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Josephson Current Flow in Pure Superconducting-Normal-Superconducting Junctions*

John Bardeen and Jared L. Johnson†
Department of Physics and Materials Research Laboratory,
University of Illinois, Urbana, Illinois
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It is shown that the Josephson current in a pure superconducting-normal-superconducting (SNS) sandwich with normal layer thickness large compared with the coherence distance can be calculated easily from the quasiparticle spectrum by use of Galilean invariance. The results agree with Ishii at $T=0$ °K and with an expression of Kulik for the bound-state contribution, but we do not find a larger contribution from states with E near Δ_0 as suggested by Kulik. The expected current decreases very rapidly with increase in temperature so that it would be necessary to go to very low temperatures to observe the effect.

By means of the Josephson effect, supercurrents can flow between two superconductors separated by a normal metal in a superconducting-normal-superconducting (SNS) sandwich. We consider the case where the normal metal is pure and the thickness of $2d$ is much larger than the coherence distance ξ_0, or $2d \gg \xi_0$. This problem has been treated recently at $T=0$ °K by Ishii,[1] who revised an earlier calculation of Kulik[2] so as to apply at the absolute zero. Ishii, who followed Kulik in calculating the Josephson current flow by a Green's-function method, found a surprisingly simple result. At $T=0$ °K, the current density J is a piece-wise periodic function of the phase difference ϕ. In each periodic region, J varies linearly with ϕ and there are discontinuous jumps when ϕ is an odd multiple of π, as illustrated in Fig. 1.

For $|\phi| < \pi$, he found that in the limit $2d \gg \xi_0$ the current density may be written

$$J(\phi) = n_{\text{eff}}\, e v_s(\phi) , \tag{1}$$

where

$$2mv_s(\phi) = \hbar\, \text{grad}\chi = \hbar\phi/2d , \tag{2}$$

$$n_{\text{eff}} = k_F^3/\pi^3 , \tag{3}$$

which differs only slightly from the usual expression for the electron density, $n_e = k_F^3/3\pi^2$. Thus, the current may be regarded as arising from a superfluid velocity given by the gradient of the phase.

Kulik and Ishii both used a model in which the effective masses and Fermi velocities are the same in both the superconductors and in the normal metal; the only difference is that the interaction constant and thus the pair potential vanishes in the normal region. For simplicity, they assume $\Delta = \Delta_0$ for $|z| > d$ and $\Delta = 0$ for $|z| < d$, as illustrated in Fig. 2. We adopt the same model.

The model used is an idealized one in that effects of the magnetic field produced by the currents are ignored so that the phase varies only in the direction normal to the junction. This situation would be difficult to attain in practice, for the area of the junction would have to be extremely small. Further, scattering of electrons is neglected except as required to bring about an equilibrium quasiparticle distribution. Nevertheless, it is important to understand physically the nature of the supercurrent flow for such an ideal problem without complications. More realistic geometries can be treated if desired.

We show that the supercurrent flow can be derived easily at all temperatures by Galilean invariance arguments. The current density decreases extremely rapidly with increasing temperature and soon attains the Josephson form

$$J = J_1 \sin\phi . \tag{4}$$

This form was derived by Kulik and has been assumed in a recent review of the theory of the current-voltage characteristics of SNS junctions by Waldram, Pippard, and Clarke.[3] They discuss the alternating currents that flow when a voltage appears across the junction as well as effects of magnetic fields.

Experimental data reported so far have been on sandwiches in which there is considerable impurity scattering in the normal region. Clarke[4] has reported on experiments on lead-copper-lead sandwiches in which the scattering free path is much less than the thickness of the normal layer. The theory of such junctions with use of the Ginzburg-Landau theory has been given by Baratoff, Blackburn, and Schwartz.[5]

In pure junctions, one may regard the flow as arising from a velocity displacement of the entire SNS system of electrons by v_s. If the SN and NS boundaries are coupled with the electrons, they would also move, but apparently such motion of the boundaries does not affect the current, since no boundary motion was assumed in Ishii's calculation. If one allows the boundaries to move with the electrons the calculation can be carried out very simply in the moving frame by Galilean invariance.

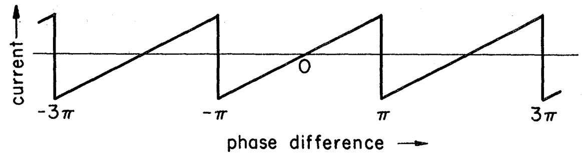

FIG. 1. Periodic variation of supercurrent density with phase difference at $T = 0$ °K.

Further, it is easy to calculate the current density at finite temperatures. We show here that the current density decreases rapidly at finite temperatures and becomes very small when $k_B T$ becomes comparable to or larger than the spacing between the bound quasiparticle states in the barrier region.

The nature of the quasiparticle states in the N region of an SNS sandwich has been discussed by Andreev,[6] McMillan,[7] Kümmel,[8] Kulik,[2] and others. A schematic diagram of the pair potential is given in Fig. 2. We are interested in states near the Fermi surface E_F. For a given k_x, k_y, parallel to the barrier, there is a k_{zF} such that

$$(\hbar^2/2m)(k_x^2 + k_y^2 + k_{zF}^2) = E_F \; . \tag{5}$$

If we take $k_z = k_{zF} + \frac{1}{2}q$, with q small, the energy relative to the Fermi energy E_F is

$$E = \hbar^2 k_{zF}\, q/2m \; . \tag{6}$$

Terms of order q^2 have been neglected.

Reflection at a pair-potential boundary changes a particle into a hole of the same quasiparticle energy. The hole state corresponds to $q \to -q$ and is below the Fermi surface. The quasiparticle eigenstates are linear combinations of particle and hole states with equal probabilities. Both particle and hole give a current in the same direction; the net quasiparticle current changes to a supercurrent at the NS boundary.

The allowed values of q are chosen so as to satisfy the boundary conditions at $z = \pm d$. If Δ were infinite, the allowed q values would be equally spaced with a separation $\delta q = \pi/d$, or

$$q = (n + \tfrac{1}{2})(\pi/d) \; . \tag{7}$$

More generally, with $\Delta = \Delta_0$ for $|z| > d$ and $\Delta = 0$ for $|z| < d$, the quasiparticle wave functions for $E < \Delta_0$ in the Nambu notation are

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} \exp[\frac{1}{2}i(qz + \eta_1)] \\ \exp[-\frac{1}{2}i(qz + \eta_1)] \end{pmatrix} e^{i\vec{k}\cdot\vec{r}} \quad \text{for } -d < z < d, \tag{8a}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} \exp[\frac{1}{2}i\eta_0^+] \\ \exp[-\frac{1}{2}i\eta_0^+] \end{pmatrix} \exp[-\alpha(z-d) + i\vec{k}\cdot\vec{r}] \quad \text{for } z > d \; , \tag{8b}$$

$$\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} \exp[\frac{1}{2}i\eta_0^-] \\ \exp[-\frac{1}{2}i\eta_0^-] \end{pmatrix} \exp[\alpha(z+d) + i\vec{k}\cdot\vec{r}] \quad \text{for } z < -d \; , \tag{8c}$$

where $k^2 = k_F^2$ in all cases. In terms of a parameter η_0, defined such that $0 < \eta_0 < \pi$, the energy is

$$E = \hbar^2 k_{zF}\, q/2m = \Delta_0 \cos\eta_0 \; . \tag{9}$$

The parameter α, which gives the exponential decay in the superconducting regions, is then

$$\alpha = (m\Delta_0/\hbar k_{zF}) \sin\eta_0 \; . \tag{10}$$

Values of η_0^+ and η_0^- may differ from η_0 and $-\eta_0$ by integral multiples of 2π. Matching solutions at the boundaries ($z = \pm d$) gives

$$\eta_0^+ = \eta_0 + 2\pi n^+ = qd + \eta_1 \; , \tag{11a}$$

$$\eta_0^- = -\eta_0 + 2\pi n^- = -qd + \eta_1 \; , \tag{11b}$$

where n^+ and n^- are integers. This implies that η_1 may be either zero or π. For $\eta_1 = 0$, we have $n^+ = n^- = n_1$ and

$$qd = \eta_0 + 2\pi n_1 \quad (\eta_1 = 0) \; . \tag{12a}$$

For $\eta_1 = \pi$, we have $n^+ = n^- + 1 = n_1$ and

$$qd = \eta_0 + \pi(2n_1 - 1) \quad (\eta_1 = \pi) \; . \tag{12b}$$

In general, $qd = \eta_0 + \pi n$. Note that for k_z positive, E is positive for q positive. A complete set of states is obtained by taking those with E positive. This requires that for k_{zF} positive, n must be a positive integer. There will be a similar set of excitations for k_z negative.

To see that we have the correct density of states, consider the case of a normal metal with periodic boundary conditions at $z = -d$ and $z = +d$, so that the period is $2d$. Again, we may consider k_z positive and k_z negative as separate states. For k_z positive, there are eigenvalues for $k_z = k_{zF} + \frac{1}{2}q$, where

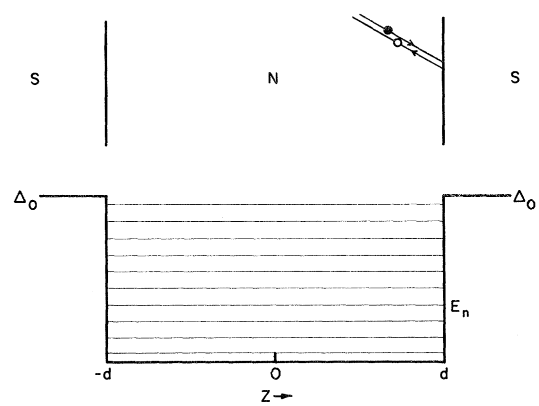

FIG. 2. Model for an SNS sandwich with abrupt pair-potential barriers at $z = -d$ and $+d$. There is an array of bound quasiparticle states in the normal region, each consisting of equal probabilities of particle and hole states. A particle is reflected into a hole at a pair-potential boundary, with no change in current.

$$qd = 2\pi n \ , \tag{13}$$

but now n may be positive or negative for states above and below the Fermi energy. The allowed states are twice as far apart as those confined by the pair-potential boundaries, but particles above and holes below the Fermi surface are counted as separate states, so that the net density of states is the same.

We are interested in the case where $d \gg \xi_0$, so that there are many levels with $E < \Delta_0$. The lowest levels are evenly spaced. When E/Δ_0 is small, η_0 is not far from $\frac{1}{2}\pi$, and we may write

$$\eta_0 = \tfrac{1}{2}\pi - \alpha_0 \ , \tag{14}$$

where α_0 is small, and

$$\cos\eta_0 = \sin\alpha_0 \simeq \alpha_0 = \hbar^2 k_{zF}\, q/(2m\Delta_0) \ . \tag{15}$$

Thus to this approximation the equation for q is

$$qd = \eta_0 + \pi n = \tfrac{1}{2}\pi - \hbar^2 k_{zF}\, q/(2m\Delta_0) + \pi n \ , \tag{16}$$

where n is an integer, or

$$qd^* = \pi(n + \tfrac{1}{2}) \ , \tag{17}$$

where

$$d^* = d + \hbar^2 k_{zF}/(2m\Delta_0) \tag{18}$$

takes into account the penetration of the quasiparticle wave functions into the superconducting regions. The quasiparticle energy is

$$E_n = \hbar^2 k_{zF}\pi(n + \tfrac{1}{2})/2md^* \tag{19}$$

Consider the case where there is a flow with velocity v_s. The quasiparticles will come into equilibrium with the lattice, which is moving at a relative velocity $-v_s$. One may take this motion into account by adding a term $v_s p_z$ to E_n, where $p_z = \hbar k_{zF}$. The probability that the state n is occupied is then $f(E_n + v_s p_z)$, where f is the Fermi function $(1 + e^{\beta E})^{-1}$.

The current density is $n_e e v_s$ plus the contribution from the quasiparticles:

$$J = n_e e v_s + \sum (e p_z/2md^*) f(E_n + v_s p_z) \ , \tag{20}$$

where the sum is over all quasiparticle states. The factor $1/2d^*$ takes into account normalization of the wave functions, including penetration into the superconducting region. Note that the quasiparticle current, representing equal contributions from particles above and holes below the Fermi surface, is independent of q and depends only on the momentum $p_z = \hbar k_{zF}$ at the Fermi surface.

At $T = 0\,^\circ$K, f will be zero until, with increasing phase difference, $v_s p_z$, with p_z negative, cancels E_n for $n = 0$, or

$$\tfrac{1}{2}\hbar p_z q_0/dm + v_s p_z = 0 \ , \tag{21}$$

giving

$$v_s = \hbar |q_0|/2m = \pi\hbar/4md^* \ . \tag{22}$$

If we ignore the small dependence of d^* on k_{zF}, this will occur at the same v_s for all k_{zF} and corresponds to a phase difference of π. The quasiparticle sum over k_x and k_y can then be carried out easily, to obtain

$$-e\sum \frac{p_{zF} f(0)}{2md^*} = -\frac{2e\hbar f(0)}{4\pi m d^*}\int_0^{k_F} (k_F^2 - k_\rho^2)^{1/2}\, k_\rho\, dk_\rho = -2f(0)\, n_e e v_s \ , \tag{23}$$

where $k_\rho^2 = k_x^2 + k_y^2$ and $n_e = k_F^3/3\pi^2$. Since $f(0) = \frac{1}{2}$, the quasiparticle current just cancels the current $n_e e v_s$:

$$J = n_e e v_s - n_e e v_s = 0 \ . \tag{24}$$

For a phase difference slightly greater than π, $f = 1$, and the current is reversed in sign:

$$J = n_e e v_s - 2n_e e v_s = -n_e e v_s \ . \tag{25}$$

As the phase difference increases beyond π, the current again will increase linearly until the next quasiparticle level ($n = 1$), corresponding to $\phi = 3\pi$, is reached, when there will be another reversal in sign. In this way we get at $T = 0\,^\circ$K the periodic structure of Ishii shown in Fig. 1.

When $\beta^{-1} = k_B T$ becomes comparable with the spacing between levels, it is necessary to sum over the partially occupied levels to obtain the current density. Rewriting the sum over states for which p_z is negative in terms of a sum over p_z positive, we have

$$J = n_e e v_s + \frac{e}{m}\sum_{p_z > 0} \frac{p_z}{2d^*}\left[f(E_n + v_s p_z) - f(E_n - v_s p_z)\right] . \tag{26}$$

The sum is over all quasiparticle states. In terms of reduced variables,

$$x = \pi\beta\hbar p_z/(2md^*) \ , \tag{27}$$

$$\gamma = v_s(4md^*/\pi\hbar) \ , \tag{28}$$

$$J = n_e e v_s + \frac{e}{\pi\beta\hbar}\sum_{p_z > 0}\left(\frac{x}{\exp\{x[n + \frac{1}{2}(1+\gamma)]\} + 1} - \frac{x}{\exp\{x[n + \frac{1}{2}(1-\gamma)]\} + 1}\right) . \tag{29}$$

If we again ignore the small dependence of d^* on k_{zF}, γ is proportional to the phase difference and is equal to unity for $\phi = \pi$:

$$\gamma = \phi/\pi \ . \tag{30}$$

If we set $\gamma = 1$, the terms in the sum over states cancel in pairs, leaving only the second term with $n = 0$. This gives just the same sum that was evaluated earlier:

$$J = n_e e v_s - \frac{e}{2\pi\beta\hbar}\sum x = 0 \ . \tag{31}$$

Thus at all temperatures, the current density is a periodic function of phase and vanishes when the phase difference is an integral multiple of π.

The sum has been evaluated numerically and analytically to give the current density as a function of the reduced variables x and γ. Let us define

$$g(x,\gamma)=\gamma+\sum_n\left(\frac{2}{\exp\{x[n+\frac{1}{2}(1+\gamma)]\}+1}-\frac{2}{\exp\{x[n+\frac{1}{2}(1-\gamma)]\}+1}\right). \qquad (32)$$

Note that $g(x,1)=0$ for all x. The current density is obtained by summing over the k_x, k_y values, which may be done by integrating over x from zero to the maximum value

$$x_F=\pi\beta\hbar p_F/(2md^*). \qquad (33)$$

It may be seen by a short calculation that

$$J=J_m\,\frac{3}{x_F^3}\int_0^{x_F}x^2g(x,\gamma)\,dx, \qquad (34)$$

where J_m is the maximum current density for $T=0$ and $\gamma=1$:

$$J_m=n_e\pi\hbar e/4md^*. \qquad (35)$$

For $n_e=10^{22}/\mathrm{cm}^3$ and $d^*=10^{-3}$ cm, $J_m\simeq10^6$ A/cm^2, a very large value.

The current density decreases very rapidly with increasing temperature, and becomes quite small when x_F is of the order of unity or less. For $x_F <\sim 5$, J is approximately of the Josephson form

$$J=J_1(x_F)\sin\pi\gamma. \qquad (36)$$

A plot of $J_1(x_F)/J_m$ as a function of x_F is given in Fig. 4. To give a typical example, take $d^*=20\xi_0$ so that

$$x_F=(\pi^2/40)\beta\Delta_0\sim 0.43\,(T_c/T) \qquad (37)$$

when $T=0.2\,T_c$, $x_F=2.15$, and $J_1/J_m\simeq10^{-3}$.

An approximate expression for $g(x,\gamma)$, valid when x is not too large, may be obtained by expanding the Fermi functions in terms of the Matsubara frequencies $\omega=(2\nu+1)\pi k_BT$. The sum over n may then be carried out. When $T\ll T_c$, the main contribution comes from the poles for $\nu=0$, $\omega=\pm\pi k_BT$. A short calculation gives

$$g(x,\gamma)=(8\pi/x)\,e^{-2\pi^2/x}\sin\pi\gamma. \qquad (38)$$

For $\gamma=\frac{1}{2}$, this limiting expression gives very good results for x as large as $2\pi^2$, where it reaches a maximum of $4/\pi e=0.458$ and then decreases for larger x. The correct expression is equal to about 0.486 for this value of x and then continues to increase to a limiting value of 0.5 as $x\to\infty$.

If we use (38), the current density is of the Josephson form (36), with J_1 calculated from (34),

$$J_1/J_m=(3/x_F^3)\int_0^{x_F}8\pi x\,e^{-2\pi^2/x}\,dx,$$

$$=24\pi\int_{y_F}^{\infty}(yx_F)^{-3}e^{-2\pi^2y}\,dy. \qquad (39)$$

If $y_F=1/x_F$ is not too small, most of the contribution to the integral will come from values of close to y_F, so that one may replace $(yx_F)^{-3}$ by unity in the integrand. In this limit, the integral is approximately

$$J_1/J_m=(12/\pi)\,e^{-2\pi^2/x_F}. \qquad (40)$$

If we insert the expression (35) for J_m, we find

$$J_1=(6n_e\hbar e/2md^*)\,e^{-4\pi d^*k_BT/\hbar v_F}$$

$$=(6n_e\hbar e/2md^*)\,e^{-4(d^*/\xi_0)(k_BT/\Delta_0)}. \qquad (41)$$

Equation (41) is identical with an expression derived by Kulik for the contribution from the bound quasiparticle states. It arises, as he says, from the quantization of the levels in the normal region. The decrease in Josephson current with increase in temperature is extremely rapid. For example, for $x_F=1$, corresponding to a spacing between the bound states equal to k_BT, J_1/J_m is about 6.7×10^{-8}.

Kulik finds another contribution from states near Δ_0 varying with temperature as $\exp(-2\pi d^*k_BT/\hbar v_F)$, which could be considerably larger than (41). However, we do not find this term and believe that a more careful analysis of the bound and scattering states is required to obtain the current when (41) is no longer valid. Great care must be taken, since the net current is a very small difference between large terms.

More generally, one would have a quasiparticle sum corresponding to (26) to evaluate the current density in the normal region, but it would be necessary to use a more accurate expression than (19) for the bound states and it would be necessary to sum over the scattering states with $E>\Delta_0$. One can estimate the range of validity of the approximations leading to (20) and (38). The correction to the sum corresponding to $g(x,\gamma)$ is at most of order $e^{-\Delta_0/k_BT}$. For (40) to be valid, this term must be small compared with the contribution to the bound states given by (38). This requires that

$$\Delta_0/k_BT>2\pi^2/x_F \qquad (42)$$

or

$$k_BT<\sim\Delta_0(\xi_0/2d^*)^{1/2}. \qquad (43)$$

Another way to state this condition is to require that

$$x_F>\pi^2(\xi_0/2d^*)^{1/2}. \qquad (44)$$

Thus for $2d^*/\xi_0$ of the order of 100, x_F would have to be of the order of unity or larger. This implies k_BT less than the order of $\frac{1}{20}\Delta_0$, or $T/T_c<0.1$.

Numerical calculations have been carried out for larger x when the approximations leading to (40)

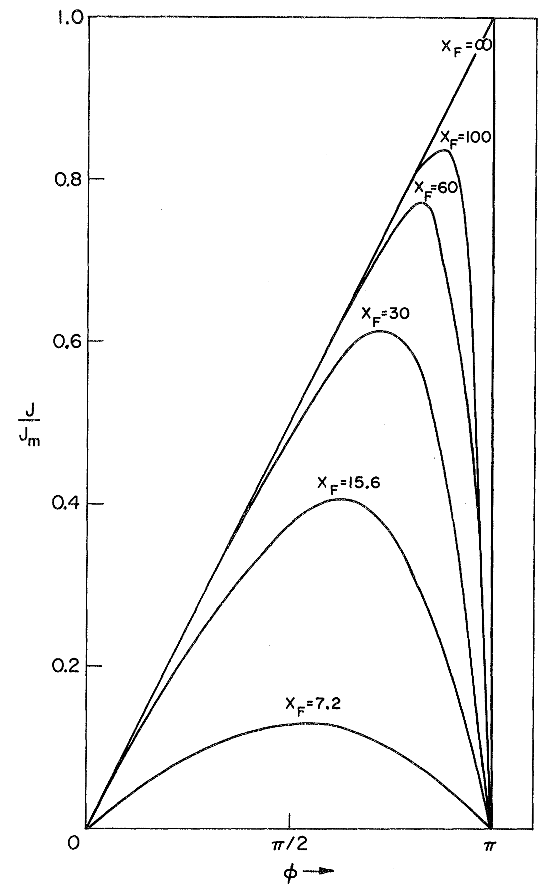

FIG. 3. Current density as a function of phase difference across an SNS sandwich. Parameter x_F is the ratio of the spacing between bound quasiparticle states and k_BT, so that $T=0\,°K$ corresponds to $x_F\to\infty$.

are no longer valid. In Fig. 3, there is a plot of J_1/J_m vs phase difference for several different values of x. The limit $T=0\,°K$, or $x=\infty$, is the linear piece-wise periodic function. Even for $x=100$ there are significant departures for phase differences near π, and for $x=7.2$ one is getting close to the sinusoidal Josephson form. In Fig. 4, we give a logarithmic plot of J/J_m for $\phi=\frac{1}{2}\pi$ ($\gamma=\frac{1}{2}$), or of J_1/J_m when the sinusoidal form is valid. The abscissa is $1/x_F$, which is proportional to T. The rapid decrease of current with increasing temperature is evident.

So far we have neglected impurity scattering. As long as the scattering is relatively small, the only effect would be to decrease the current by a factor equal to the probability that an electron go from one superconducting region to the other without being scattered. This applies to scattering by the NS boundaries as well as scattering within the N region itself.

These results indicate that to observe Josephson effects in thick SNS sandwiches ($2d\gg\xi_0$) it would be necessary to go to temperatures very small compared with T_c. The currents decrease very rapidly with increase in temperature, but decrease much more slowly with increase in thickness of the normal layer than would be expected by the proximity effect. The effect is due to quantization of the quasiparticle states in the normal region.

Note added in proof. In order to estimate the errors involved in the use of (32), we have evaluated numerically a more exact expression for $g(x,\gamma)$. The bound states contribute

$$g_b(x_1,\gamma_1)=\gamma_1+\sum_n \frac{d}{d^*}\left(\frac{2}{\exp[x_1(n+\pi^{-1}\eta_0+\frac{1}{2}\gamma_1)]}-\frac{2}{\exp[x_1(n+\pi^{-1}\eta_0-\frac{1}{2}\gamma_1)]}\right), \quad (45)$$

where x_1 and γ_1 differ from x and γ [Eqs. (27) and (28)] in that d^* is replaced by d:

$$x_1=\pi\beta\hbar p_s/2md\,,\quad \gamma_1=v_s\,4md/\pi\hbar\,, \quad (46)$$

and d^* now depends on the energy $[E(n)=\Delta_0\cos\eta_0]$ of the nth level:

$$d^*/d=1+\hbar p_s/2md\,\Delta_0\sin\eta_0\,. \quad (47)$$

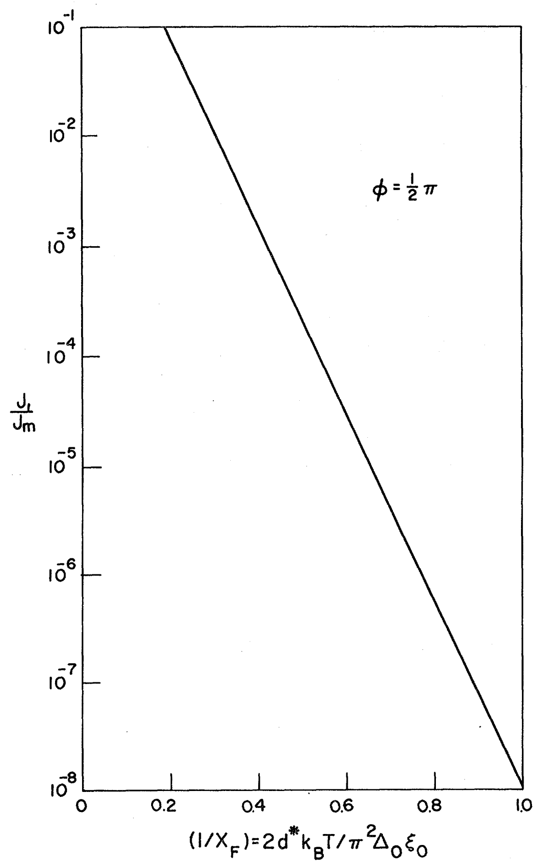

FIG. 4. Current density for a phase difference of $\frac{1}{2}\pi$ across an SNS sandwich as a function of reduced temperature.

The sum is over the discrete number of bound states.

To (45) we must add the contribution from the scattering states with $E > \Delta$. As shown in the Appendix, the scattering states have resonance energies peaked when qd is an integral multiple of π. The levels are sharp when $E = \Delta$ but gradually broaden to a uniform continuum with increasing energy. Bounds can be set by taking as limiting cases a set of discrete levels with energies

$$E(n) = \hbar p_z q/2m = \pi\hbar p_z n/2md \ , \tag{48}$$

and a uniform continuum with a density equal to that of bulk normal metal.

Numerical calculations were made for wells containing 5, 10, 20, and 40 bound levels. Even for as few as five bound levels, we find that (32) is accurate to within 5% for $x > 2$. For smaller values of x (higher temperature), the scattering states become more important and the current decreases less rapidly with increasing temperature than indicated in Fig. 4. Errors involved in the use of (32) are less than 3% for 10 levels and $x > 2$ and 5% for 20 levels and $x > 1$. No significant errors could be found for 40 levels and $x > 1$. Thus the plot shown in Fig. 4, based on (32) and (38), should be reasonably accurate for most values of the parameters.

APPENDIX : EFFECT OF SCATTERING STATES

As pointed out by Kulik,[2] scattering states with $E > \Delta_0$ can contribute to the Josephson current. We need to find the effective density of states in energy per unit volume in the normal region as compared with that in uniform normal metal. It is necessary to take account of scattering from quasiparticles to quasiholes at the step-function boundaries of Fig. 2. For $k_{zF} > 0$, one can get a complete set of states from (i) quasiparticles incident from the left-hand side, partially reflected at $z = -d$ and transmitted as quasiparticles in the superconducting region for $z > d$, and (ii) quasiholes incident from the right-hand side, partially reflected at $z = d$ and transmitted as quasiholes in the region $z < -d$.

For (i), we have in the left-hand-side region (at $z = -d$),

$$\begin{pmatrix} u \\ v \end{pmatrix} = A_1 \begin{pmatrix} e^{(1/2)\eta_2} \\ e^{-(1/2)\eta_2} \end{pmatrix} + B_1 \begin{pmatrix} e^{-(1/2)\eta_2} \\ e^{(1/2)\eta_2} \end{pmatrix} \ , \tag{A1}$$

where A_1 and B_1 are proportional to the amplitudes of the incident and reflected waves and

$$\cosh\eta_2 = E/\Delta \ . \tag{A2}$$

In the normal region ($-d < z < d$),

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} C_1 e^{(1/2)iqz} \\ D_1 e^{-(1/2)iqz} \end{pmatrix} \ . \tag{A3}$$

In the right-hand-side region (at $z = d$), we have

$$\begin{pmatrix} u \\ v \end{pmatrix} = T_1 \begin{pmatrix} e^{(1/2)\eta_2} \\ e^{-(1/2)\eta_2} \end{pmatrix} \ . \tag{A4}$$

A common factor $e^{i\vec{k}\cdot\vec{r}}$ has been surpressed in (A1)–(A4). The incident wave may be normalized such that A_1 is real and

$$A_1^2 (e^{\eta_2} + e^{-\eta_2}) = 1 \ . \tag{A5}$$

Matching solutions at $z = -d$ and $z = d$, we find that the mean-square amplitude in the normal region is

$$|C_1|^2 + |D_1|^2 = |T_1|^2 (e^{\eta_2} + e^{-\eta_2}) = \frac{(e^{\eta_2} - e^{\eta_2})^2}{e^{2\eta_2} - 2\cos 2qd + e^{-2\eta_2}} \ . \tag{A6}$$

An equivalent expression would be obtained from quasiholes incident from the right-hand side. Thus (A6) gives the ratio of the density of states in energy in the normal region as compared with that in the superconducting regions.

If we average (A6) over the phase $2qd$, we find

$$\langle |C_1|^2 + |D_1|^2 \rangle = \frac{(e^{\eta_2} - e^{-\eta_2})^2}{e^{2\eta_2} - e^{-2\eta_2}} = \tanh\eta_2 = \frac{(E^2 - \Delta^2)^{1/2}}{E} \ , \tag{A7}$$

which is just the usual ratio between bulk-normal and superconducting regions. If we divide (A6) by $\tanh\eta_2$, we find that the density in the normal region is changed by the presence of the superconducting regions by the factor

$$r = \frac{e^{2\eta_2} - e^{-2\eta_2}}{e^{2\eta_2} - 2\cos 2qd + e^{-2\eta_2}} \ , \tag{A8}$$

which is equal to unity when averaged over the phase $2qd$.

Note that r is a maximum when $2qd = 2\pi n$ (n, an integer) and a minimum when $2qd = \pi(2n+1)$. The maximum

$$r_{\max} = \frac{e^{\eta_2} + e^{-\eta_2}}{e^{\eta_2} - e^{-\eta_2}} = \frac{E}{(E^2 - \Delta^2)^{1/2}} \tag{A9}$$

corresponds to a resonant scattering condition.[7] The resonance is sharp when $E = \Delta$ and is reasonably narrow when E is close to Δ. The spacing between the resonance levels is approximately the same as that between the bound states. Thus, the bound states for $E < \Delta$ go into a series of resonance levels for $E > \Delta$, which gradually broaden into a uniform continuum.

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