constant of the doped sample. In Fig. 4, the curves using Eq. (4) are drawn for $C^f = C$. This means we have assumed that the line shape does not change with Mg doping. Experimentally, however, we have observed a slight change in the line shape from Lorentzian towards Gaussian as the Mg doping is increased. Unfortunately this change in line shape is very difficult to take into account quantitatively in Eq. (4). Because of this and the line-shape problem even for the undoped $KMnF_3$,[15] the qualitative agreement shown in Fig. 4 is considered satisfactory. We also note that if the line were Gaussian for a doped sample (definitely an extreme limit) and we adjust C to fit the linewidth for $KMnF_3$, the ratio $C/C^f = 1.33$. In this case the theoretical curves in Fig. 4 would be considerably lower than the experimental points. Therefore, it seems reasonable that the discrepancy between the theoretical curves and the experimental points may mainly be due to the change in the line shape with Mg doping. Consequently because of the line-shape difficulties we cannot choose between different values of β in Fig. 4 and the qualitative agreement may be all one should expect at present.

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Scattering Theory in the Heisenberg Ferromagnet

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It is proved that ideal spin waves (magnons with Bose statistics) describe the asymptotic behavior for $t \to \pm\infty$ of certain collective excitations in the infinite Heisenberg ferromagnet.

It has been shown by Bloch[1] and Dyson[2] that the thermodynamic behavior of a three-dimensional Heisenberg ferromagnet for $T \downarrow 0$ is correctly described by an ideal Bose gas of magnons. The relation of the ideal many-spin-wave states to physically realizable states of the ferromagnet has been illuminated by remarks of Dyson and Watts[2]: They should correspond to scattering situations "in which a finite number of spin waves interacts in an infinite and otherwise empty lattice." In this note we shall apply the time-dependent scattering theory of quantum spin systems[3] to show how in the Hilbert space $\mathcal{H}$ of the infinite Heisenberg ferromagnet Fock spaces $\mathcal{H}_\pm$ of magnons are imbedded, and that on $\mathcal{H}_\pm$ the Hamiltonian of this model is diagonal.

We consider an infinite cubic crystal, where every atom is labeled by a lattice vector $\vec{x}\in\mathbf{Z}^3$. For each $\vec{x}\in\mathbf{Z}^3$ there is a two-dimensional state space $\mathbf{C}^2(\vec{x})$ and spin operators $S_i(\vec{x})$ satisfying

$$[S_i(\vec{x}),\ S_j(\vec{y})]=i\textstyle\sum_k \epsilon_{ijk}S_k(\vec{x})\,\delta(\vec{x}-\vec{y})$$

and

$$\vec{S}(\vec{x})\cdot\vec{S}(\vec{x})=\tfrac{3}{4}\ .$$

In the incomplete tensor product

$$\mathcal{H}=\otimes_{\vec{x}}^{\varphi}\ \mathbf{C}^2(\vec{x}) \tag{1}$$

with respect to $\varphi=\otimes\ \varphi'(\vec{x})=|0\rangle$, the Heisenberg Hamiltonian H is defined by[4]

$$H\textstyle\prod_i S_+(\vec{x}_i)\,|\,0\rangle=[K,\prod_i S_+(\vec{x}_i)]|\,0\rangle\ ,$$
$$K=L\sum_{\vec{x}}S_3(\vec{x})-\tfrac{1}{2}J\sum_{\vec{x},\vec{a}}\vec{S}(\vec{x})\cdot\vec{S}(\vec{x}+\vec{a})\ . \tag{2}$$

Here $L\geq 0$, $J>0$, and the summation extends over all $\vec{x}\in\mathbf{Z}^3$ and the six nearest neighbors $\vec{a}$ of $\vec{0}$. H has a natural self-adjoint extension such that

$$A\in\mathfrak{A}\to\alpha_t(A)=e^{iHt}A\,e^{-iHt}$$

is the time-evolution automorphism of the quasi-local algebra $\mathfrak{A}$, associated with K in the φ representation.[4] $H\geq 0$ is reduced and bounded on every eigenspace $\mathcal{H}_n$ of the spin-deviation operator $\sum[S_3(\vec{x})+\frac{1}{2}]$. An orthonormal basis of $\mathcal{H}$ is given by the states

$$|\{m(\vec{x})\}\rangle=\textstyle\prod_{\vec{x}}S_+(\vec{x})^{m(\vec{x})}\,|\,0\rangle\ , \tag{3}$$

where

$$m(\vec{x})\in\{0,1\}\quad\text{and}\quad\textstyle\sum m(\vec{x})<\infty\ .$$

Besides $\mathcal{H}$ we introduce the Fock space $\mathfrak{F}$ with vacuum $|0)$, where magnon creation and annihilation operators $a^*(\vec{x})$, $a(\vec{x})$ $(\vec{x}\in\mathbf{Z}^3)$ act irreducibly:

$$[a(\vec{x}),\ a^*(\vec{y})]=\delta(\vec{x}-\vec{y}),\quad [a(\vec{x}),\ a(\vec{y})]=0,\quad a(\vec{x})\,|0)=0$$

for all $\vec{x}\in\mathbf{Z}^3$. An orthonormal basis of $\mathfrak{F}$ is given by all

$$|\{m(\vec{x})\})=[\textstyle\prod m(\vec{x})!]^{-1/2}\prod a^*(\vec{x})^{m(\vec{x})}\,|0)\ , \tag{4}$$

where $m(\vec{x})=0,1,2,\ldots$ and $\sum m(\vec{x})<\infty$.

Let T be the contraction $\mathfrak{F}\to\mathcal{H}$, defined by linear extension of

$$T|\{m(\vec{x})\})=\begin{cases}0\text{ if some }m(\vec{x})>1\\ |\{m(\vec{x})\}\rangle\ \text{ otherwise}\ .\end{cases} \tag{5}$$

Let $B=\{\vec{k}\in\mathbf{R}^3:|k_i|\leq\pi,\ i=1,2,3\}$ be the first Brillouin zone. Then $\mathfrak{F}$ is isomorphic to the Fock space $\mathfrak{F}(B)$ over $L^2(B)$: For every $f=\{f_n\}\in\mathfrak{F}(B)$ with

$$(f|f)=\sum_{n=0}^{\infty}(2\pi)^{-3n}\int_{B^n}d\vec{k}_1\cdots d\vec{k}_n\,|f_n(\vec{k}_1\cdots\vec{k}_n)|^2<\infty\ , \tag{6}$$

let

$$\tilde{f}_n(\vec{x}_1\cdots\vec{x}_n)=(2\pi)^{-3n}\int_{B^n}d\vec{k}_1\cdots d\vec{k}_n f_n(\vec{k}_1\cdots)\times\exp(i\textstyle\sum\vec{x}_i\cdot\vec{k}_i)\ ,$$

and define

$$|f)=\sum_{n=0}^{\infty}\sum_{\vec{x}_1\cdots\vec{x}_n}\tilde{f}_n(\vec{x}_1\cdots\vec{x}_n)(n!)^{-1/2}\times a^*(\vec{x}_1)\cdots a^*(\vec{x}_n)\,|0)\ . \tag{7}$$

Let $E(\vec{k})$ be the spin-wave energy:

$$E(\vec{k})=L+\tfrac{1}{2}J\textstyle\sum_{\vec{a}}(1-e^{i\vec{k}\cdot\vec{a}})\ . \tag{8}$$

A self-adjoint magnon Hamiltonian H_0 is defined on $\mathfrak{F}$ by

$$e^{-iH_0t}\,|f)=|f^t)\ ,$$
$$f_n^t(\vec{k}_1\cdots\vec{k}_n)=f_n(\vec{k}_1\cdots\vec{k}_n)\exp\Big(-it\sum_{i=1}^{n}E(\vec{k}_i)\Big). \tag{9}$$

Observe that formula (26) of Ref. 2 does not define a linear operator K_0 on $\mathcal{H}$ by $K_0\,|a\rangle=H|a\rangle-\sum Q_{ab}|b\rangle$.

We shall show that Te^{-iH_0t} is asymptotically isometric from $\mathfrak{F}$ to $\mathcal{H}$:

Theorem 1. For all $|f)\in\mathfrak{F}$,

$$\lim_{t\to\pm\infty}(f^t|T^*T|f^t)=(f|f). \tag{10}$$

Proof. Since Te^{-iH_0t} is a contraction from every $\mathfrak{F}_n\to\mathcal{H}_n$, it is sufficient to prove (10) for a dense set $f_n\in L^2(B^n)_{\text{sym}}$. Now, we have

$$(f_n|f_n)-(f_n^t\,|T^*T|\,f_n^t)=\textstyle\sum|\tilde{f}_n^t(\vec{x}_1\cdots\vec{x}_n)|^2\ , \tag{11}$$

where the sum extends over all $(\vec{x}_1\cdots\vec{x}_n)\in\mathbf{Z}^{3n}$ with $\vec{x}_i=\vec{x}_k$ for some $i\neq k$. Equation (11) can be majorized by

$$\tfrac{1}{2}n(n-1)\sum_{\vec{x},\vec{x}_3\cdots\vec{x}_n}|\tilde{f}_n^t(\vec{x},\vec{x},\vec{x}_3\cdots\vec{x}_n)|^2$$
$$=\tfrac{1}{2}n(n-1)(2\pi)^{-3(n+1)}\int_{B^{n-2}}d\vec{k}_3\cdots d\vec{k}_n\times\int_D d\vec{k}\,d\vec{l}\,d\vec{m}\,e^{itE(\vec{k},\vec{l},\vec{m})}\times f_n(\tfrac{1}{2}\vec{k}+\vec{l},\ \tfrac{1}{2}\vec{k}-\vec{l},\vec{k}_3\cdots\vec{k}_n)^*f_n\times(\tfrac{1}{2}\vec{k}+\vec{m},\tfrac{1}{2}\vec{k}-\vec{m},\vec{k}_3\cdots\vec{k}_n)\ , \tag{12}$$

where

$$D=\{(\vec{k},\vec{l},\vec{m})\in\mathbf{R}^9:\tfrac{1}{2}\vec{k}\pm\vec{l}\in B,\ \tfrac{1}{2}\vec{k}\pm\vec{m}\in B\}\ ,$$

$$E(\vec{k},\vec{l},\vec{m}) = E(\tfrac{1}{2}\vec{k}+\vec{l}) + E(\tfrac{1}{2}\vec{k}-\vec{l}) - E(\tfrac{1}{2}\vec{k}+\vec{m}) - E(\tfrac{1}{2}\vec{k}-\vec{m})$$

$$= 2J\sum_{i=1}^{3}\cos(\tfrac{1}{2}k_i)\left[\cos l_i - \cos m_i\right] . \quad (13)$$

Let $f_n(\tfrac{1}{2}\vec{k}+\vec{l}, \tfrac{1}{2}\vec{k}-\vec{l}, \vec{k}_3 \cdots \vec{k}_n) \in \mathfrak{D}$ be symmetric and vanish in a neighborhood of $\{\nabla_l E(\vec{k},\vec{l},\vec{m}) = \vec{0}\}$. For such f_n, which form a dense set in $L^2(B^n)_{\mathrm{sym}}$, Eq. (11) is $O(|t|^{-N})$ for all N. QED

Theorem 2. For all $|f) \in \mathfrak{F}$,

$$\underset{t\to\pm\infty}{s\lim}\, e^{iHt}\,T e^{-iH_0 t}\,|f) = \Omega_\pm\,|f) \quad (14)$$

exists and satisfies

$$\Omega_\pm^*\,\Omega_\pm = 1 , \quad (15)$$

$$e^{iHt}\,\Omega_\pm = \Omega_\pm\, e^{iH_0 t} . \quad (16)$$

Proof. Again, we need only to consider a dense set of $f_n \in L^2(B^n)_{\mathrm{sym}}$. Now we have

$$\frac{d}{dt}\, e^{iHt}\, T e^{-iH_0 t}\,|f_n) = i\, e^{iHt}\, T|g_n^t)$$

$$= i/(n!)^{1/2}(2\pi)^{3n}\int d\vec{k}_1 \cdots d\vec{k}_n f_n(\vec{k}_1\cdots\vec{k}_n)\exp\left(-it\sum_{i=1}^{n} E(\vec{k}_i)\right)$$

$$\times e^{iHt}\left(H - \sum_{i=1}^{n} E(k_i)\right)\tilde{S}_+(\vec{k}_1)\cdots\tilde{S}_+(\vec{k}_n)\,|0\rangle , \quad (17)$$

where by Ref. 2

$$\tilde{S}_+(\vec{k}) = \sum_{\vec{x}} S_+(\vec{x})\, e^{i\vec{k}\cdot\vec{x}}$$

and

$$g_n^t(\vec{k}_1\cdots\vec{k}_n) = \sum_{1\le i<j\le n}\int d\vec{l}\, V(\vec{k}_i,\vec{k}_j,\vec{l})$$

$$\times f_n(\cdots\vec{k}_i + \tfrac{1}{2}\vec{l}\cdots\vec{k}_j - \tfrac{1}{2}\vec{l}\cdots)\exp\{it[\cdots E(\vec{k}_1) + \ldots E(\vec{k}_i + \tfrac{1}{2}\vec{l}) + \ldots E(\vec{k}_j - \tfrac{1}{2}\vec{l})\cdots]\}, \quad (18)$$

$$V(\vec{p},\vec{q},\vec{l}) = J(2\pi)^{-3}\sum_{\vec{a}}\cos[\vec{a}\cdot(\vec{p}-\vec{q})/2]\{\cos[\vec{a}\cdot(\vec{p}+\vec{q})] - \cos(\vec{a}\cdot\vec{l})\} . \quad (19)$$

For a dense set of f_n the $L^2(B^n)$ norm of (18) is $O(|t|^{-N})$ for all N, and therefore $\| T|g_n^t)\| = O(|t|^{-N})$. This leads to (14), while (15) and (16) are immediate consequences of (10) and (14) (see Ref. 5). QED

Hence $\mathcal{H}$ contains Fock spaces $\mathcal{H}_\pm = \Omega_\pm\mathfrak{F}$ of scattering states, whose asymptotic incoming and outgoing configurations are characterized by noninteracting wave packets of magnons:

$$\lim_{t\to\pm\infty}\| e^{-iHt}\,\Omega_\pm\,|f) - T e^{-iH_0 t}\,|f)\| = 0 , \quad (20)$$

and which diagonalize H. An isometric scattering operator S is defined by $S\Omega_+\,|f) = \Omega_-\,|f)$. It is known[6] that $\mathcal{H} \ne \mathcal{H}_\pm$ and that a multichannel scattering situation occurs. Our methods can be generalized to bound-state scattering and to a large class of other quantum-spin systems.[3]

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