Resistive Transition in a Superconducting Filament*

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The contribution of thermal fluctuations to the conductivity of a narrow superconducting filament is examined. A Fokker-Planck equation based on time-dependent Ginzburg-Landau theory is used to describe the system. A variational solution of this equation yields an upper bound for the resistivity as a function of temperature in the limit of small electric field. The result is accurate above T_c and appears to be a better estimate of the resistivity at and just below T_c than that given by previous calculations.

I. INTRODUCTION

Much interest has recently been shown in describing the transition between normal and superconducting states for narrow filaments. These filaments of superconducting material can be regarded as one dimensional if the cross-sectional dimensions are much less than $\xi(T)$, the Ginzburg-Landau coherence length. Thermal fluctuations of the order parameter are primarily responsible for the resistive properties of the sample in such transitions.

Most of the earlier calculations describing these resistive properties have been reviewed in the recent paper by Tucker and Halperin[1] (TH). For temperatures below T_c the problem was first treated by Langer and Ambegaokar[2] (LA). Their theory was improved by McCumber and Halperin[3] (MH). Above T_c, Aslamazov and Larkin[4] first determined the additional conductivity due to fluctuating pairs. A phenomonological version of their theory (closely related to the point of view we shall take in this paper) was presented by Abrahams and Woo.[5] This solution was restricted to temperatures well above T_c in the limit of small electric field. Schmid and Hurault[6] succeeded in generalizing this result to the case of finite fields by employing a linearized time-dependent Ginzburg-Landau (TDGL) theory. Masker, Marčelja, and Parks[7] (MMP) attempted to extend the theory throughout the entire transition region in the limit of small field by including the fourth-order term in a TDGL equation in a self-consistent Hartree approximation. Below T_c, their results disagree with LA-MH, and experimental evidence[8] appears to favor the latter. Tucker and Halperin have subsequently treated the fourth-order term, representing interaction between fluctuations, in a self-consistent Hartree-Fock approximation. The TH calculations extend the high-temperature resistivity down closer to T_c for finite fields. Their results disagree with MMP, particularly for clean samples (long mean free path).

The present paper is an attempt to extend the linearized TDGL results to lower temperatures in the limit of small field by a variational method. We make no special Hartree-like approximations in treating the fourth-order term in the free energy, but we determine only the linear response of the system to an applied field. In Sec. II we formulate the problem starting from a TDGL equation and develop a variational expression for the conductivity in terms of functional integrals. In Sec. III we evaluate this expression using a simple but nontrivial trial function. In Sec. IV we discuss the results and compare them with previous work.

We shall assume the validity of the (nonlinear) TDGL theory for calculating the resistivity near T_c. Questions involving this assumption are beyond the scope of this paper.

II. BASIC FORMULATION

The standard Ginzburg-Landau expressions for the free energy and supercurrent density are

$$F(\{\psi\},T)=\int d^3r\left[a\left|\psi(\vec{\mathbf{r}})\right|^2+\tfrac{1}{2}b\left|\psi(\vec{\mathbf{r}})\right|^4+\delta\left|\left(\frac{1}{i}\vec{\nabla}+\frac{2e}{\hbar c}\vec{\mathbf{A}}\right)\psi(\vec{\mathbf{r}})\right|^2\right], \quad (1)$$

$$\vec{\mathbf{j}}_s(\vec{\mathbf{r}})=-\frac{2e}{\hbar}\delta\left[\psi^*(\vec{\mathbf{r}})\left(\frac{1}{i}\vec{\nabla}+\frac{2e}{\hbar c}\vec{\mathbf{A}}\right)\psi(\vec{\mathbf{r}})+\text{c.c.}\right]. \quad (2)$$

Following the notation of Tucker and Halperin,

$$a=a_0[(T-T_c)/T_c], \qquad \delta=\hbar^2/2m,$$

and a_0 and b are constants independent of temperature.

As derived from microscopic theory,[9] the TDGL theory predicts that for small slow variations of $\{\psi\}$, the time dependence of the order parameter will be described by

$$\hbar\gamma\left(\frac{\partial}{\partial t}-\frac{2ie}{\hbar}V\right)\psi(t)$$

$$= -\left[a + b|\psi(t)|^2 + \delta\left(\frac{1}{i}\vec{\nabla} + \frac{2e}{\hbar c}\vec{A}\right)^2\right]\psi(\vec{r},t), \quad (3)$$

where $V = V(\vec{r},t)$ represents the electrochemical potential and

$$\gamma = \pi A_0/8k_B T_c .$$

Assuming $V = 0$, we have

$$\hbar\gamma \frac{\partial\psi(\vec{r},t)}{\partial t} = -\frac{\delta F(\psi,T)}{\delta\psi^*(\vec{r},t)} . \quad (4)$$

This TDGL equation describes the decay of the average order parameter towards its equilibrium value. The recent fluctuation theories, e.g., LA, MH, and TH, are based on the assumption that (4) can be generalized to a Fokker-Planck equation governing the behavior of $\rho\{\psi\}$, the probability distribution in the space of functions $\psi(\vec{r})$. The Fokker-Planck equation is conventionally written in the form of a continuity equation:

$$\frac{\partial\rho}{\partial t}\{\psi\} = -\sum_q \left(\frac{\partial J_q}{\partial\psi_q} + \frac{\partial J_q^*}{\partial\psi_q^*}\right) \equiv \mathfrak{L}\rho\{\psi\} . \quad (5)$$

The components of the probability current J are

$$J_q = -\Gamma\left(\beta\frac{\partial F\{\psi\}}{\partial\psi_q^*}\rho\{\psi\} + \frac{\partial\rho\{\psi\}}{\partial\psi_q^*}\right) , \quad (6)$$

where ψ_q is the qth Fourier coefficient of ψ, and

$$\psi(\vec{r},t) = \frac{1}{\Omega^{1/2}}\sum_q \psi_q(t)\, e^{i\vec{q}\cdot\vec{r}} . \quad (7)$$

In (7), Ω is the volume of the system, and in (6),

$$\Gamma = 1/\beta\hbar\gamma \quad \text{and} \quad \beta = 1/k_B T .$$

Equation (5) defines the operation $\mathfrak{L}$. The distribution ρ is normalized to unity:

$$\int \delta\psi\,\rho\{\psi\} = 1 ,$$

where the symbol $\int\delta\psi$ denotes integration over the space of complex functions $\psi(\vec{r})$.

Equations (5) and (6) may be justified most easily by noting that J_q has been chosen to describe forced diffusion in the potential F. To recover (4), one can take the first moment of Eq. (5), i.e., multiply (5) by $\psi(\vec{r})$ and integrate over $\psi(\vec{r})$. The resulting expression reduces to (4) in the case that $\rho\{\psi\}$ is sharply peaked in ψ space. More systematically, the Fokker-Planck equation (5) may be derived from the TDGL theory by adding a Langevin force to the right-hand side of Eq. (4). This force is a model for the heat bath (e.g., phonons in the metal), which drives the thermal fluctuations of the order parameter. This point of view has been discussed in some detail by MH.

The equilibrium solution of (6) is simply

$$\rho_0\{\psi\} = (1/Z_0)\, e^{-\beta F\{\psi\}} ,$$

where the partition function Z_0 is defined as

$$Z_0 = \int \delta\psi\, e^{-\beta F\{\psi\}} .$$

In a "steady-state" or constant-current solution of (6), the current will be carried by modes which propagate with velocities

$$\vec{v}_q = (\hbar/m)\left[\vec{q} + (2e/\hbar c)\vec{A}\right] .$$

The current density $\vec{j}_s$ from (2) will be given by

$$\vec{j}_s = (2e/\Omega)\sum_q \vec{v}_q |\psi_q|^2 . \quad (8)$$

With our choice of gauge, $\vec{A} = c\vec{E}t$ depends upon time. In the steady state the only explicit time dependence of $\rho\{\psi\}$ will occur via the quantities $\vec{v}_q$: Thus we can write

$$\mathfrak{L}\rho = \frac{\partial\rho}{\partial t} = \sum_q \frac{\partial\rho}{\partial\vec{v}_q}\cdot\frac{\partial\vec{v}_q}{\partial t} = \frac{e\vec{E}}{m}\cdot\left(\sum_q \frac{\partial\rho}{\partial\vec{v}_q}\right) .$$

In the limit of small electric field, we write $\rho = \rho_0 + \rho_1 + \cdots$, where ρ_1 represents the first-order correction to the equilibrium density. Since $\mathfrak{L}\rho_0 = 0$, we have, to first order in $\vec{E}$,

$$\mathfrak{L}\rho_1 = \frac{e\vec{E}}{m}\cdot\left(\sum_q \frac{\partial\rho_0}{\partial\vec{v}_q}\right) = -\frac{\beta e^{-\beta F}}{Z_0}\frac{\Omega}{2}(\vec{E}\cdot\vec{j}_s) . \quad (9)$$

Equation (9) is our basic time-independent transport equation. (The explicit time dependence remaining in $\vec{v}_q$ may be transformed away by a simple translation in q space, leaving $\vec{v}_q = \hbar\vec{q}/m$.) It remains to solve (9) for ρ_1 and then to evaluate the conductivity

$$\sigma_s = |\langle\vec{j}_s\rangle|/|\vec{E}| = (1/|\vec{E}|)\left|\int \delta\psi\rho_1\vec{j}_s\right| . \quad (10)$$

To accomplish this, we employ the Kohler variational method,[10] which we reformulate for our purpose as follows.

Choosing two real functions $f(\psi)$ and $g(\psi)$, we define the inner product

$$(g,f) \equiv \int \delta\psi\, e^{+\beta F\{\psi\}} g\{\psi\} f\{\psi\} .$$

The product $(g, \mathfrak{L}f)$ will then be given by

$$(g,\mathfrak{L}f) = +\Gamma\sum_q \int \delta\psi e^{+\beta F} g\{\psi\} \times\left[\frac{\partial}{\partial\psi_q}\left(\beta\frac{\partial F}{\partial\psi_q^*}f + \frac{\partial f}{\partial\psi_q^*}\right) + \text{c.c.}\right] .$$

Since functional integration is equivalent to integration over all ψ_q, we can integrate by parts to obtain

$$(g,\mathfrak{L}f) = -\Gamma\sum_q \int \delta\psi\, e^{-\beta F} \times\left(\frac{\partial}{\partial\psi_a}(e^{\beta F} g\{\psi\})\frac{\partial}{\partial\psi_q^*}(e^{\beta F} f\{\psi\}) + \text{c.c.}\right) , \quad (11)$$

a result symmetric in f and g. Also, we have

$$(f,\mathfrak{L}f) = -2\Gamma\sum_q \int \delta\psi\, e^{-\beta F}\left|\frac{\partial}{\partial\psi_q}(e^{\beta F} f\{\psi\})\right|^2 \leq 0 . \quad (12)$$

From (9) we can write

$$(\rho_1, \mathcal{L}\rho_1) = \int \delta\psi\, e^{+\beta F}\rho_1, \mathcal{L}\rho_1 = -\frac{\beta\Omega}{2Z_0}\vec{E}\cdot\left(\int\psi\rho_1\vec{j}_s\right)$$

$$= -\frac{\beta\Omega}{2Z_0}\vec{E}\cdot\langle\vec{j}_s\rangle . \tag{13}$$

If we can evaluate $(\rho_1, \mathcal{L}\rho_1)$, we can therefore determine σ_s.

To estimate ρ_1, we choose a trial function $\tau\{\psi\}$ satisfying

$$\left(\tau, \mathcal{L}\tau + \frac{e^{-\beta F}}{Z_0}\frac{\beta\Omega}{2}(\vec{E}\cdot\vec{j}_s)\right) = 0 . \tag{14}$$

From (13), we have

$$\left(\tau, \mathcal{L}\rho_1 + \frac{e^{-\beta F}}{Z_0}\frac{\beta\Omega}{2}\vec{E}\cdot\vec{j}_s\right) = 0 ,$$

so that $(\tau, \mathcal{L}\tau) = (\rho_1, \mathcal{L}\tau) = (\tau, \mathcal{L}\rho_1)$. Combining this with (12), we have

$$((\rho_1 - \tau), \mathcal{L}(\rho_1 - \tau)) = (\rho_1, \mathcal{L}\rho_1) - (\tau, \mathcal{L}\tau) \leq 0 .$$

We conclude that

$$-(\rho_1, \mathcal{L}\rho_1) \geq -(\tau, \mathcal{L}\tau),$$

therefore, we must choose τ to maximize $-(\tau, \mathcal{L}\tau)$, while satisfying the constraint imposed by (14).

III. EVALUATION OF CONDUCTIVITY

We choose for our trial function τ the simple form

$$\tau = (e^{-\beta F}/Z_0)(\textstyle\sum_q C_q|\psi_q|^2) . \tag{15}$$

From (12), we have

$$(\tau, \mathcal{L}\tau) = -\frac{2\Gamma}{Z_0^2}\int \delta\psi\, e^{-\beta F}\sum_q C_q^2|\psi_q|^2 \equiv P .$$

Also from (8),

$$\frac{\beta\Omega}{2Z_0}\vec{E}\cdot(\tau, e^{-\beta F}\vec{j}_s)$$

$$= \frac{\beta\hbar}{mZ_0^2}e\vec{E}\cdot\sum_{q,q'}\vec{q}'\int \delta\psi\, e^{-\beta F}\, C_q|\psi_q|^2|\psi_{q'}|^2 \equiv M .$$

Defining

$$\frac{1}{Z_0}\int \delta\psi\, e^{-\beta F}|\psi_q|^2 \equiv \langle n_q\rangle ,$$

$$\frac{1}{Z_0}\int \delta\psi\, e^{-\beta F}|\psi_q|^2|\psi_{q'}|^2 \equiv \langle n_q n_{q'}\rangle ,$$

and

$$\vec{M}_0 \equiv \frac{\beta\hbar}{m}e\vec{E} ,$$

we have

$$P = -\frac{2\Gamma}{Z_0}\sum_q C_q^2\langle n_q\rangle$$

and

$$M = \frac{\vec{M}_0}{Z_0}\cdot\sum_q\sum_{q'}\vec{q}'\, C_q\langle n_q n_{q'}\rangle .$$

We determine C_q to maximize P, with the constraint $P + M = 0$ from (14). The result is

$$C_q = \frac{\vec{M}_0}{2\Gamma}\cdot\sum_{q'}\vec{q}'\frac{\langle n_q n_{q'}\rangle}{\langle n_q\rangle}$$

and, consequently,

$$(\tau, \mathcal{L}\tau) = -\frac{1}{2\Gamma Z_0}\sum_q\frac{(\vec{M}_0\cdot\sum_{q'}\langle n_q n_{q'}\rangle\vec{q}')^2}{\langle n_q\rangle} .$$

Up to this point our formalism has been quite general with regard to dimensionality. For the one-dimensional case which is of interest here, we can write the last equation in the form

$$-(\tau, \mathcal{L}\tau) = +\frac{M_0^2}{2\Gamma Z_0}\sum_q\frac{(\sum_{q'} q'\langle n_q n_{q'}\rangle)^2}{\langle n_q\rangle}$$

$$\leq \frac{\beta\Omega}{2Z_0}E\langle j_s\rangle .$$

Thus

$$\sigma_s \geq \frac{\beta e^2\hbar^2}{\Gamma m^2\Omega}\sum_q\frac{(\sum_{q'} q'\langle n_q n_{q'}\rangle)^2}{\langle n_q\rangle} . \tag{16}$$

We are able to evaluate the quantities Z_0, $\langle n_q\rangle$, and $\langle n_q n_{q'}\rangle$ exactly only in one dimension. To do this we introduce Green's-function notation, defining

$$G(\psi_1, \psi_0; L) \equiv \int_{\psi_0(0)}^{\psi_1(L)}\delta\psi\, e^{-\beta F\{\psi\}} ,$$

where the functional integration is performed over all complex functions $\psi(x)$ such that $\psi(0) = \psi_0$ and $\psi(L) = \psi_1$ and for one dimension,

$$F\{\psi\} = d^2\int_0^L dx\left(\frac{\hbar^2}{2m}\left|\frac{d\psi}{dx}\right|^2 + a|\psi|^2 + \frac{b}{2}|\psi|^4\right) .$$

Here d^2 is the cross-sectional area of the sample. It has been shown elsewhere[11] that G will satisfy the differential equation

$$\frac{\partial G}{\partial L} = -\beta d^2(a|\psi_1|^2 + \tfrac{1}{2}b|\psi_1|^4)G$$

$$+\frac{m}{2\beta d^2\hbar^2}\left(\frac{\partial^2 G}{\partial u_1^2} + \frac{\partial^2 G}{\partial v_1^2}\right),$$

where $\psi_1 = u_1 + iv_1$. The Green's function can therefore be constructed as a sum of eigenfunctions $\chi_\lambda(\psi)$:

$$G(\psi_1, \psi_0; L) = \textstyle\sum_\lambda \chi_\lambda(\psi_1)\chi_\lambda^*(\psi_0)e^{-\lambda L} . \tag{17}$$

χ_λ will satisfy the equation

$$-\lambda\chi_\lambda(\psi) = -d^2\beta(a|\psi|^2 + \tfrac{1}{2}b|\psi|^4)\chi_\lambda$$

$$+\frac{m}{2\beta d^2\hbar^2}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)\chi_\lambda . \tag{18}$$

The normalization of χ is chosen as

$$\int\int du\, dv\, |\chi_\lambda|^2 = 1$$

so that

$$\lim_{L\to 0} G(\psi_1, \psi_0; L) = \delta(\psi_1 - \psi_0)\,.$$

We can evaluate functional integrals by expressing them in terms of Green's functions and then employing (17) and (18). For example,

$$Z_0 = \int \delta\psi\, e^{-\beta F} = \int d^2\psi_0\, G(\psi_0, \psi_0; L)$$

$$= \sum_\lambda \int d^2\psi_0 |\chi_\lambda(\psi_0)|^2 e^{-\lambda L} = \sum_\lambda e^{-\lambda L}\,.$$

In the limit of large L, the lowest eigenvalue will dominate, leaving $Z_0 \simeq e^{-\lambda_0 L}$.

Next we can evaluate

$$\langle n_q\rangle = \frac{1}{Z_0}\int \delta\psi\, e^{-\beta F}|\psi_q|^2$$

$$= \frac{d^2}{Z_0 L}\int \delta\psi\, e^{-\beta F}\int_0^L dx\,\psi(x)\, e^{-iqx}\int_0^L dy\,\psi^*(y)\, e^{+iqy}$$

$$= \frac{d^2}{Z_0}\int d^2\psi_1 \int d^2\psi_0 \int_0^L dx\,\psi_1(x)\, e^{-iqx}\,\psi_0^*(0)$$

$$\times G(\psi_1, \psi_0; x)\, G(\psi_0, \psi_1; L - x)$$

$$= \frac{d^2}{Z_0}\sum_{\lambda,\lambda'}\left(\int d^2\psi_1 \chi_\lambda(\psi_1)\,\psi_1 \chi_{\lambda'}^*(\psi_1)\right)$$

$$\times\left(\int d^2\psi_0 \chi_{\lambda'}(\psi_0)\,\psi_0^* \chi_\lambda^*(\psi_0)\right)$$

$$\times\int_0^L dx\, e^{-iqx}\, e^{-\lambda x}\, e^{-\lambda'(L-x)}\,.$$

Defining $M_{\lambda\lambda'} = \int d^2\psi \chi_\lambda(\psi)\,\psi\chi_{\lambda'}^*(\psi)$, we have

$$\langle n_q\rangle = \frac{d^2}{Z_0}\sum_{\lambda,\lambda'}|M_{\lambda\lambda'}|^2\left(\frac{e^{-\lambda L} - e^{-\lambda' L}}{-iq + \lambda' - \lambda}\right). \tag{19}$$

Taking the limit of large L, we obtain a sum of terms with factors $e^{-(\lambda-\lambda_0)L}$. These terms will become vanishingly small unless $\lambda = \lambda_0$. Defining $\epsilon = \lambda - \lambda_0$, we can reduce (19) to

$$\lim_{L\to\infty}\langle n_q\rangle = d^2\sum_\lambda |M_{\lambda\lambda_0}|^2\,\frac{2\epsilon}{\epsilon^2 + q^2}\,. \tag{20}$$

The product $\langle n_q n_{q'}\rangle$ can be evaluated by a longer but analogous procedure. The results simplify, however, in the limit of large L; the product averages are uncorrelated:

$$\langle n_q n_{q'}\rangle = \begin{cases}\langle n_q\rangle\langle n_{q'}\rangle & q \neq q' \\ 2\langle n_q\rangle^2 & q' = q\,.\end{cases}$$

Consequently, the summation over q' in (16) reduces to

$$\sum_{q'} q'\langle n_q n_{q'}\rangle = q\langle n_q\rangle^2$$

and (16) becomes

$$\sigma_s \geq \frac{\beta e^2\hbar^2}{\Gamma m^2\Omega}\sum_q q^2\langle n_q\rangle^3\,. \tag{21}$$

The fact that products like $\langle n_q n_{q'}\rangle$ are uncorrelated makes (21) a much more general result than the simplicity of our chosen trial function (15) might imply. It turns out that trial functions containing higher powers of $|\psi_q|^2$ and products like $|\psi_q|^2|\psi_{q'}|^2$ reduce to (15) and reproduce (21). We have not, however, considered functions with terms like $\psi_{q_1}\psi_{q_2}\psi_{q_3}^*\psi_{q_4}^*$, which will have a nonvanishing average if $q_1 + q_2 = q_3 + q_4$.

To evaluate (20) for $\langle n_q\rangle$, we must solve (18) for χ and λ and determine the appropriate matrix elements. This equation is the Schrödinger equation for a particle in a circularly symmetric two-dimensional anharmonic potential well. The eigensolutions must be determined numerically. We can simplify (18) by introducing scaled variables R and θ and scaled eigenfunction η:

$$Re^{i\theta} = \left(\frac{\beta^2 d^4\hbar^2 b}{m}\right)^{1/6}\psi\,,$$

$$\eta_\kappa = \frac{1}{2}\left(\frac{m^2 b}{\beta d^2\hbar^4}\right)^{1/3}\chi_\lambda\,,$$

so that the eigenvalue equation becomes

$$\kappa\eta_\kappa = (AR^2 + R^4)\,\eta_\kappa$$

$$-\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial}{\partial\theta^2}\right)\eta_\kappa\,, \tag{22}$$

where

$$\kappa \equiv 2\left(\frac{\beta\hbar^4 d^2}{m^2 b}\right)^{1/3}\lambda \quad\text{and}\quad A \equiv 2\left(\frac{\hbar^2}{\beta d^2 m b^2}\right)^{1/3} a\,.$$

The angular dependence of η in (22) will be simply $e^{il\theta}$ (l = integer), leaving a radial equation to solve for η and κ:

$$\kappa\eta_{\kappa,l} = (AR^2 + R^4)\,\eta_{\kappa,l}$$

$$-\left(\frac{d^2}{dR^2} + \frac{1}{R}\frac{d}{dR}\right)\eta_{\kappa,l} + \frac{l^2}{R^2}\,\eta_{\kappa,l}\,. \tag{23}$$

Two independent numerical methods were used to solve this equation with the aid of a computer. In the first, we set initial conditions for η at the origin and at large R, based on the required asymptotic behavior. We then stepped R in small increments from both ends, using (23) to evaluate the change in η by a Taylor series expansion. At some central value of R, the logarithmic derivatives of η from above and below were compared. The eigenvalue κ was varied until these derivatives matched. Once the correct eigenvalues were determined, the matrix elements $M_{\lambda\lambda_0}$ which appear in (20) were computed by numerical integration. In the second approach, η was approximated as a polynomial function. Equation (23) provides a recursion relationship from which the coefficients of successive powers could be constructed. κ was

then varied until η became small at a selected value of R outside the potential well. The matrix elements were calculated from these η's as definite integrals of products of polynomials.

The solutions of (23) will obviously depend upon A, our temperature-dependent parameter. We have solved the equation for values of A throughout the transition region using (20) and (21) to evaluate σ_s. For temperatures well above T_c, A is large and the quartic term in (23) has a negligible effect on κ and η. In this region the eigenvalue equation reduces to an exactly soluble harmonic-oscillator problem. We then obtain

$$|M_{\lambda\lambda_0}|^2 = \left(\frac{m}{\beta d^4\hbar^2 b}\right)^{1/3} A^{-1/2}\,\delta_{\lambda\lambda_1},$$

$$\lambda_n = \frac{1}{2}\left(\frac{m^2 b}{\beta\hbar^4 d^2}\right)\kappa_n,$$

and (20) reduces to

$$\langle n_q\rangle = d^2\left(\frac{m}{\beta^2 d^4\hbar^2 b}\right)^{1/3} A^{-1/2}\frac{2\epsilon_1}{\epsilon_1^2+q^2},$$

recalling that $\epsilon = \lambda - \lambda_0$. The conductivity from (21) will be

$$\sigma_s \geq \frac{\beta d^8 e^2\hbar^2}{\Gamma m^2\Omega}\,\frac{8m}{\beta^2 d^4\hbar^2 b}\,A^{-3/2}\sum_q \frac{\epsilon_1^3 q^2}{(\epsilon_1^2+q^2)^3}.$$

In the limit of large L, q becomes a continuous variable and the summation becomes an integral:

$$\sum_q \frac{q^2\epsilon_1^3}{(\epsilon_1^2+q^2)^3} \to \frac{L}{2\pi}\int_{-\infty}^{\infty}\frac{q^2 dq\,\epsilon_1^3}{(\epsilon_1^2+q^2)^3} = \frac{L}{16}.$$

Thus

$$\sigma_s \simeq \frac{e^2}{2\beta\Gamma m b}A^{-3/2} \quad (A \gg 1), \tag{24}$$

which corresponds to the Azlamazov-Larkin result.

The numerical results which we obtain for the resistivity for general A are plotted in Fig. 1. In most cases, only a single matrix element contributed to the answer; below T_c, a second term provided small corrections. Summations over q were in each case evaluated in the continuum limit. Certain eigenvalues κ of (23) are graphed in Fig. 2 as a function of A. These eigenvalues are associated with the eigenfunctions producing nonvanishing $M_{\lambda\lambda_0}$.

IV. CONCLUSIONS

Our variational calculation provides an upper bound for the resistivity throughout the critical region. Comparison with the results of TH and MMP is given graphically in Fig. 1 for dirty and clean samples. Our calculations agree closely with TH in both cases down to T_c. Below T_c, the TH results exceed our upper bound; Tucker and Halperin had estimated that their results were unreliable in this region. The results of TH and MMP differ noticeably above T_c and our close agreement with TH is further support for their results. At temperatures well below T_c, in the

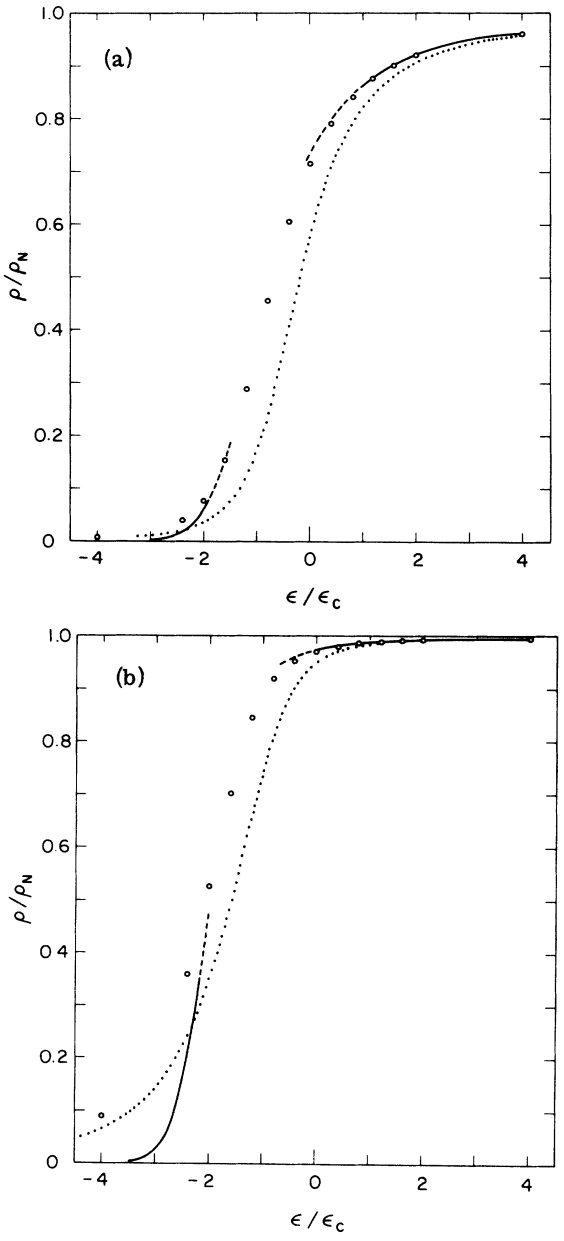

FIG. 1. Resistivity $\rho = (1/\rho_s + 1/\rho_n)^{-1}$ is plotted vs (ϵ/ϵ_c). ϵ_c is the standard scaling factor for the temperature variable $\epsilon = (T - T_c)/T_c$. (ϵ/ϵ_c) is related to the variable A used in this paper by $A = 4^{2/3}(\epsilon/\epsilon_c)$. The results of TH, MMP, and MH are based on the zero-current limits given in TH (Appendix C). TH and MH are shown as solid curves, MMH is the dotted curve, and the circles represent our results. (a) ρ vs ϵ/ϵ_c in the dirty limit, $l \ll \epsilon_0$. (b) $l = 10\xi_0$ for the clean limit. The scaling factor ϵ_c is mean free path dependent, causing the difference between the graphs.

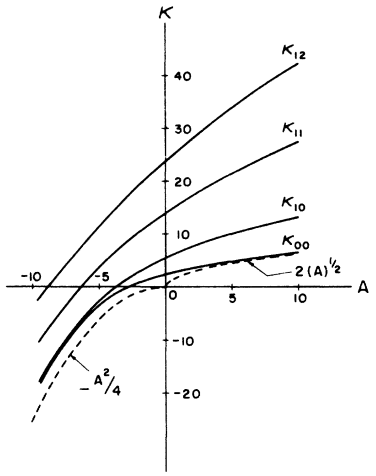

FIG. 2. Eigenvalues κ of Eq. (23) are graphed vs A in the transition region. The ground-state eigenvalue κ_{00} and the three lowest eigenvalues for $l=1$ are given. (Since $M_{\lambda\lambda_0}$ vanishes unless $l=1$, only these values of k were calculated.) Two additional curves are sketched to indicate asympototic behavior. For A large and positive, the quartic term has negligible effect. In this region eigenvalues should be approximately proportional to $2\sqrt{A}$. For A large and negative, the depth of the potential well, $A^2/4$, sets a scale for the eigenvalues. The proximity of κ_{00} and κ_{01} below $A=-4$ illustrates the sma small effect of the angular-momentum term there.

region where LA-MH results are considered reliable, our upper bound lies considerably above their curve, indicating the inadequacy of our solution here. In the region just below T_c, however, our results appear to provide a smooth link between LA-MH below and TH above, completing the description of the resistivity throughout the transition region.

The accuracy of our results is difficult to determine directly. The principal limitation of our method is the choice of trial function. As the conductivity increases at lower temperatures, our simple choice for τ is less likely to provide an adequate description of the system's behavior. Correction terms for our results are difficult to obtain, because they require a more complicated trial function. In particular, it seems to be extremely difficult to construct a trial function which corresponds to the physical situation at $T < T_c$ as described by the LA-MH theories. The accuracy of the numerical calculations is easier to evaluate. The principal limitation of both methods is their iterative nature; errors are compounded as successive terms are calculated. Both approaches work best above T_c, where the eigenfunction changes less abruptly and dies off at smaller R. The errors of these two methods should be independent of one another. The numbers computed agree to a high degree of accuracy. The limitations of the theory are essentially unrelated to this aspect of the problem.

In general, we expect our results to remain valid to lower temperatures than TH or MMP, since the zero-field fluctuation density is included exactly here, and only the response to applied field is approximated. The graphs suggest that this is the case.

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