PHYSICAL REVIEW B VOLUME 50, NUMBER 14 1 OCTOBER 1994-II

Quantum Heisenberg model with long-range ferromagnetic interactions

H. Nakano and M. Takahashi
Institute for Solid State Physics, University of Tokyo, Roppongi, Tokyo 106, Japan

(Received 13 June 1994)

A modified spin-wave theory is applied to the one- and two-dimensional quantum Heisenberg model with long-range ferromagnetic interactions proportional to r^{-p} $[\mathcal{H}=-\frac{1}{2}\sum_{i\neq j}(J_0/r_{ij}^p)\boldsymbol{S}_i\cdot\boldsymbol{S}_j]$. It is shown that for $d<p<2d$ there exists a phase transition at finite temperatures; the critical temperature is estimated. The susceptibility and the specific heat are obtained at low temperatures. Our results for $d=1$, $p=2$, and $S=\frac{1}{2}$ agree with the exact solution of the Haldane-Shastry model. For $d=1$, we find a scaling relation $\alpha+1=\gamma^{-1}$ when $2<p<3$.

There has been much interest taken in models with long-range interactions proportional to the inverse square of the distance between sites ever since Haldane and Shastry obtained the eigenstates and eigenvalues of the $S=\frac{1}{2}$ Heisenberg chain with a r^{-2} exchange (HS model).[1,2] Haldane discussed the thermodynamics of the $S=\frac{1}{2}$ model and obtained the susceptibility, which is exponentially divergent at low temperatures;[3] less is known about the general interaction r^{-p}. It has been shown that the Ising chain with long-range ferromagnetic interactions (LRFI's) decaying as $1/r^p$ has a phase transition for $1<p\leq 2$ and that for $p>2$ no phase transition of the model exists.[4–6] Regarding the d-dimensional classical Heisenberg model with LRFI's ($d=1,2$), the existence of a phase transition for $d<p<2d$ and the nonexistence for $p\geq 2d$ are known.[7,8] Simulations for the classical case were performed using the Monte Carlo method[9] and the renormalization group approach to the n-vector model was used.[10,11] Also, rigorous bounds for the correlation functions in the disordered phase are known.[12]

For the investigation of the Heisenberg model with nearest-neighbor interaction (NNH model), many methods have been developed. One of them is the modified spin-wave (MSW) theory, proposed by one of the authors.[13] The MSW theory is valid even in dimensions less than three and results from it agree with the exact results from the Bethe ansatz method.[14,15] Recently, we made the first attempt to apply this MSW theory to a case with LRFI's.[16]

In this paper we consider the quantum Heisenberg model with long-range ferromagnetic interactions decaying as $1/r^p$. The model in one dimension and that on a two-dimensional square lattice are studied. Its Hamiltonian with a periodic boundary condition is written as

$$\mathcal{H}=-\frac{1}{2}\sum_{\mathbf{m}}\sum_{\mathbf{n}}{}' J(\mathbf{n})\boldsymbol{S}_{\mathbf{m}}\cdot\boldsymbol{S}_{\mathbf{m}+\mathbf{n}}, \qquad (1)$$

where the prime means the exclusion of the origin. The strength of coupling $J(\mathbf{n})$ is defined by $\sum_l J_0|Ll+\mathbf{n}|^{-p}$, where L is such that L^d is the size of the system. Note that $J(\mathbf{n})$ tends to $J_0/|\mathbf{n}|^p$ in the thermodynamic limit and the condition $p>d$ is necessary for sensible thermodynamics. This model in the limit $p\to\infty$ is the NNH model; the one-dimensional model for $p=2$ and $S=\frac{1}{2}$ is the HS model. We will now discuss a MSW theory of this model and study its properties at low temperatures. After a formulation of the MSW theory, the terms up to the second order in the Bose operators of the transformed Hamiltonian are considered as the first step of this approximation. It is then shown that in the region $p\geq 2d$ there is no phase transition at finite temperatures. The nonzero critical temperature in the region $d<p<2d$ is estimated. The temperature dependence of the susceptibility and of the specific heat are calculated; in the one-dimensional case, a previously unknown scaling relation is obtained for $2<p<3$. Next, the terms up to the fourth order in Bose operators are considered and a set of equations is derived. The procedure by which physical quantities are obtained is shown for the case of the HS model; the susceptibility and the specific heat are analytically calculated.

First of all, the Holstein-Primakoff (HP) transformation $[S^x_{\mathbf{m}}+iS^y_{\mathbf{m}}=(2S-a^\dagger_{\mathbf{m}}a_{\mathbf{m}})^{1/2}a_{\mathbf{m}},\ S^x_{\mathbf{m}}-iS^y_{\mathbf{m}}=a^\dagger_{\mathbf{m}}(2S-a^\dagger_{\mathbf{m}}a_{\mathbf{m}})^{1/2},\ S^z_{\mathbf{m}}=S-a^\dagger_{\mathbf{m}}a_{\mathbf{m}}]$ is applied to the Hamiltonian (1). Note here that $\boldsymbol{S}_{\mathbf{m}}\cdot\boldsymbol{S}_{\mathbf{m}}=S(S+1)$ and that $[a_{\mathbf{m}},a^\dagger_{\mathbf{n}}]=\delta_{\mathbf{m},\mathbf{n}}$. After an expansion in S^{-1}, we use the Fourier transformation $[a_{\mathbf{m}}=(1/\sqrt{N})\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{m}}a_{\mathbf{k}},\ a^\dagger_{\mathbf{m}}=(1/\sqrt{N})\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{m}}a^\dagger_{\mathbf{k}}]$ to change from coordinate space to momentum space, where N is the number of sites. Then, the Hamiltonian is rewritten as $\mathcal{H}=E_0+\mathcal{H}_2+\mathcal{H}_4+O(S^{-1})$, where $E_0=-(1/2)NS^2\eta(0)$, $\mathcal{H}_2=\sum_{\mathbf{k}}a^\dagger_{\mathbf{k}}a_{\mathbf{k}}S\{\eta(0)-\eta(\mathbf{k})\}$, and

$$\begin{aligned}\mathcal{H}_4=\frac{1}{8N}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3}&a^\dagger_{\mathbf{k}_1}a^\dagger_{\mathbf{k}_2}a_{\mathbf{k}_3}a_{\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3}\\&\times\big[\eta(\mathbf{k}_1)+\eta(-\mathbf{k}_2)+\eta(-\mathbf{k}_3)+\eta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3)\\&-2\eta(\mathbf{k}_1-\mathbf{k}_3)-\eta(\mathbf{k}_3-\mathbf{k}_2)-\eta(\mathbf{k}_2-\mathbf{k}_3)\big].\end{aligned}$$

Here $\eta(\mathbf{k})\equiv\sum'_{\mathbf{n}}J(\mathbf{n})e^{i\mathbf{k}\cdot\mathbf{n}}$.

We define $\langle\mathcal{O}\rangle$ as the expectation value of an operator $\mathcal{O}$ for the state $|\{n_{\mathbf{k}}\}\rangle$, i.e., $\langle\mathcal{O}\rangle\equiv\langle\{n_{\mathbf{k}}\}|\mathcal{O}|\{n_{\mathbf{k}}\}\rangle$, where $|\{n_{\mathbf{k}}\}\rangle\equiv\prod_{\mathbf{k}}(n_{\mathbf{k}}!)^{-\frac{1}{2}}(a^\dagger_{\mathbf{k}})^{n_{\mathbf{k}}}|0\rangle$, which are eigenstates of $\mathcal{H}_2$. Thus, the energy E, defined to be $\langle\mathcal{H}\rangle$, is $E_0+\langle\mathcal{H}_2\rangle+\langle\mathcal{H}_4\rangle$, where $\langle\mathcal{H}_2\rangle=\sum_{\mathbf{k}}\langle n_{\mathbf{k}}\rangle S\{\eta(0)-\eta(\mathbf{k})\}$ and

$$\langle\mathcal{H}_4\rangle = -\frac{1}{2N}\Bigg(\sum_{\mathbf{k}}(\langle n_{\mathbf{k}}^2\rangle - \langle n_{\mathbf{k}}\rangle)\Big\{\eta(0)-\eta(\mathbf{k})\Big\} + \sum_{\mathbf{k},\mathbf{k}'(\mathbf{k}\neq\mathbf{k}')}\langle n_{\mathbf{k}}n_{\mathbf{k}'}\rangle\Big\{\eta(0)+\eta(\mathbf{k}-\mathbf{k}')-\eta(\mathbf{k})-\eta(-\mathbf{k}')\Big\}\Bigg) .$$

Here, the number operator $n_{\mathbf{k}}$ is defined by $a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$. The magnetization in the z direction is given by $SN - \sum_{\mathbf{k}} a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}$, which in conventional spin-wave theory is divergent in dimensions less than three. To avoid this divergence, we impose the zero-magnetization condition given by $SN - \sum_{\mathbf{k}}\tilde{n}_{\mathbf{k}} = 0$, where $\tilde{n}_{\mathbf{k}} \equiv \langle n_{\mathbf{k}}\rangle$. This modification makes the spin-wave theory valid even in one and two dimensions.[13]

We assume here that $\tilde{n}_{\mathbf{k}}$ is the Bose distribution; the entropy and the free energy are respectively written as (entropy) $= \sum_{\mathbf{k}}\{(1+\tilde{n}_{\mathbf{k}})\ln(1+\tilde{n}_{\mathbf{k}}) - \tilde{n}_{\mathbf{k}}\ln\tilde{n}_{\mathbf{k}}\}$ and $F = E - T \times$ (entropy). Note that $\langle n_{\mathbf{k}}^2\rangle$ is given by $2\tilde{n}_{\mathbf{k}}^2 + \tilde{n}_{\mathbf{k}}$. We want to know the $\tilde{n}_{\mathbf{k}}$ which minimizes the free energy under the constraint condition of zero magnetization. Therefore, we introduce a Lagrange multiplier μ and minimize the quantity $W = F - \mu(\sum_{\mathbf{k}}\tilde{n}_{\mathbf{k}} - SN)$ with respect to $\tilde{n}_{\mathbf{k}}$. From $\partial W/\partial\tilde{n}_{\mathbf{k}} = 0$, the Bose distribution $\tilde{n}_{\mathbf{k}} = [e^{\beta(\varepsilon(\mathbf{k})-\mu)} - 1]^{-1}$ is reproduced, where $\varepsilon(\mathbf{k}) = \partial E/\partial\tilde{n}_{\mathbf{k}}$ and $\beta = T^{-1}$. From $\partial W/\partial\mu = 0$, we have the self-consistent condition (SCC) $S = (1/N)\sum_{\mathbf{k}}\tilde{n}_{\mathbf{k}}$ which determines the nonzero chemical potential μ. Using rotational averaging,[13] we obtain the static susceptibility (χ) when we have no Bose condensation, i.e., $\chi = (\beta/3N)\sum_{\mathbf{k}}(\tilde{n}_{\mathbf{k}}^2 + \tilde{n}_{\mathbf{k}})$. When we have Bose condensation (BC), we cannot determine the nonzero chemical potential μ, which means spontaneous magnetization. Instead, we can determine the critical temperature T_c ($= 1/\beta_c$) from $S = (1/N)\sum_{\mathbf{k}}[e^{\beta_c\varepsilon(\mathbf{k})} - 1]^{-1}$.

We first consider the two terms up to quadratic order in the Bose operators of the HP transformed Hamiltonian, i.e., $\mathcal{H} = E_0 + \mathcal{H}_2$.

The model in one dimension can now be investigated. To calculate the behavior of $\varepsilon(k)$, we use the property of the Bose-Einstein integral function $F(p,v)$ defined by $[1/\Gamma(p)]\int_0^{\infty}x^{p-1}(e^{x+v}-1)^{-1}dx$ near $v \sim 0$,

$$F(p,v) = \frac{(-v)^{p-1}}{(p-1)!}\left(\sum_{r=1}^{p-1}\frac{1}{r} - \ln v\right) + \sum_{n\neq p-1}\frac{\zeta(p-n)}{n!}(-v)^n ,$$

when p is a positive integer, and

$$F(p,v) = \Gamma(1-p)v^{p-1} + \sum_{n=0}^{\infty}[\zeta(p-n)/n!](-v)^n ,$$

otherwise, where $\zeta(p)$ is Riemann's zeta function and $\Gamma(p)$ is the gamma function. The dispersion $\varepsilon(k)$ in the thermodynamic limit $N\to\infty$ is

$$2J_0S\sum_{n=1}^{\infty}\frac{1-\cos(kn)}{n^p} \simeq \begin{cases} J_0S\zeta(p-2)k^2 & (p>3) \\ -J_0Sk^2\ln k & (p=3) \\ J_0S\omega_1(p)k^{p-1} & (1<p<3) \end{cases} \tag{2}$$

for positive and small k, where $\omega_d(p)$ is defined by $\pi^d d^{2-p}[\Gamma(p/d)]^{-d}/\sin[\pi(p-d)/2]$ for $d = 1$, 2.

In the region $p \geq 2$, we have no BC which breaks the SCC and therefore the system has no finite-temperature phase transition. In this region, we can determine the nonzero μ from the SCC, finding it to be

$$\frac{-1}{\beta\mu} \simeq \begin{cases} 4\zeta(p-2)S^3\beta J_0 & (p>3) \\ \left(S^p[\kappa(p)]^{p-1}\omega_1(p)\beta J_0\right)^{\frac{1}{p-2}} & (2<p<3) \\ \exp(\beta J_0S^2\pi^2) & (p=2), \end{cases}$$

where $\kappa(p) \equiv (p-1)\sin[\pi/(p-1)]$. The susceptibility can now be determined and we obtain

$$\chi = \begin{cases} \frac{2S^4\zeta(p-2)}{3}\beta^2J_0 & (p>3) \\ \frac{p-2}{3J_0(p-1)}[\omega_1(p)]^{\frac{1}{p-2}}\left\{S^2\kappa(p)\beta J_0\right\}^{\frac{p-1}{p-2}} & (2<p<3) \\ \frac{1}{3J_0S\pi^2}\exp(\beta J_0S^2\pi^2) & (p=2). \end{cases}$$

In the region $1<p<2$, on the other hand, we have BC. Thus, the nonzero critical temperature can be estimated from the constant terms of the Bose-Einstein integral function in the SCC with $\mu=0$. Near $p\sim2$, we have

$$T_{\mathrm{c}} = J_0S\omega_1(p)[\pi S(p-1)/\lambda(\tfrac{1}{2-p})]^{p-1} \sim J_0S^2\pi^2(2-p), \tag{3}$$

where $\lambda(p) \equiv \Gamma(\frac{p}{p-1})\zeta(\frac{p}{p-1})$. We note that for any $k(>0)$, $\varepsilon(k)$ goes to $\varepsilon(\pi)$ as $p \to 1^+$. The critical temperature near $p \sim 1$ can then be estimated for $S = \frac{1}{2}$ to be

$$T_c = 2(1-2^{-p})\zeta(p)J_0/\ln 3. \tag{4}$$

The free energy per site of the one-dimensional model in the thermodynamic limit is written as $f = e_0 + S\mu + (T/\pi)\int_0^{\varepsilon(\pi)}(dk/d\varepsilon)\ln(1-e^{-\beta\varepsilon-v})d\varepsilon$, where $E_0/N \to e_0$ $[= -J_0S^2\zeta(p)\ ]$ as $N \to \infty$. From the dispersion relation (2), the free energy is found at low temperatures to be

$$\frac{f-e_0}{T} \simeq \begin{cases} -[\zeta(3/2)/\sqrt{4\pi S\zeta(p-2)}]\sqrt{T/J_0} & (p>3) \\ -[\lambda(p)/\pi][S\omega_1(p)]^{\frac{1}{1-p}}(T/J_0)^{\frac{1}{p-1}} & (1<p<3). \end{cases}$$

We can now calculate the specific heat per site (c) and for S=1/2 we have

$$c \simeq \begin{cases} \{3\zeta(3/2)/[4\sqrt{2\pi\zeta(p-2)}]\}\sqrt{T/J_0} & (p>3) \\ \frac{p\lambda(p)}{\pi(p-1)^2}[\omega_1(p)/2]^{\frac{1}{1-p}}(T/J_0)^{\frac{1}{p-1}} & (1<p<3). \end{cases}$$

We now consider the case when the model is on a two-dimensional square lattice. First, we consider the sum $S_p(\mathbf{k})$, defined by $\sum_{\mathbf{m}}' e^{-i\mathbf{k}\cdot\mathbf{m}}|\mathbf{m}|^{-p}$, to get the dispersion for small k ($= |\mathbf{k}|$). Lee and Bagchi[17] showed that $S_p(\mathbf{k})\Gamma(p/2)(4/\pi)^{p/2}$ is

$$4\sum_{\mathbf{m}}\Phi_{-\frac{p}{2}}\left(\frac{|\mathbf{k}-2\pi\mathbf{m}|^2}{\pi}\right) - \frac{2}{p} + \sum_{\mathbf{m}}{}' e^{-i\mathbf{k}\cdot\mathbf{m}}\Phi_{\frac{p}{2}-1}\left(\frac{\pi|\mathbf{m}|^2}{4}\right),$$

where $\Phi_m(x) \equiv \int_1^{\infty}e^{-\nu x}\nu^m d\nu$. Moreover, Glasser obtained $S_p(\pi,\pi) = 4\left(2^{1-p}-1\right)\zeta(p)\beta(p)$ and $S_p(0,0) = 4\zeta(p)\beta(p)$, where $\beta(p) \equiv \sum_{n=1}^{\infty}(-1)^{n-1}/(2n-1)^p$.[18] Using these results, we get for $\varepsilon(\mathbf{k})/J_0S$, when $k \simeq 0$ and in the thermodynamic limit,

$$\sum_{\mathbf{m}}{}'\frac{1-\cos(\mathbf{k}\cdot\mathbf{m})}{|\mathbf{m}|^p} \simeq \begin{cases} \zeta(\frac{p}{2}-1)\beta(\frac{p}{2}-1)k^2 & (p>4) \\ \frac{1}{2}\pi k^2\ln(1/k) & (p=4) \\ \omega_2(p)k^{p-2} & (2<p<4). \end{cases}$$

For $p > 4$, we can use the SCC: $S = \int_{1BZ} \tilde{n}_{\mathbf{k}}(2\pi)^{-2}d\mathbf{k}$, where the region of integration is the first Brillouin zone (1BZ), to determine the nonzero chemical potential μ, because we then have no BC. Hence, we can determine the nonzero chemical potential, obtaining $-\beta\mu = \exp\left(4\pi\beta J_0 S^2 \zeta(\frac{p}{2}-1)\beta(\frac{p}{2}-1)\right)$. We can now calculate the susceptibility in the same way as the one-dimensional case; we find that $\chi = \exp\left(4\pi\beta J_0 S^2\zeta(\frac{p}{2}-1)\beta(\frac{p}{2}-1)\right)/[12J_0S\zeta(\frac{p}{2}-1)\beta(\frac{p}{2}-1)]$.

In the region $2<p<4$, BC breaks the SCC of zero magnetization. We can again use the SCC, with $\mu=0$, to estimate T_c of BC as in the one-dimensional case. The critical temperature near $p\sim 4$ is estimated to be

$$T_c = J_0 S \omega_2(p)[2\pi(p-2)S/\lambda(\tfrac{2}{4-p})]^{\frac{p-2}{2}} \ . \tag{5}$$

Noting that as $p \to 2$, $\varepsilon(\mathbf{k})$ goes to $\varepsilon(\pi,\pi)$ which equals $8J_0S(1-2^{-p/2})\zeta(p/2)\beta(p/2)$, we can obtain for the critical temperature for $S = 1/2$ near $p \sim 2$

$$T_c = 4J_0(1-2^{-p/2})\zeta(p/2)\beta(p/2)/\ln 3. \tag{6}$$

The free energy per site in the two-dimensional case is now given by $f = e_0 + S\mu - T\int_{1BZ}(2\pi)^{-2}\ln(1+n_{\mathbf{k}})d\mathbf{k}$. Thus, we obtain for the low-temperature behaviors of the free energy and of the specific heat

$$\frac{f-e_0}{T} = \begin{cases} -\pi/[24S\zeta(\frac{p}{2}-1)\beta(\frac{p}{2}-1)](T/J_0) & (p>4)\ , \\ -\frac{\lambda(p/2)}{4\pi}[S\omega_2(p)]^{\frac{2}{2-p}}(T/J_0)^{\frac{2}{p-2}} & (2<p<4), \end{cases}$$

$$c = \begin{cases} \pi/[12S\zeta(\frac{p}{2}-1)\beta(\frac{p}{2}-1)](T/J_0) & (p>4)\ , \\ \frac{p\lambda(p/2)}{2\pi(p-2)^2}[S\omega_2(p)]^{\frac{2}{2-p}}(T/J_0)^{\frac{2}{p-2}} & (2<p<4), \end{cases}$$

respectively.

The case of the two-dimensional model for $p = 4$, which is marginal, as there is the problem as to whether or not the SCC can determine the nonzero chemical potential, is now examined, concluding that there is no transition. To reach this conclusion, first we use spherical polar coordinates, then the right-hand side of the SCC is estimated at low temperatures to be $(1/2\pi)\int_0^{\beta\varepsilon(\pi,\pi)} k(dk/dx)(e^{x+v}-1)^{-1}dx$, where $x \equiv \beta\varepsilon$ and $v \equiv -\beta\mu$. We divide the integral region into two parts. One is the region from 0 to x_0, where k and x are sufficiently small; there, we can make the approximation $k(dk/dx) \simeq 2/[\pi S\beta J_0\ln(1/x)]$. The other is the region from x_0 to $\beta\varepsilon(\pi,\pi)$, where this approximation for the state density is insufficient. At low temperatures the contribution from the former region is dominant; then, we consider the following improper integral for $v = 0$:

$$\int_0^{x_0}\frac{dx}{\ln(1/x)}\frac{1}{e^x-1}. \tag{7}$$

If this improper integral is divergent, we can obtain the nonzero chemical potential without BC; if not, we have BC. It is easy to check that $\lim_{x\to 0} x^{1-\delta}\left(\ln(\frac{1}{x})\right)[(e^x-1)/x]=0$ for $0<\delta<1$ and therefore (7) is divergent. Hence, we reach the conclusion that the two-dimensional model for $p=4$ has no phase transition at finite temperatures.

It is reasonable to consider that our results obtained when we approximate $\mathcal{H}$ by $E_0+\mathcal{H}_2$ are qualitatively correct, because both Haldane's susceptibility[3] and our corresponding χ are exponentially divergent at low temperatures and because the region $d<p<2d$, where a finite-temperature phase transition exists, is the same as the region in Fröhlich and co-workers' discussion regarding the phase transition of the classical case by means of the reflection positivity method.[8] It is a striking result to obtain results for the region $2<p<3$, as here we get a new scaling relation $\alpha+1=\gamma^{-1}$, in the $d=1$ case. Here, α and γ are the exponents of the specific heat and of the susceptibility, respectively, at the zero-temperature phase transition. We have no other results of T_c to compare with ours. We have obtained an unexpected gap of T_c at $p=4$ in the $d=2$ case. The behaviors of γ for the $d=1$ case, T_c for the $d=1$ case, and T_c for the $d=2$ case are shown in Fig. 1.

Next, we consider the terms up to quartic order in the Bose operators of the HP transformed Hamiltonian, i.e., $\mathcal{H} = E_0 + \mathcal{H}_2 + \mathcal{H}_4$. Therefore, because the dispersion relation is given by $\varepsilon(\mathbf{k}) = \partial E/\partial\tilde{n}_{\mathbf{k}}$, we have

$$\varepsilon(\mathbf{k}) = \sum_{\mathbf{n}}{}' J(\mathbf{n})\{1-\cos(\mathbf{k}\cdot\mathbf{n})\} \times\left[S - \frac{1}{N}\sum_{\mathbf{k}'}\tilde{n}_{\mathbf{k}'} + \frac{1}{N}\sum_{\mathbf{k}'}\tilde{n}_{\mathbf{k}'}\cos(\mathbf{k}'\cdot\mathbf{n})\right]. \tag{8}$$

We can make the Fourier expansion of the even function $\tilde{n}_{\mathbf{k}}$ with respect to each component of $\mathbf{k}$; the Fourier coefficient $f(\mathbf{m})$ is written by

$$f(\mathbf{m}) = \int_{1BZ}(d\mathbf{k}/\pi^d)\tilde{n}_{\mathbf{k}}\cos(\mathbf{k}\cdot\mathbf{m}). \tag{9}$$

The dispersion $\varepsilon(\mathbf{k})$ in the limit $N\to\infty$ is then given by

$$\varepsilon(\mathbf{k}) = (J_0/2^d)\sum_{\mathbf{n}}{}'[1-\cos(\mathbf{k}\cdot\mathbf{n})]|\mathbf{n}|^{-p}f(\mathbf{n}), \tag{10}$$

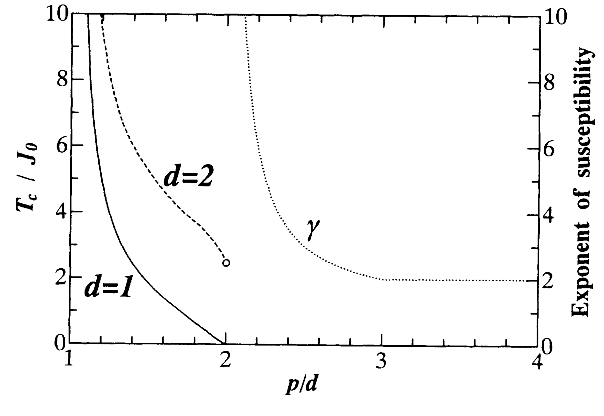

FIG. 1. The exponent of the susceptibility for the case of $d = 1$ and the critical temperature of the $S = \frac{1}{2}$ model for $d = 1$ and $d = 2$. The exponent is represented by the dotted line. The solid line and the dashed line denote T_c for $d = 1$ and for $d = 2$, respectively. The two results (3) and (4) are linked numerically; the two results (5) and (6) are also linked. Note that at $p = 4$ in the two-dimensional case there is no transition at any finite temperatures.

where we have used the SCC in this limit, i.e., $S=f(0)/2^d$.

Hence, the problem of calculating physical quantities in a disordered phase is reduced to obtaining the dispersion and the nonzero chemical potential which satisfy Eqs. (9) and (10) and the equation of Bose distribution under the SCC. For arbitrary p, we can obtain $\varepsilon(\mathbf{k})$ and v from the two equations $S(2\pi)^d = \int_{1\mathrm{BZ}} d\mathbf{k}[e^{\beta\varepsilon(\mathbf{k})+v} - 1]^{-1}$ and

$$\varepsilon(\mathbf{k}) = \frac{J_0}{(2\pi)^d}{\sum_{\mathbf{n}}}' \int_{1\mathrm{BZ}} d\mathbf{q}\frac{[1-\cos(\mathbf{k}\cdot\mathbf{n})]\cos(\mathbf{q}\cdot\mathbf{n})}{|\mathbf{n}|^p(e^{\beta\varepsilon(\mathbf{q})+v}-1)} . \tag{11}$$

The dispersion and the chemical potential which are obtained from these two equations then give physical quantities.

Fortunately, we can employ an analytical treatment for the case of the HS model. Equation (10) for d=1 and for p=2 is differentiated twice, giving the differential equation $d^2\varepsilon/dk^2=J_0\tilde{n}_k-J_0S$. The introduction of a function $g(k)=d\varepsilon/dk$ and the integration of the differential equation give

$$\frac{dk}{d\varepsilon} = \left[\frac{2J_0}{\beta}\ln\left|\frac{1-e^{-\beta\varepsilon-v}}{1-e^{-v}}\right| - 2J_0S\varepsilon\right]^{-1/2} , \tag{12}$$

where the initial conditions $g=\varepsilon=0$ at $k=0$ are used. Equation (12) is used to change an integral variable from k to ε in the integrations.

The substitution of π for k in Eqs. (10) and (12) gives the chemical potential as $v \simeq \exp(-\beta J_0 S^2\pi^2/2)$ at low temperatures. Using Eq. (12), the low-temperature susceptibility of the HS model is calculated for $S = 1/2$ to be $\chi \simeq (\beta/6)\sqrt{2/(\beta J_0\pi)}\exp(\beta J_0\pi^2/8)$. The dominant term in the specific heat per site of the HS model is found to be $c \simeq (2/3)(T/J_0)$ for $S = 1/2$.

Our susceptibility, when we take $E_0 + \mathcal{H}_2 + \mathcal{H}_4$ as $\mathcal{H}$, agrees well with Haldane's. The only difference is the constant factor 2/3, which we can improve in Schwinger-boson mean-field theory,[19] since in that theory, χ is calculated from $(\beta/2N)\sum_{\mathbf{k}}(\tilde{n}_{\mathbf{k}}^2 + \tilde{n}_{\mathbf{k}})$.

In summary, we have studied one- and two-dimensional quantum Heisenberg ferromagnets with a $1/r^p$ exchange, using modified spin-wave theory. We have shown the existence of a phase transition at finite temperatures for $d < p < 2d$ and estimated its critical temperature. We have also calculated the low-temperature behavior of the susceptibility and that of the specific heat; a scaling relation $\alpha+1=\gamma^{-1}$ has been obtained for $2<p<3$ in the one-dimensional case.

This work was supported in part by Grants-in-Aid for Scientific Research on Priority Areas, "Computational Physics as a New Frontier in Condensed Matter Research" (Area No. 217), and "Molecular Magnetism" (Area No. 228), from the Ministry of Education, Science and Culture, Japan.

---

[1] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988).

[2] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).

[3] F. D. M. Haldane, Phys. Rev. Lett. **66**, 1529 (1991).

[4] F. J. Dyson, Commun. Math. Phys. **12**, 91 (1969); **12**, 212 (1969); **21**, 269 (1971).

[5] B. Simon, J. Stat. Phys. **26**, 307 (1981).

[6] J. Fröhlich and T. Spencer, Commun. Math. Phys. **84**, 87 (1982).

[7] G. S. Joyce, J. Phys. **2**, 1531 (1969).

[8] J. Fröhlich, R. Israel, E. H. Lieb, and B. Simon, Commun. Math. Phys. **62**, 1 (1978).

[9] S. Romano, Phys. Rev. B **40**, 4970 (1989); **46**, 5420 (1992).

[10] M. E. Fisher, S. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).

[11] M. Suzuki, Phys. Lett. **42A**, 5 (1972).

[12] K. R. Ito, J. Stat. Phys. **29**, 747 (1982).

[13] M. Takahashi, Prog. Theor. Phys. Suppl. **87**, 233 (1986).

[14] M. Takahashi and M. Yamada, J. Phys. Soc. Jpn. **54**, 2808 (1985).

[15] M. Yamada and M. Takahashi, J. Phys. Soc. Jpn. **55**, 2024 (1986).

[16] H. Nakano and M. Takahashi, J. Phys. Soc. Jpn. **63**, 926 (1994).

[17] M. H. Lee and A. Bagchi, Phys. Rev. B **22**, 2645 (1980).

[18] M. L. Glasser, J. Math. Phys. **14**, 409 (1973).

[19] D. P. Arovas and A. Auerbach, Phys. Rev. B **38**, 316 (1988).