PHYSICAL REVIEW B VOLUME 5, NUMBER 3 1 FEBRUARY 1972

Boson Method in Superconductivity: Time-Dependent Theory

L. Leplae, F. Mancini, and H. Umezawa

Department of Physics, University of Wisconsin—Milwaukee, Milwaukee, Wisconsin 53201

(Received 26 April 1971; revised manuscript received 6 July 1971)

The boson method in superconductivity, developed in previous articles, is extended in order to take into account time-dependent phenomena. Starting from the equations of motion for the Heisenberg electron field we have constructed a formulation of superconductivity in which gauge invariance and current conservation are implicitly satisfied. Previously we showed how, by means of invariant transformations called boson transformations, we can generate a space-dependent order parameter; in this way the space-dependent properties of a superconductor are described and the problem of solving the Gor'kov equations is bypassed. In the present article we generalize the concept of boson transformations letting them generate a space- and time-dependent order parameter; in this way we propose a formulation of the theory of superconductivity in which space- and time-dependent effects are taken into account. As an illustration of the method, in the last part of this work we study a system consisting of two weakly coupled superfluid liquids, and we give a theoretical explanation of the experimental fact that the system will present dynamical stability at the harmonics and subharmonics of a fundamental frequency.

I. INTRODUCTION

In the last few years considerable attention has been paid to time-dependent phenomena in superconductivity. These problems are essentially associated with spatial and temporal variations of the order parameter Δ; known examples are the motion of flux lines in type-II superconductors, ac Josephson effects, etc. The theoretical description of these nonstationary phenomena has presented a much harder problem than the static case did. One basic tool in the theory of superconductivity is the Gor'kov equations, which are supposed to be able to describe both stationary and nonstationary phenomena. Unfortunately, owing to their nonlocal nature, these equations are not easy to solve. In stationary situations the form of the Gor'kov equations is such that it is possible to expand the kernels of the integral equations in powers of Δ/T_c; following this procedure Gor'kov showed[1] how in the range of temperatures close to the critical temperature his equations reduce to the coupled Ginzburg-Landau (GL) equations[2] for the vector potential $\vec{A}$ and the order parameter. Subsequently, this method was extended to lower temperatures, and some generalized GL equations were derived.[3]

In the case of nonstationary problems the first attempt was to generalize the GL equations to the case in which the order parameter is time dependent and to derive an "extra" GL equation for the charge density and the electric field. Unfortunately such an extension is not straightforward and many different time-dependent GL equations have been proposed,[4] all being based on the assumption that the space and time variations are sufficiently slow so that the order parameter Δ is close to its equilibrium value Δ_0. Many of these time-dependent GL equations were derived from a microscopic theory by following a procedure similar to the one in the static case.

More recently this problem has been reexamined by Gor'kov and Eliashberg.[5] These authors pointed out that all the time-dependent GL equations derived so far have a range of applicability not immediately related to the physical range of interest. This is due to the fact that in nonstationary cases it is impossible to make an expansion of the kernels of the Gor'kov equations in powers of Δ in the whole range of temperature. Owing to this reason it is very difficult to reduce the Gor'kov equations to local differential equations. The possibility of obtaining generalized GL equations with a range of applicability valid for all the important frequencies exists only in the case that the superconductivity effects are relatively small. This is the case, for example, of a superconductor with a large concentration of paramagnetic impurities.[5] So far, theoretical study of superconductors in presence of time-varying fields has not reached any definitive conclusion.

In this paper we extend the boson method in superconductivity, developed in previous papers,[6,7] by taking into account the time-dependent effects. The approach follows the general lines of Ref. 7. Starting from the BCS model we have expressed most of the important observables in terms of the quasifermion field ϕ and the quasiboson field B. These results have been obtained by solving the equations of motion for the electron field ψ (here ψ is a Heisenberg operator) for a space-independent superconductor: The ground state is assumed position independent so that the order parameter $\Delta = \langle \psi_\dagger \psi_\dagger \rangle$ is space independent. To move from the space-independent solutions to the space-dependent ones we introduced an invariant transformation, the "boson transformation," under which the field equations stay invariant while the ground-state expectation values[8] of various observables are allowed to change. Through this transformation the space-dependent order parameter and the corresponding solutions of the Gor'kov equations are obtained simultaneously. An application of the boson method in a space-dependent problem has been made[7,9] to study the structure of vortex lines in type-II superconductors. There we compute the distribution of current and magnetic field in the whole domain, up to the center of the flux line, and our results are valid for a wide range of the Ginzburg parameter $\kappa = \lambda/\xi$.

The advantage presented by the boson method is particularly clear in this example. As we have already mentioned, all the different GL equations[2,3] proposed in the static case are valid under the requirement that all the quantities vary very slowly in the range of the coherence length and that the electrodynamics be local. Naturally these limitations restrict the analysis to London superconductors ($\kappa \gg 1$) and to distances far from the center of the flux line. If one wants to remove these limitations then one must resort to the Gor'kov equations, or equivalently to the Bogoliubov equations,[10] and try to solve them in a self-consistent manner. Recent attempts[11] have been made in this direction, but the principal difficulty of solving the equations in a self-consistent way has not been avoided, and practically only particular models can be solved by means of numerical calculations.

Our approach to time-dependent situations is the generalization of the method described above; we look for a transformation which leaves invariant the field equations of the electron field, but modifies the energy of the ground state; previously we have restricted ourselves to the case in which this transformation keeps the order parameter time independent, while in this paper we let the boson transformation generate a time-dependent order parameter. As a result the space- and time-dependent properties of a superconductor can be derived simply by operating invariant transformations on the operator form of the corresponding quantities expressed in terms of quasielectron and boson fields. The generator of these invariant transformations is expressed in terms of a function $f(\vec{x}, t)$ which satisfies the equations given in Sec. II. Once these equations are solved according to the boundary conditions appropriate to the problem under study, most of the properties of the system are known; in particular an expression for the space- and time-dependent order parameter is given.

The problem of solving the Gor'kov equations is not present in the boson method since *we do not need* these equations; the space and time dependences is introduced through the boson transformations. This situation turns out to be extremely advantageous in the cases, illustrated previously, where it is not possible to reduce the Gor'kov equations to a system of nonlinear differential equations of the GL type. Furthermore it is also well known[10,12] that a theory of superconductivity based on the Hartree-Fock approximation is not gauge invariant and the current is not conserved, while in the boson method the gauge invariance and current conservation are guaranteed by the presence of the boson field.

In order to illustrate our formulation we have applied the boson method to the study of two superfluid systems which are weakly coupled together. In the present paper we study the case of two superfluid liquids joined by a weak link, in a subsequent paper[13] we extend our investigation to the study of alternating supercurrents in weakly coupled superconductors.[14] In these applications, starting from the microscopic equations, we give a simple derivation of the experimental fact that the system will present dynamical stability only at frequencies which are harmonic and subharmonic to a fundamental frequency. In other approaches one must resort to more complicated arguments in order to give a theoretical explanation of this experimental situation.

In the case of superconductivity the Josephson formulation does not give a full description of the experimental facts. It has been argued[15] that this formulation, based on a perturbative treatment of the tunneling Hamiltonian, may be adequate to describe only junctions thick enough that the tuneling process can be treated by lowest-order perturbation theory, so that its application to point-contact junctions might not be appropriate. This is certainly the case; however, there exists a class of experimental phenomena that are common to systems of two superconductors joined by a weak link, independently of how the link is realized. Therefore, it is our opinion that all the phenomena occurring in weakly coupled superconductors should be described by a unified theory; the formulation based on the phenomenological argument of the tunneling Hamiltonian seems not to give a general description.

In the framework of the boson method a study of the static Josephson effect has been already made[16]; our approach gives a general description of this effect and shows how the Josephson formulation corresponds only to a particular solution of our equations; in other words the results of the usual theory are valid only for certain specific geometrical configurations. One of the experimental consequences of this generalization is the fact that the Fraunhofer diffraction pattern for the net current crossing the junction does not follow necessarily the form $(\sin x)/x$, predicted by Josephson. The advantages presented from the boson method become more transparent in the study of time-dependent situations. According to the Josephson formulation when two superconductors coupled together by a junction are maintained at a relative potential V_0, an alternating supercurrent with a frequency $\nu_J = (2eV_0)/2\pi$ will flow between the two superconductors. If the junction is placed in a microwave field with frequency ν, the Josephson formulation predicts that a dc current in the form of a sharp step will appear in the dc I-V characteristic at dc voltages V_0 which satisfy the relation $\nu = n\nu_J$, where n is an integer; furthermore, according to this formulation, the amplitude of this dc current should be a Bessel function of V, where V is the amplitude of the external field. The presence of these steps in the I-V characteristic has been experimentally observed in many type of junctions,[17] however, in the case of thin-film bridges and in point-contact junctions[18] steps appear also at dc voltages given by $m\nu = n\nu_J$, where m and n are both integers. The variation of I with respect to V has been also experimentally measured[19]; the behavior as a Bessel function is verified only in approximate way, more generally from the data of Ref. 19 we can deduce only a periodic nature. All these experimental facts show with enough clarity that the Josephson formulation does not cover all the experimental situation and one needs a more general theory.[15] By applying the methods described in this paper we give in Ref. 13 a general description of the electromagnetic properties of a system of two weakly coupled superconductors. Our formulation confirms most of the experimental facts: dc currents are shown to appear at voltages V_0 which correspond to harmonic or subharmonic of the fundamental frequency ν_J; the amplitude of this dc current is a periodic function of V. We have not studied in detail the form of our solutions; by applying additional boundary conditions which take into account the geometry of the system and the properties of the materials we should be able to fix the shape of some undetermined parameters that enter in the expression of the alternating current; in particular we might verify the experimental fact that n/m is usually an integer number in the case of tunnel junctions and is a rational number in the case of point-contact junctions and thin-film bridges. Our expression of the net current crossing the junction contains a "Fraunhofer factor" which shows that some lines in the I-V characteristic should be missing; this circumstance can be verified experimentally and once that this is confirmed we can obtain, as it is shown in Ref. 13, detailed informations about the shape and the velocity of the flux lines present in the junction.

In the case of superfluidity experiments have been performed on a system of two baths of superfluid helium which are weakly coupled through a small orifice. When the system is coupled to an external field oscillating in time the experiments show that the system will present dynamical stability only for certain values of the difference in the helium head. These effects are analogous to the ac Josephson effects in superconductivity. Theoretically, however, the situation is not so similar; in the case of superfluidity one does not have an equivalent of the Josephson equation (this equation is derived using phenomenological arguments together with the Maxwell equations) and the usual theoretical explanation is based on an intuitive picture. In the present paper we apply the time-dependent boson formulation to the study of these phenomena that have place in superfluid liquids. In Sec. V we show how the condition of stability comes naturally from the microscopic theory by requiring that the system be in resonance with the external perturbation. We do not study in detail the solutions, deferring this problem to a future paper, however, we show how in principle we can generate solutions that correspond to the usual physical picture that is used to explain the experimental results.

Summarizing, in this paper we have extended the boson method in superconductivity to the case in which time-dependent effects are considered; in this way we propose a formulation of the theory of superconductivity in which nonstationary phenomena can be described. At the present stage of our formulation we have not taken into account the presence of impurities and the interaction of the Heisenberg electron field with the lattice phonons; in some problems these effects can affect in a sensible way the dynamics of the system, so that the present shape of the boson method may be incomplete in describing situations at temperature close to the critical temperature. Extension of the method is under study. In order to verify and to illustrate the method we have applied the formulation to the study of the phenomena that occur when two superfluids are weakly coupled together. In Sec. V of the present paper we consider the case of two superfluid liquids, while a subsequent paper[13] is devoted to the case of two superconductors.

II. BOSON TRANSFORMATION

For the sake of clarity, let us first recall briefly the results obtained in previous papers. We have derived the operator form of some of the observables in terms of quasiparticles. In particular the Hamiltonian, the density, and the current operators take the form

$$H = \sum_k E_k (\alpha_{k\uparrow}^\dagger \alpha_{k\uparrow} + \alpha_{k\downarrow}^\dagger \alpha_{k\downarrow}) + \sum_l \omega_l B_l^\dagger B_l \,, \qquad (2.1)$$

$$\rho(\vec{x}, t) = \rho^{(1)}(\vec{x}, t) + \rho^{(2)}(\vec{x}, t) , \quad (2.2)$$

$$\vec{j}(\vec{x}, t) = \vec{j}^{(1)}(\vec{x}, t) + \vec{j}^{(2)}(\vec{x}, t) . \quad (2.3)$$

Here $\rho^{(1)}$ and $\vec{j}^{(1)}$ denote the quasifermion part and $\rho^{(2)}$ and $\vec{j}^{(2)}$ denote the boson part; α_k, $\alpha_k^\dagger$ and B_l, $B_l^\dagger$ are annihilation and creation operators of the quasifermion ϕ and of the boson B, respectively.[20] The expression of ω_l, the energy spectrum of the boson field, has been given in Ref. 6. If we concentrate our attention only on the boson part, since the fermion part does not play an important role in the boson method, the explicit expression of the observables in Eqs. (2.1)–(2.3) is given by

$$H(B) = \frac{1}{2} \int d^3x [\pi^2(\vec{x}, t) + v_0^2 \vec{\nabla} B(\vec{x}, t) \cdot \vec{\nabla} B(\vec{x}, t)] + \frac{e^2\eta^2}{2} \iint d^3x\, d^3y \, \frac{\pi(\vec{x}, t)\pi(\vec{y}, t)}{|\vec{x} - \vec{y}|} , \quad (2.4)$$

$$\rho^{(2)}(\vec{x}, t) = -\eta(\vec{\nabla})\pi(\vec{x}, t) , \quad (2.5)$$

$$\vec{j}^{(2)}(\vec{x}, t) = v_0^2 \eta(\vec{\nabla}) \vec{\nabla} B(\vec{x}, t) . \quad (2.6)$$

Here v_0, the boson velocity, is given by $v_0 = v_F/\sqrt{3}$; the temperature-dependent coefficient $\eta(\vec{\nabla})$ is defined in Ref. 6; η means $\eta(0)$. The last term in the Eq. (2.4) is present only in the case of charged superconductors. π is the canonical conjugate of B and can be put in the form

$$\pi(\vec{x}, t) = \frac{\partial b(\vec{x}, t)}{\partial t} , \quad (2.7)$$

where

$$b(\vec{x}, t) = B(\vec{x}, t) \quad \text{in neutral superconductors} , \quad (2.8)$$

$$b(\vec{x}, t) = -\frac{1}{4\pi} \int d^3y \, \frac{e^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x} - \vec{y}|} \vec{\nabla}_y^2 B(\vec{y}, t) \quad \text{in charged superconductors} . \quad (2.9)$$

The quantity μ is given by $\mu = (4\pi)^{1/2} e\eta$. The boson field satisfies the field equation

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \vec{\nabla}^2\right) B(\vec{x}, t) = 0 \quad \text{in neutral superconductors} , \quad (2.10)$$

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \vec{\nabla}^2\right) B(\vec{x}, t) - \frac{\mu^2 v_0^2}{4\pi} \times \int d^3y \, \frac{1}{|\vec{x} - \vec{y}|} \vec{\nabla}_y^2 B(\vec{y}, t) = 0 \quad \text{in charged superconductors} . \quad (2.11)$$

From Eqs. (2.7)–(2.11) the following important relation

$$\frac{\partial}{\partial t} \pi(\vec{x}, t) = v_0^2 \vec{\nabla}^2 B(\vec{x}, t) \quad (2.12)$$

is valid for neutral and charged superconductors. Another important result is that the electron fields can be expressed in terms of the quasiparticle fields as

$$\psi_{\uparrow,\downarrow} = e^{[i/\eta(\vec{\nabla})]B} F(\phi_{\uparrow,\downarrow}, \vec{\nabla} B, \pi) , \quad (2.13)$$

where F is a certain function.

Now let us consider the transformation induced by the following generator:

$$N_f = \int d^3x [\dot{f}(\vec{x}, t)\eta(\vec{\nabla}) b(\vec{x}, t) - f(\vec{x}, t)\eta(\vec{\nabla})\pi(\vec{x}, t)] , \quad (2.14)$$

where $\dot{f} \equiv \partial f/\partial t$, and let us study the conditions under which this transformation leaves invariant the equations of motion. This condition is that the generator will be time independent[7]; i.e., $\dot{N}_f = 0$.

A. Neutral Superconductors

Using Eqs. (2.7), (2.8), and (2.12) we find

$$\dot{N}_f = \int d^3x [\ddot{f}(\vec{x}, t)\eta(\vec{\nabla}) B(\vec{x}, t) - v_0^2 f(\vec{x}, t)\eta(\vec{\nabla})\vec{\nabla}^2 B(\vec{x}, t)] . \quad (2.15)$$

Next we consider the commutator

$$[\dot{N}_f, \pi(\vec{y}, t)] = i \int d^3x [\ddot{f}(\vec{x}, t) - v_0^2 f(\vec{x}, t)\vec{\nabla}^2] \eta(\vec{\nabla}) d(\vec{x} - \vec{y}) , \quad (2.16)$$

where

$$id(\vec{x} - \vec{y}) = [B(\vec{x}, t), \pi(\vec{y}, t)] . \quad (2.17)$$

Since $d(\vec{x} - \vec{y})$ is a short-range function we can integrate by parts and write Eq. (2.16) in the form

$$[\dot{N}_f, \pi(\vec{y}, t)] = i \int d^3x [\ddot{f}(\vec{x}, t) - V_0^2 \vec{\nabla}^2 f(\vec{x}, t)] \eta(\vec{\nabla}) d(\vec{x} - \vec{y}) . \quad (2.18)$$

On the other hand it is obvious that $\dot{N}_f$ commutes with ϕ and B, therefore it follows from Eq. (2.18) that $\dot{N}_f$ acts as a null operator in the algebra of operators in $D(L)$,[21] if $f(\vec{x}, t)$ is a solution of the equation

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \vec{\nabla}^2\right) f(\vec{x}, t) = 0 . \quad (2.19)$$

B. Charged Superconductors

Proceeding as in the neutral case, making use of Eqs. (2.7), (2.9), and (2.12), it is easy to show that $\dot{N}_f$ will be a null operator in the algebra of operators in $D(L)$ if $f(\vec{x}, t)$ satisfies the following equation:

$$v_0^2 \vec{\nabla}^2 f(\vec{x}, t) + \frac{1}{4\pi}\int d^3y \frac{e^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} \vec{\nabla}_y^2 \ddot{f}(\vec{y}, t) = 0 . \tag{2.20}$$

Since $e^{-\mu|\vec{x}-\vec{y}|}/|\vec{x}-\vec{y}|$ is a function of finite range, the integration by parts is permitted, so that Eq. (2.20) can be put in the differential form

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \vec{\nabla}^2 + \omega_p^2\right) \vec{\nabla}^2 f(\vec{x}, t) = 0 . \tag{2.21}$$

Here ω_p is the plasma frequency

$$\omega_p^2 = \mu^2 v_0^2 . \tag{2.22}$$

The general form of solutions of Eq. (2.21) can be written as

$$f = f_1 + f_2 , \tag{2.23}$$

where f_1 satisfies the plasma equation

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \vec{\nabla}^2 + \omega_p^2\right) f_1(\vec{x}, t) = 0 , \tag{2.24}$$

while f_2 satisfies the Laplace equation

$$\vec{\nabla}^2 f_2(\vec{x}, t) = 0 . \tag{2.25}$$

It should be noted that the time-dependent behavior of f_2 is arbitrary. This reflects the feature of the energy spectrum of the boson field (cf. Ref. 6); the behavior of ω_l in the domian of extremely small momentum ($l \leq 1/L$, where L is the linear size of the system) depends on the shape of the surface of the system. Intuitively speaking, the time dependence of f_2 depends almost entirely on the boundary conditions. Indeed the Laplace equation corresponds to the Klein-Gordon equation with maximum signal velocity being infinite: The infinite nature of the maximum signal velocity is a manifestation of the Coulomb effect.[22] Note that, when the Coulomb potential does not exist, f satisfies Eq. (2.19) which is the Klein-Gordon equation with maximum signal velocity v_0.

Summarizing, we define the boson transformation as the transformation induced by the generator N_f in Eq. (2.14), where the function $f(\vec{x}, t)$ stands for a solution of Eq. (2.19) in the neutral case, and Eqs. (2.23)–(2.25) in the charged case.

III. GROUND-STATE EXPECTATION VALUES OF THE OBSERVABLES

A. Neutral Superconductors

Under the boson transformation the quasiparticle fields transform in the following way:

$$\phi_{\uparrow,\downarrow} \to \phi_{\uparrow,\downarrow} ,$$

$$B(\vec{x}, t) \to B_f(\vec{x}, t) = B(\vec{x}, t) + \int d^3y f(\vec{y}, t)\eta(\vec{\nabla}_y) d(\vec{x}-\vec{y}) , \tag{3.1}$$

$$\pi(\vec{x}, t) \to \pi_f(\vec{x}, t) = \pi(\vec{x}, t) + \int d^3y \dot{f}(\vec{y}, t)\eta(\vec{\nabla}_y) d(\vec{x}-\vec{y}) . \tag{3.2}$$

It is easy to see that the transformations (3.1) and (3.2) leave Eqs. (2.10) and (2.7) invariant, as is expected from the fact that the boson transformations are invariant transformations. Recalling Eqs. (2.2), (2.3), (2.5), and (2.6) we find that the boson transformation induces the following ground-state density:

$$\rho(\vec{x}, t) \equiv \langle \rho_f \rangle = -\eta^2 \int d^3y\, c(\vec{x}-\vec{y}) \dot{f}(\vec{y}, t) \tag{3.3}$$

and the following ground-state current density:

$$\vec{j}(\vec{x}, t) \equiv \langle \vec{j}_f \rangle = \eta^2 v_0^2 \int d^3y\, c(\vec{x}-\vec{y}) \vec{\nabla} f(\vec{y}, t) . \tag{3.4}$$

Here the correlation function $c(\vec{x}-\vec{y})$ is defined by

$$c(\vec{x}-\vec{y}) = \frac{\eta(\vec{\nabla}_x)\eta(\vec{\nabla}_y)}{\eta^2} d(\vec{x}-\vec{y}) . \tag{3.5}$$

This function has a range of ξ, the coherence length, and it is normalized to unity[7]:

$$\int d^3x\, c(\vec{x}-\vec{y}) = 1 . \tag{3.6}$$

From Eq. (2.19), the conservation law is immediately verified:

$$\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = \eta^2 \int d^3y\, c(\vec{x}-\vec{y}) \left(v_0^2 \vec{\nabla}^2 - \frac{\partial^2}{\partial t^2}\right) f(\vec{y}, t) = 0 . \tag{3.7}$$

From Eqs. (2.4), (3.1), and (3.2) we find that the boson transformation modifies the c-number part of the Hamiltonian; the ground-state energy after the transformation being given by

$$W \equiv \langle H_f \rangle = \frac{1}{2\eta^2}\int d^3x \left(\rho^2(x, t) + \frac{1}{v_0^2} \vec{j}(\vec{x}, t)\cdot\vec{j}(\vec{x}, t)\right) . \tag{3.8}$$

It should be noted that this effect does not contradict the previous result that the operator N_f is time independent. This was discussed in Appendix A of Ref. 7.

B. Charged Superconductors

Taking into account Eqs. (2.7), (2.9), and (2.12) we can easily see that under the transformation induced by N_f, the fields transform as

$$\phi_{t,i} \rightarrow \phi_{t,i} \, ,$$

$$B(\vec{x}, t) \rightarrow B_f(\vec{x}, t) = B(\vec{x}, t) + \int d^3y \, f(\vec{y}, t) \eta(\vec{\nabla}_y) \, d(\vec{x}-\vec{y}), \tag{3.9}$$

$$\pi(\vec{x}, t) \rightarrow \pi_f(\vec{x}, t) = \pi(\vec{x}, t) - \frac{1}{4\pi} \iint d^3y \, d^3z \, \frac{e^{-\mu|\vec{y}-\vec{z}|}}{|\vec{y}-\vec{z}|} \times \eta(\vec{\nabla}_y) \, d(\vec{x}-\vec{y}) \vec{\nabla}_z^2 \dot{f}(\vec{z}, t) \, , \tag{3.10}$$

$$\psi_{t,i} \rightarrow \psi_{f\,t,i} = e^{[i/\eta(\vec{\nabla})]B_f} \, F(\phi_{t,i}, \vec{\nabla} B_f, \pi_f) \, . \tag{3.11}$$

We can show that these transformations leave Eqs. (2.11) and (2.7) invariant, as is expected from the fact that the boson transformations are invariant transformations.

The effect of the electromagnetic field is easily taken into account[7] observing that gauge-invariance considerations require that $\vec{\nabla} B$ is replaced everywhere by

$$\vec{\nabla} B(\vec{x}, t) \rightarrow \vec{\nabla} B(\vec{x}, t) - e \int d^3y \, \vec{A}_T(\vec{y}, t) \eta(\vec{\nabla}_y) \, d(\vec{x}-\vec{y}) \, , \tag{3.12}$$

where $\vec{A}_T$ is the transverse part of the vector potential of the electromagnetic field. Here the electromagnetic field is the total field, which contains the external fields and the fields due to persistent currents. The scalar part of the vector potential, due to self-consistent effects, is taken into account by the Coulomb potential introduced in the Hamiltonian.[6] The scalar part of the vector potential, due to external fields, can be introduced by the same argument of gauge invariance, requiring that $\dot{B}$ always appears in the combination

$$\frac{\partial}{\partial t} B(\vec{x}, t) - \int d^3y \, \phi_{\text{ext}}(\vec{y}, t) \eta(\vec{\nabla}_y) \, d(\vec{x}-\vec{y}) \, , \tag{3.13}$$

where ϕ_{ext} is the external scalar potential. However, since ϕ_{ext} does not penetrate into the metal, in the following we shall not consider its effect.

Then Eqs. (2.2), (2.5), and (3.10) lead to the following ground-state charge density:

$$\rho(\vec{x}, t) \equiv \langle \rho_f \rangle = \frac{\eta^2 e}{4\pi} \int d^3y \, c(\vec{x}-\vec{y}) \int d^3z \, \frac{e^{-\mu|\vec{y}-\vec{z}|}}{|\vec{y}-\vec{z}|} \, \vec{\nabla}^2 \dot{f}(\vec{z}, t). \tag{3.14}$$

This expression can be also put in the form

$$\rho(\vec{x}, t) = -\eta^2 e \int d^3y \, c(\vec{x}-\vec{y}) \, \dot{f}(\vec{y}, t) + \frac{\mu^2 \eta^2 e}{4\pi} \int d^3y \, c(\vec{x}-\vec{y}) \int d^3z \, \frac{e^{-\mu|\vec{y}-\vec{z}|}}{|\vec{y}-\vec{z}|} \, \dot{f}(\vec{z}, t), \tag{3.15}$$

where the last term represents the modification due to the Coulomb potential. The ground-state charge current follows from Eqs. (2.3), (2.6), (3.9), and (3.12):

$$\vec{J}(\vec{x}, t) = e\eta^2 v_0^2 \left[\int d^3y \, c(\vec{x}-\vec{y}) \vec{\nabla} f(\vec{y}, t) - e \int d^3y \, c(\vec{x}-\vec{y}) \vec{A}_T(\vec{y}, t) \right] . \tag{3.16}$$

Let us note that, as $\vec{A}_T$ is a transverse vector, i.e., $\vec{\nabla} \cdot \vec{A}_T = 0$, the conservation law is easily verified, since $f(\vec{x}, t)$ satisfies Eq. (2.20):

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = e\eta^2 \int d^3y \, c(\vec{x}-\vec{y}) \left(v_0^2 \vec{\nabla}^2 f(\vec{y}, t) + \frac{1}{4\pi} \int d^3z \, \frac{e^{-\mu|\vec{y}-\vec{z}|}}{|\vec{y}-\vec{z}|} \, \vec{\nabla}^2 \ddot{f}(\vec{z}, t) \right) = 0 \, . \tag{3.17}$$

Another important result is that through the boson transformation we can obtain the expression for the order parameter, which is now space and time dependent. In fact making use of Eqs. (3.9)–(3.11) we obtain

$$\Delta(\vec{x}, t) \equiv \langle \psi_f \, , \psi_{f\,i} \rangle = e^{i\theta(\vec{x}, t)} \, |\Delta(\vec{x}, t)| \, , \tag{3.18}$$

where

$$\theta(\vec{x}, t) = 2 \int d^3y \, d(\vec{x}-\vec{y}) f(\vec{y}, t) \, , \tag{3.19}$$

$$|\Delta(\vec{x}, t)| = \langle 0 | F^2(\phi, \vec{\nabla} B_f - e \int d^3y \, \vec{A}_T(\vec{y}, t) \eta(\vec{\nabla}_y) \, d(\vec{x}-\vec{y}), \pi_f) \, | 0 \rangle \, . \tag{3.20}$$

Finally the boson transformation induces the following ground-state energy:

$$W = \frac{1}{2e^2\eta^2} \int d^3x \left(\rho^2(\vec{x}, t) + \frac{1}{v_0^2} \, \vec{J}(\vec{x}, t) \cdot \vec{J}(\vec{x}, t) \right) . \tag{3.21}$$

IV. EQUATIONS FOR THE VECTOR POTENTIAL $\vec{A}$

We shall now combine the results obtained in Sec. III B with the Maxwell equations in order to derive equations for the vector potential $\vec{A}$. The Maxwell equations can be written as[23]

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \, , \tag{4.1}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \, , \tag{4.2}$$

$$\vec{\nabla} \times \vec{H} = 4\pi \vec{J} + \frac{\partial \vec{E}}{\partial t} \, . \tag{4.3}$$

Now let us write

$$\vec{A} = \vec{A}_T + \vec{A}_L \, , \tag{4.4}$$

where $\vec{A}_L$ is the longitudinal component of $\vec{A}$, and

$$\vec{j}=\vec{j}_1+\vec{j}_2\,, \tag{4.5}$$

where

$$\vec{j}_1(\vec{x},t)=\eta^2 v_0^2\int d^3y\, c(\vec{x}-\vec{y})\,\vec{\nabla}f_1(\vec{y},t)\,, \tag{4.6}$$

$$\vec{j}_2(\vec{x},t)=\eta^2 v_0^2\int d^3y\, c(\vec{x}-\vec{y})\,\vec{\nabla}f_2(\vec{y},t)\,. \tag{4.7}$$

Recalling that f_1 and f_2 are solutions of Eqs. (2.24) and (2.25), respectively, and using (4.2), we can rewrite Eq. (4.3) separating the longitudinal and the transverse part, as

$$\frac{\partial^2}{\partial t^2}\vec{A}_L(\vec{x},t)=4\pi e\,\vec{j}_1(\vec{x},t)\,, \tag{4.8}$$

$$\left(\frac{\partial^2}{\partial t^2}+\vec{\nabla}\times\vec{\nabla}\times\right)\vec{A}_T(\vec{x},t)+4\pi e^2\eta^2 v_0^2\int d^3y\, c(\vec{x}-\vec{y})\times\vec{A}_T(\vec{y},t)=4\pi e\,\vec{j}_2(\vec{x},t)\,. \tag{4.9}$$

At first let us study Eq. (4.8). This reads as

$$\frac{\partial^2}{\partial t^2}\vec{A}_L(\vec{x},t)=4\pi e\eta^2 v_0^2\int d^3y\, c(\vec{x}-\vec{y})\,\vec{\nabla}f_1(\vec{y},t)\,. \tag{4.10}$$

On the other hand, since f_1 is a solution of Eq. (2.24), the following relation is true:

$$v_0^2\vec{\nabla}f_1(\vec{x},t)+\frac{1}{4\pi}\int d^3y\,\frac{e^{-\mu|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\,\vec{\nabla}\ddot{f}_1(\vec{y},t)=0\,. \tag{4.11}$$

Substituting (4.11) in (4.10) we find

$$\frac{\partial^2}{\partial t^2}\vec{A}_L(\vec{x},t)=-e\eta^2\int d^3y\, c(\vec{x}-\vec{y})\int d^3z\,\frac{e^{-\mu|\vec{y}-\vec{z}|}}{|\vec{y}-\vec{z}|}\,\vec{\nabla}\ddot{f}_1(\vec{z},t)\,. \tag{4.12}$$

Now the solution of this last equation is obtained by adding a particular solution to the general solution of the associated homogeneous equation

$$\ddot{\vec{A}}_L=0\,. \tag{4.13}$$

Therefore we find that

$$\vec{A}_L(\vec{x},t)=-e\eta^2\int d^3y\, c(\vec{x}-\vec{y})\int d^3z\,\frac{e^{-\mu|\vec{y}-\vec{z}|}}{|\vec{y}-\vec{z}|}\times\vec{\nabla}f_1(\vec{z},t)+\vec{\alpha}(\vec{x})\,t+\vec{\beta}(\vec{x})\,, \tag{4.14}$$

where $\vec{\alpha}$ and $\vec{\beta}$ are unknown functions of $\vec{x}$ only. The solution (4.14) gives

$$\vec{\nabla}\cdot\vec{E}=\vec{\nabla}\cdot\left(-\frac{\partial\vec{A}}{\partial t}\right)=4\pi\rho+\vec{\nabla}\cdot\vec{\alpha}\,,$$

where use was made of Eqs. (4.2) and (3.14). By comparing this result with the Maxwell equation (4.1) we find that $\vec{\nabla}\cdot\vec{\alpha}=0$. This means that $\vec{\alpha}$ is a transverse vector, though $\vec{A}_L$ is longitudinal; therefore $\vec{\alpha}=0$. The other unknown function $\vec{\beta}(\vec{x})$ can be put equal to zero by choosing a special gauge.

Let us now consider Eq. (4.9). The general solution of this equation is given by

$$\vec{A}_T=\vec{A}_T^{(1)}+\vec{A}_T^{(2)}\,, \tag{4.15}$$

where $\vec{A}_T^{(1)}$ is the general solution of the associated homogeneous equation and $\vec{A}_T^{(2)}$ is a particular solution:

$$\left(\frac{\partial^2}{\partial t^2}+\vec{\nabla}\times\vec{\nabla}\times\right)\vec{A}_T^{(1)}(\vec{x},t)+\frac{1}{\lambda_L^2}\int d^3y\, c(\vec{x}-\vec{y})\,\vec{A}_T^{(1)}(\vec{y},t)=0\,, \tag{4.16}$$

$$\left(\frac{\partial^2}{\partial t^2}+\vec{\nabla}\times\vec{\nabla}\times\right)\vec{A}_T^{(2)}(\vec{x},t)+\frac{1}{\lambda_L^2}\int d^3y\, c(\vec{x}-\vec{y})\,\vec{A}_T^{(2)}(\vec{y},t)=4\pi e\,\vec{j}_2(\vec{x},t)\,. \tag{4.17}$$

Here λ_L is the London penetration depth:

$$\frac{1}{\lambda_L^2}=4\pi e^2\eta^2 v_0^2\simeq\frac{4\pi n_s e^2}{m}\,. \tag{4.18}$$

Equation (4.16) shows that $\vec{A}_T^{(1)}$ is induced only by external fields, and is nothing else than a generalized London equation in which the displacement current $\partial\vec{E}/\partial t$, and the nonlocal relation between the persistent current and the vector potential are taken into account. To solve Eq. (4.17) we proceed in an analogous way as in Ref. 7 by using the following expansion:

$$\int d^3y\, c(\vec{x}-\vec{y})\,\vec{A}_T^{(2)}(\vec{y},t)=[1-G\,\vec{\nabla}\times\vec{\nabla}\times]\,\vec{A}_T^{(2)}(\vec{x},t)+O(\xi^4/\lambda_L^4)\,, \tag{4.19}$$

where G is a constant length of the order of ξ^2, defined by

$$G=\tfrac{2}{3}\,\pi\int_0^\infty r^4 c(r)\,dr\,. \tag{4.20}$$

Substituting the expansion (4.19) into the Eq. (4.17) we find that

$$\vec{A}_T^{(2)}(\vec{x},t)=4\pi e\int d^4y\, D(x-y)\,\vec{j}_2(y)\,, \tag{4.21}$$

where $D(x-y)$ is the Green's function, solution of the equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\lambda_G^2}{\lambda_L^2}\nabla^2+\frac{1}{\lambda_L^2}\right)D(x-y)=\delta^{(4)}(x-y) \tag{4.22}$$

and

$$\lambda_G^2 = \lambda_L^2 - G \ . \tag{4.23}$$

At this point it will be useful to summarize the results we have obtained. In the case of a charged superconductor the boson transformation induces the following charge density:

$$\rho(\vec{x}, t) = \frac{e\eta^2}{4\pi} \int d^3y\, c(\vec{x}-\vec{y}) \int d^3z \, \frac{1}{|\vec{y}-\vec{z}|} \times e^{-\mu|\vec{y}-\vec{z}|} \vec{\nabla}^2 \dot{f}_1(\vec{z}, t) \tag{4.24}$$

and the following charge current:

$$\vec{J}(\vec{x}, t) = e\,\vec{j}_1(\vec{x}, t) + \vec{J}_T(\vec{x}, t) \ , \tag{4.25}$$

where $\vec{J}_T$, the transversal component of the current, is given by

$$\vec{J}_T(\vec{x}, t) = 4\pi e \int d^4y\, F(x-y)\, \vec{j}_2(y) \ , \tag{4.26}$$

$$F(x-y) = \delta^{(4)}(x-y) - \frac{1}{\lambda_L^2} (G\,\nabla^2 + 1)\, D(x-y) \ . \tag{4.27}$$

The vector potential is given by $\vec{A} = \vec{A}_T + \vec{A}_L$, where $\vec{A}_T$ is given by Eq. (4.21) and

$$\vec{A}_L(\vec{x}, t) = -e\eta^2 \int d^3y\, c(\vec{x}-\vec{y}) \int d^3z \, \frac{1}{|\vec{y}-\vec{z}|} \times e^{-\mu|\vec{y}-\vec{z}|} \vec{\nabla} f_1(\vec{z}, t) \ . \tag{4.28}$$

The electric and magnetic fields can be obtained by the usual relations $\vec{E} = -\partial \vec{A}/\partial t$, $\vec{H} = \vec{\nabla} \times \vec{A}$. It should be noted that in Eq. (4.25) we have not taken into account the Meissner current, since it does not penetrate into the metal. Also, Eqs. (4.21) and (4.26) are valid in the approximation in which we neglect effects of order $(\xi/\lambda_L)^4$.

V. ANALOG OF THE ac JOSEPHSON EFFECT IN SUPERFLUID LIQUIDS

Owing to the remarkable similarity between the properties of the superconducting state of a metal and the superfluid phase of liquid helium, it is generally believed that an analog of the Josephson effect should be observable in superfluid helium.[24] Unfortunately at the present time it is not known experimentally how to realize a Josephson junction for superfluid helium, so that many effects (for example, the diffraction pattern of the net current crossing the junction) which are observed in superconductivity cannot be seen in superfluid helium. However, to observe quantum interference effects one does not need a Josephson junction but only a weak link between the two superconductors: The connection can be also realized by a point contact. Obviously such a weak link also can be obtained for superfluid helium and one expects to observe effects analogous to the ac Josephson effect in superconductors. Anderson and Dayem[18] performed an experiment in which the tunnel junction is replaced by a thin-film superconducting bridge which separates the two superconductors. They observed steps in the I-V characteristic at dc voltages given by $m2eV_0 = n\omega_1$. The exact analog of the Anderson-Dayem experiment for a superconductor was made by Richards and Anderson[25] in superfluid helium; more recently similar experiments with several refinements and clearer results have been repeated by Khorana and Chandrasekhar[26] and by Richards.[27]

Two baths of superfluid helium at the same temperature are weakly coupled through a small orifice (diameter $\simeq 10^{-3}$ cm); near the orifice in the bath is placed a quartz crystal oscillator. A difference d in the helium head will produce a difference $\Delta\mu$ in the chemical potential: This is the equivalent of the difference in the chemical potential induced by the dc voltage V_0 in superconductivity. The equivalent of the alternating voltage is played by the quartz crystal, which acts as an ultrasonic transducer producing sound waves.

In this section we shall apply the formulation developed for the case of a neutral superconductor to the study of the system described above. By neutral superconductor we mean a hypothetical system (of the BCS type) of uncharged Fermi particles with short-range interaction. The properties of such a system are closely related to those of superfluid helium,[28] and we expect that the boson formulation of a neutral superconductor will give a good description of the properties of superfluid helium.

A. Oscillator Turned Off

Let us suppose that the transducer is turned off and a head difference d is created between the two baths. In these conditions the system can be schematized as two superfluids connected together by a horizontal junction of negligible thickness, at whose edges there is a potential difference given by the the pressure difference

$$\Delta p = \rho g d \ . \tag{5.1}$$

Here ρ is the density and g is the acceleration due to gravity; this pressure difference creates a difference between the chemical potential of the two systems, given by

$$\Delta\mu = mgd \ , \tag{5.2}$$

where m is the mass of the helium atom. The experiments show that once the head difference is created there is a constant flow through the orifice which tends to equalize the levels.

For a superfluid system our results of Sec. III show that the boson transformation induces a ground-state density

$$\rho(\vec{x}, t) = -\eta^2 \int d^3y\, c(\vec{x}-\vec{y}) \dot{f}(\vec{y}, t) \tag{5.3}$$

and a ground-state current

$$\vec{j}(\vec{x}, t) = \eta^2 v_0^2 \int d^3y\, c(\vec{x}-\vec{y}) \vec{\nabla} f(\vec{y}, t) \,. \tag{5.4}$$

Here f, the phase of the condensate wave function, satisfies the equation

$$\left(\frac{\partial^2}{\partial t^2} - v_0^2 \nabla^2\right) f(\vec{x}, t) = 0 \,. \tag{5.5}$$

In order to determine the observable quantities of the system, we must solve Eq. (5.5) with the boundary conditions appropriate to the problem under consideration. Choosing the origin of the system of Cartesian coordinates at the center of the barrier and the z axis in the direction of the gravitational field, we shall look for a solution of Eq. (5.5) satisfying the following boundary conditions: (i) f is a function only of z and t and is denoted by $f(z, t)$. (ii) The density and the current are finite at any point z, t. Defining

$$\phi(t) = f(0^+, t) - f(0^-, t) \tag{5.6}$$

we shall further require that (iii) $\phi(t)$ is periodic in time with modulus 2π; i.e.,

$$\phi(t+T) = \phi(t) + 2\pi l \quad (l = \text{integer}) \,; \tag{5.7}$$

(iv) $\phi(t)$ is controlled by the difference in the chemical potential

$$\frac{\partial \phi}{\partial t} = \Delta\mu \,. \tag{5.8}$$

The general solution of Eq. (5.5) satisfying the boundary conditions (i)–(iv) is given by

$$f(z, t) = A^{(i)} t + B^{(i)} z + D + \sum_{ln} F_{ln}^{(i)} \sin\left(\frac{\omega_{ln}}{v_0} z + \gamma_n^{(i)}\right) \sin(\omega_{ln} t) \,, \tag{5.9}$$

where $i = 1$ for $z > 0$ and $i = 2$ for $z < 0$ and

$$\omega_{ln} = (n/l)\, mgd \,, \tag{5.10}$$

$$A^{(1)} - A^{(2)} = mgd \,. \tag{5.11}$$

In writing the solution (5.9) we have required also that $\phi(0) = 0$ and that the current be an odd function of t. The constants $B^{(i)}$, D, $F_{ln}^{(i)}$, and $\gamma_n^{(i)}$ $(i = 1, 2)$ should be fixed by requiring more restrictive boundary conditions. Substituting the solution (5.9) in Eq. (5.4) we find the following expression for the current:

$$\vec{j}(z, t) = \vec{j}_1(z, t) + \vec{j}_2(z, t) \,, \tag{5.12}$$

$$\vec{j}_1(z, t) = \hat{z} \sum_{l,n} j_{ln}(z) \sin\omega_{ln} t \,, \tag{5.13}$$

$$\vec{j}_2(z, t) = \eta^2 v_0^2 \hat{z} \left[B^{(1)} \int_{-\infty}^{z} \bar{c}(r)\, dr + B^{(2)} \int_{z}^{\infty} \bar{c}(r)\, dr\right] , \tag{5.14}$$

where $\hat{z}$ is the unit vector in the z direction and where $\bar{c}$ is defined by

$$\bar{c}(z - z') = \int_{-\infty}^{+\infty} dx' \int_{-\infty}^{+\infty} dy'\, c(\vec{r} - \vec{r}') \tag{5.15}$$

and

$$j_{ln}(z) = \eta^2 v_0^2 \omega_{ln} \left[F_{ln}^{(1)} \int_0^{\infty} \bar{c}(z - z') \cos\left(\frac{\omega_{ln}}{v_0} z' + \gamma_n^{(1)}\right) dz' + F_{ln}^{(2)} \int_{-\infty}^{0} \bar{c}(z - z') \cos\left(\frac{\omega_{ln}}{v_0} z' + \gamma_n^{(2)}\right) dz'\right] . \tag{5.16}$$

The expression (5.13) represents an oscillating flow with a spectrum of frequencies given by

$$\nu_{ln} = \frac{n}{l} \frac{mgd}{2\pi} \,. \tag{5.17}$$

The net flow crossing the junction is given by

$$\langle \vec{j} \rangle = \frac{1}{T} \int_0^T \vec{j}(0, t)\, dt = \vec{j}_2(0) = \hat{z}\, \eta^2 v_0^2 \frac{B^{(1)} + B^{(2)}}{1} \,. \tag{5.18}$$

The expression (5.18) agrees with the experimental result of a constant flow through the orifice which tends to equalize the levels. Let us note that in deriving this result we have neglected the dependence of the head difference on time. This dependence is negligible; it is appreciable only for very small d ($d < 1$ mm); if such dependence is taken into account we find that for very small d the flow is no longer constant, but linear in time; this result agrees with the experimental result (see Fig. 3 reported by Khorana[26]).

The density can be computed in an analogous way by substituting the solution (5.9) in the expression (5.3).

B. Oscillator Turned On

When the transducer is turned on the quartz crystal vibrates with a frequency ν_1; this effect creates at the junction a difference in the chemical potential given by $\Delta\mu = (V_1/2\pi\nu_1)\cos(2\pi\nu_1 + \theta_1)$. The experiments show that under these conditions the system will present dynamical stability at values of d which satisfy the relation

$$mgd \doteq (n_1/n_2)\, 2\pi\nu_1 \,, \tag{5.19}$$

where n_1 and n_2 are integers. In the presence of this external field Eq. (5.5) should be modified and $f(\vec{x}, t)$ will satisfy an inhomogeneous equation, where the external effect acts as a source. We then solve Eq. (5.5) under the boundary conditions given in Sec. V A, where now the boundary condition (iv) is modified by requiring that

$$\frac{\partial \phi}{\partial t} = mgd + \frac{V_1}{2\pi\nu_1}\cos(2\pi\nu_1 t + \theta_1) \ . \quad (5.20)$$

The general solution of Eq. (5. 5) satisfying the boundary conditions (i)–(iii) is given by

$$f(z, t) = A^{(i)} t + \sum_n F_n^{(i)} \sin\left(\frac{2\pi n}{v_0 T} z + \gamma_n^{(i)}\right) \times \sin\left(\frac{2\pi n}{T} t + \beta_n^{(i)}\right) \quad (5.21)$$

for a given integer l, with

$$A^{(1)} - A^{(2)} = 2\pi l/T \ . \quad (5.22)$$

In Eq. (5. 21) we have disregarded a constant term, which does not have any physical relevance. Also we have not considered terms linear in z; as we have seen in Sec. V A these terms produce a constant flow of current in the z direction. Since we are principally interested in stationary states we do not take these terms into account; however, these linear terms will be responsible for the transition between one resonance state to the next one.

The boundary condition (5. 20) requires that

$$A^{(1)} - A^{(2)} = mgd \ , \quad (5.23)$$

$$\nu_1 = p/T \quad (p = \text{integer}) \ , \quad (5.24)$$

$$F_n^{(1)} \sin\gamma_n^{(1)} - F_n^{(2)} \sin\gamma_n^{(2)} = V_1 \delta_{np} \ . \quad (5.25)$$

By comparing Eqs. (5. 22)–(5. 24) we see that the values of d are controlled by the frequency of the oscillator through the relation

$$mgd = (l/p)\, 2\pi\nu_1 \ . \quad (5.26)$$

Summarizing, we thus find that the system will present stationary states in resonance with the external field when and only when Eq. (5. 26) is satisfied.

Let us close this section by remarking that we have taken a very simple solution of Eq. (5. 5). One could be more sophisticated by considering more complicated solutions. In particular one solution that corresponds to an interesting physical picture is the following. As we have already mentioned we can generate from the Laplace equation $\nabla^2 f = 0$ a solution that describes a static array of vortices; from this we can easily obtain[29] a solution of Eq. (5. 5) which corresponds to the case in which an array of quantized vortices is crossing the junction with a constant velocity. When the vortex crossing is synchronized with the external effect we have a stable situation; this synchronization will be present at harmonic and subharmonic frequencies. The physical picture that corresponds to this solution has been previously presented by other authors[27,30,31] in order to explain the experimental results obtained in liquid helium.

VI. CONCLUSIONS

The formulation developed in this paper shows the powerful method of investigation that is contained in the boson transformations.[5] The motivation of this approach is based on the fact that a system is described by certain equations of motion, and all the possible phenomena correspond to solutions of the same equations of motion, solved under different boundary conditions. From this it follows that all the possible states of a system must be connected among themselves by transformations which leave invariant the equations of motion but may modify the expectation value of some of the observables. In other words each of these invariant transformations corresponds to a different realization of the original equations of motion.

When we apply the boson method to superconductivity, first we solve the equations of motion in the case in which the order parameter is constant; once that the problem is solved for this situation we look for invariant transformations (boson transformations) which permit us to consider different situations in which the order parameter can vary. In this way, starting from the original microscopic equations, we have constructed a formulation of the theory of superconductivity in which the persistent currents are space and time dependent.

The convenience of this approach resides essentially in two points. The first point is that the boson method gives a unified formulation of superconductivity which describes many different phenomena; the second point is that the study of these phenomena is made in principle very simple since each of them corresponds to a particular invariant transformation. In other words we do not apply the boundary conditions to the original equations of motion, but to the choice of the invariant transformation.

The practical advantages of this formulation are presented in the last part of the present paper and in Ref. 13. Starting from the original microscopic field equations and without introducing any phenomenological assumption we succeeded in constructing a general theory which describes most of the experimental phenomena that are observed when two superfluid systems are weakly coupled together. Our results show that the Josephson formulation may be valid only in some limiting cases, but in a rigorous sense it does not seem to give a general and consistent description of the physical processes.

In conclusion we would like to remark that, as we have already mentioned, there exist in the literature many attempts to generalize the GL equations in order to describe situations in which the order parameter is varying in the time. If these approaches correspond to a basic theory of time-dependent phenomena in superconductivity they should be able to describe the ac Josephson effect without making use of any phenomenological assumption.

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[20]Let us note that B and all the other field operators π, ϕ, ψ are different operators in the case of a neutral or of a charged superconductor; however, for simplicity of notation we shall use the same symbols, and we shall point out where the results are different.

[21]As it is mentioned in Ref. 5, the support of our algebra is the three-dimensional domain $D(L)$, defined as the set of points $\vec{x}$ such that $|\vec{x}| \leq L$.

[22]It should be noted that we have not taken into account relativistic effects which induce retardation in the electron-electron potential.

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