values $\langle a_{m_1}^\dagger a_{m_2} \rangle$ and $\langle b_{m_1}^\dagger b_{m_2} \rangle$ to be different than zero only if $m_1 = m_2$. We also allow an anomolous matrix element $\langle a_{m_1}^\dagger b_{m_2} \rangle$ to be nonzero if $m_1 = m_2 + 1$ for the case of charge 1 excitations, and if $m_1 = m_2 - 1$ for charge $-1$. The resulting HF Hamiltonian takes the form

$$
\begin{aligned}
H^{\rm HF} = &-g\mu_B B \sum_m [b_m^\dagger b_m - a_m^\dagger a_m] \\
&+ \sum_m U^H(m)[a_m^\dagger a_m + b_m^\dagger b_m] \\
&- \sum_m [U_a^{\rm ex}(m) a_m^\dagger a_m + U_b^{\rm ex}(m) b_m^\dagger b_m] \\
&- \sum_m [U_\pm^{\rm sk}(m) a_m^\dagger b_{m\pm1} + U_\pm^{{\rm sk}\,*}(m) b_{m\pm1}^\dagger a_m],
\end{aligned}
$$

where

$$
\begin{aligned}
U^H(m) &= \sum_{m_1} V_{mm_1mm_1}[\langle a_{m_1}^\dagger a_{m_1}\rangle + \langle b_{m_1}^\dagger b_{m_1}\rangle - 1], \\
U_c^{\rm ex}(m) &= \sum_{m_1} V_{mm_1m_1m}\langle c_{m_1}^\dagger c_{m_1}\rangle, \\
U_\pm^{\rm sk}(m) &= \sum_{m_1} V_{m,m_1,m_1\pm1,m\pm1}\langle a_{m_1}^\dagger b_{m_1\pm1}\rangle.
\end{aligned} \tag{5}
$$

where $c$ is either $a$ or $b$. From the form of $H^{\rm HF}$, it may be seen that its eigenstates will be of the form in Eq. (4).

Having determined $H^{\rm HF}$, it is apparent that its ground state is of the form in Eqs. (1), with

$$
\begin{aligned}
u_m &= \frac{U_\pm^{\rm sk}(m)}{\sqrt{\epsilon(m)^2 + |U_\pm^{\rm sk}(m)|^2}}, \\
v_m &= \frac{\epsilon(m)}{\sqrt{\epsilon(m)^2 + |U_\pm^{\rm sk}(m)|^2}},
\end{aligned} \tag{6}
$$

where $\epsilon(m) = \frac{1}{2}\{\epsilon_a(m) - \epsilon_b(m) + \left[(\epsilon_a(m) - \epsilon_b(m))^2 + 4|U_\pm^{\rm sk}(m)|^2\right]^{1/2}\}$ and $\epsilon_c(m) = s_c g\mu_B B + U^H(m) + U_c^{\rm ex}(m)$, with $s_c = 1$ for $c = a$, $s_c = -1$ for $c = b$. Finally, we can compute the order parameters from the $u_m$ and $v_m$ by noting

$$
\begin{aligned}
&\langle a_m^\dagger a_m\rangle = |u_m|^2, \quad \langle b_m^\dagger b_m\rangle = |v_{m\mp1}|^2, \\
&\langle a_m^\dagger b_{m\pm1}\rangle = u_m^* v_m.
\end{aligned}
$$

The HF approximation is accomplished by iterating this with Eqs. (5) and (6), until a self-consistent solution is obtained. The spin and charge densities, relative to the ground state, are given by $\rho_s(\vec r) = \frac{1}{2}\sum_m |\phi_m(\vec r)|^2[\langle a_m^\dagger a_m\rangle - \langle b_m^\dagger b_m\rangle + 1]$ and $\rho(\vec r) = \sum_m |\phi_m(\vec r)|^2[\langle a_m^\dagger a_m\rangle + \langle b_m^\dagger b_m\rangle - 1]$, respectively. Our results are shown in Figs. 1 and 2, keeping angular momentum states up to $m = 240$, for $\tilde g$ as small as 0.001.

Finally, it is instructive to contrast the results for the Coulomb interaction with those of the hardcore[10] model. We find in our Hartree-Fock calculations that skyrmion spin textures of any size, when appropriately projected onto the lowest Landau level,[8] are solutions of the Hartree-Fock equations at $g = 0$. (However for any nonzero $g$ the only stable solutions of the Hartree-Fock equations correspond to spin $\frac{1}{2}$ quasiparticles.) To understand the degeneracy, it is helpful to note that[5] $E_{\rm ST} = \sum_M V_M(-)^M(2M+1)$ for $g = 0$, where $V_M$ are the Haldane pseudopotentials,[10] *independent* of the size scale of the spin texture. It may also be shown that $\epsilon_H = \sum_M V_M$ and $\epsilon_X = \sum_M (-)^{M+1} V_M$, so that

$$
\begin{aligned}
\mu_{\rm QP}^+ - \mu_{\rm ST}^+ = \mu_{\rm ST}^- - \mu_{\rm QP}^- &= \sum_M V_M 2M(-)^{(M+1)} \\
&= 2V_1 - 4V_2 + \cdots .
\end{aligned}
$$

In the hardcore model, all the pseudopotentials except $V_0$ are set to zero, leading directly to the degeneracy. Physically, the degeneracy arises from the fact that spins of all the electrons in the spin-texture state point in the same direction at all points in space, even though this direction depends on position. In the hardcore model, only particles at the same position in space can interact. Since Pauli exclusion forbids any two particles with the same spin to be at the same position, there is no contribution to the energy from the spin texture.

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