# Lattice Thermal Conductivity and High-Field Electrical and Thermal Magnetoconductivities of Tungsten†

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Measurements of the high-field transverse electrical and thermal conductivities $\sigma_{xx}$ and $\kappa_{xx}$ of a high-purity tungsten single crystal are presented for the temperature range 1.5–6 K. The magnetic field dependences of the conductivities conform excellently to the predictions of high-field semiclassical magnetoresistance theory, provided that thermal conduction by the lattice is taken into account. The results show that the lattice thermal conductivity is proportional to $T^2$, as expected for a pure metal in which the phonons are scattered principally by the conduction electrons. The temperature dependence of the high-field electrical conductivity $\sigma_{xx}$, and the corresponding electronic contribution $(\kappa_e)_{xx}$ to the thermal conductivity $\kappa_{xx}$ are also measured. Theoretical expressions for these quantities are derived from semiclassical magnetoresistance theory, allowing estimates to be made of the temperature dependence associated with possible low-temperature scattering mechanisms. Difficulties in interpreting previous zero-field measurements in terms of electron-electron or electron-phonon scattering are discussed.

## I. INTRODUCTION

In a previous paper we reported measurements of the low-temperature electrical and thermal resistivities of tungsten in the absence of a magnetic field.[1] Like many other transition metals, $\rho$ and $WT$ ($\rho$ and $W$ are the electrical and thermal resistivities, respectively) exhibit a predominantly $T^2$ dependence at low temperatures, suggesting that the dominant low-temperature scattering mechanism is electron-electron scattering. However, only a small amount of independent evidence exists to support this interpretation.[2] Therefore, we felt that it would be useful to investigate the temperature dependence of the electrical and thermal conductivities of tungsten in a strong magnetic field to provide further information about the low-temperature scattering mechanisms.

In this paper we present measurements of the transverse electrical and thermal conductivities $\sigma_{xx}$ and $\kappa_{xx}$ of a high-purity tungsten crystal oriented with the magnetic field along the [001] direction. The measurements span a range of temperature from 1.5 to 6 K and a range of magnetic field strength from 2.7 to 18.6 kG. Extensive use of high-field semiclassical magnetoresistance theory[3–5] is made to separate the lattice and electronic components $\kappa_g$ and $(\kappa_e)_{xx}$ from the total thermal conductivity $\kappa_{xx}$, and to interpret the temperature dependences of the electronic conductivities $\sigma_{xx}$ and $(\kappa_e)_{xx}$ at high fields.

To provide the framework for analyzing these measurements the relevant aspects of the high-field theory are presented in Sec. II. In Sec. II A we summarize the magnetic field dependence of the conductivities predicted by the high-field theory. As shown by Lifshitz, Azbel, and Kaganov (LAK),[3,4] these results reflect the topology of the Fermi surface and do not depend upon the nature of the scattering mechanism. In Secs. II B and II C we review and extend the LAK treatment to investigate the temperature dependence of $\sigma_{xx}$ and $(\kappa_e)_{xx}$, which, unlike the magnetic field dependence, is determined by the scattering mechanisms. In particular, it is shown that at high fields, with the magnetic field directed along a high-symmetry axis, each scattering mechanism contributes in a strictly additive fashion to $\sigma_{xx}$ and $(\kappa_e)_{xx}$. In this respect, high-field measurements of $\sigma_{xx}$ and $(\kappa_e)_{xx}$ should be less ambiguous than zero-field measurements of $\rho$ and $W$, for which the additivity of the contributions of different scattering mechanisms (Matthiessen's rule) is only approximate. In addition, simple expressions for the contributions of various scattering mechanisms to $\sigma_{xx}$ and $(\kappa_e)_{xx}$ are derived, allowing estimates of the temperature dependence of each contribution to be made. In Sec. III, the experimental results are presented, and this is followed by a discussion of these results in Sec. IV. Difficulties in interpreting both the previous zero-field results and the present high-field results in a consistent manner in terms of electron-electron or electron-phonon scattering are discussed. Lastly, a comparison of the results reported in this paper with similar recent measurements in tungsten by Long is made.

## II. THEORY

Section II A consists of a general consideration of the magnetic field dependence of the high-field electrical and thermal conductivity tensors $\overleftrightarrow{\sigma}(H)$ and $\overleftrightarrow{\kappa}_e(H)$. The discussion is specialized to those cases in which the magnetic field $\vec{H}$ is oriented along a high-symmetry direction (threefold, fourfold, or

sixfold symmetry axis), and to metals, such as tungsten, in which the sheets of the Fermi surface are closed. It should be emphasized that these results are applicable to any scattering mechanism and do not depend upon the existence of a relaxation time. In Sec. II B the explicit dependence of the conductivity tensor elements $\sigma_{xx}$ and $(\kappa_e)_{xx}$ upon the nature of the scattering mechanism is considered. Finally, in Sec. II C, the relationship of the high-field conductivities to the zero-field resistivities is explored, and estimates of the temperature dependences of the high-field conductivities are made for several scattering mechanisms.

### A. High-Field Conductivity Tensors $\vec{\vec{\sigma}}(H)$ and $\vec{\vec{\kappa}}_e(H)$

It is simplest to discuss first the field dependence of the electrical conductivity tensor $\vec{\vec{\sigma}}(H)$. With the magnetic field along a high-symmetry direction, the conductivity tensor $\vec{\vec{\sigma}}(H)$ assumes the simple form: $\sigma_{xx}(H)=\sigma_{yy}(H)$, $\sigma_{yx}(H)=-\sigma_{xy}(H)$, and $\sigma_{xz}(H)=\sigma_{zx}(H)=\sigma_{yz}(H)=\sigma_{zy}(H)=0$. These relations, coupled with the Onsager relation $\sigma_{ij}(H)=\sigma_{ji}(-H)$, require that $\sigma_{xx}(H)$ and $\sigma_{xy}(H)$ be even and odd functions of $H$, respectively. According to semiclassical magnetoresistance theory,[3] if the Fermi surface is closed, the tensor elements $\sigma_{xx}$ and $\sigma_{xy}$ have the following asymptotic form at high fields[6]:

$$\sigma_{xx}\sim a_{xx}(T)/H^2, \qquad \sigma_{xy}\sim (n_e-n_h)ec/H+a_{xy}(T)/H^3,$$

where $n_e$ and $n_h$ are the number of electrons and holes, respectively, and $a_{xx}(T)$ and $a_{xy}(T)$ are generally temperature-dependent quantities that depend upon the nature of the scattering processes in the metal. Since tungsten is compensated ($n_e=n_h$), it follows that $\sigma_{xy}\sim a_{xy}(T)/H^3$. At high fields, inversion of the conductivity tensor gives

$$1/\rho_{xx}\sim\sigma_{xx}\sim a_{xx}(T)/H^2\,, \tag{1}$$

where $\vec{\vec{\rho}}(H)$ is the electrical resistivity tensor. Equation (1) is valid at fields strong enough such that $|a_{xx}H/a_{xy}|^2\gg 1$, or in terms of directly measurable quantities $|\rho_{xx}|^2\gg|\rho_{xy}|^2$.

The thermal conductivity tensor $\kappa_e(H)$ can be treated in a similar manner.[4,7] At high fields the tensor elements $(\kappa_e)_{xx}$ and $(\kappa_e)_{xy}$ have the following asymptotic form:

$$(\kappa_e)_{xx}\sim A_{xx}(T)/H^2\,,$$

$$(\kappa_e)_{xy}\sim L_0T(n_e-n_h)ec/H+A_{xy}(T)/H^3\,,$$

where $L_0=2.44\times10^{-8}\ \mathrm{W\,\Omega/K^2}$ is the Lorenz number and $A_{xx}(T)$ and $A_{xy}(T)$ are generally temperature-dependent quantities that depend upon the nature of the scattering processes in the metal. These quantities and their electrical counterparts obey the Wiedemann-Franz law if the scattering is elastic; that is, $A_{xx}/Ta_{xx}=L_0$ and $A_{xy}/Ta_{xy}=L_0$. If the scattering is not elastic, these ratios will generally be larger than the Lorenz number $L_0$.

At this point only thermal conduction by the electrons has been considered. In fact, some heat is conducted by the phonons (lattice) in addition to that conducted by the electrons. Assuming that the two conduction mechanisms are independent, the conductivity tensors for electron and phonon conduction simply add to give the total conductivity tensor $\vec{\vec{\kappa}}(H)$. Thus, at high fields in a compensated metal, we have $\kappa_{xx}\sim\kappa_g(T)+A_{xx}(T)/H^2$ and $\kappa_{xy}\sim A_{xy}(T)/H^3$, where $\kappa_g$ is the thermal conductivity of the lattice. Letting $\vec{\vec{W}}(H)$ be the thermal resistivity tensor,[8] inversion of the thermal conductivity tensor at high fields gives

$$\frac{1}{W_{xx}T}\sim\frac{\kappa_{xx}}{T}\sim\frac{\kappa_g}{T}+\frac{A_{xx}(T)/T}{H^2}\,. \tag{2}$$

Expressions for $a_{xx}(T)$ in Eq. (1) and $A_{xx}(T)/T$ in Eq. (2) are derived in Sec. II B, showing the explicit dependence of these two quantities upon the scattering mechanism.

### B. High-Field Expressions for $\sigma_{xx}$ and $(\kappa_e)_{xx}$

In a metal with several valence electrons, portions of the Fermi surface will generally be distributed among one or more bands and will therefore occupy one or more Brillouin zones. The portion of the Fermi surface belonging to a single band may consist of one or more surfaces or sheets, which we assume to be closed. In the presence of a magnetic field $\vec{H}$ along the $z$ axis, an electron on one of these sheets will move on the orbit formed by the intersection of this sheet with a plane perpendicular to the magnetic field. Following the semiclassical theory of LAK, the motion of the electron in momentum space can be described in terms of its component of wave vector $k_z$ along the magnetic field, its energy $\epsilon$, and a coordinate $\phi$ that measures its position on the orbit. For a given orbit (specified by $k_z$ and $\epsilon$) $\phi$ is defined by

$$\phi=-\omega_c(\hbar c/eH)\int dk_{\parallel}/v_{\perp}\,, \tag{3}$$

calculated from an arbitrary point on the orbit. In this expression, $dk_{\parallel}$ is an element of arc along the orbit taken in the direction of motion and $v_{\perp}$ is the component of velocity perpendicular to the magnetic field. ($v_{\perp}$ is taken to be positive if directed outward from the orbit.) The cyclotron frequency $\omega_c$ is defined such that one full orbit corresponds to a change of $\phi$ by $2\pi$. In general, $\omega_c$ will have a different value for each orbit and therefore will be a function of $k_z$ and $\epsilon$.

In the semiclassical theory, the electric and magnetic fields are assumed not to cause interband transitions,[9] but interband scattering is allowed. However, because the inclusion of interband scattering into the theory only complicates the formalism without altering the principal results, we shall

consider initially only intraband scattering in this section. Thus, each band can be treated independently. For simplicity we assume that the band under consideration contains only a single sheet of the Fermi surface. Finally, at the end of this section we discuss the results that are obtained when these restrictions are relaxed.

In the presence of an electric field $\vec{E}$ along the $x$ axis, in addition to the magnetic field $\vec{H}$ along the $z$ axis, the steady-state distribution function $f$ will deviate by an amount $eE_x\eta_E$ from its value $f_0$ at equilibrium. In terms of $\eta_E$ and the orbit variables $k_z$, $\epsilon$, and $\phi$, the linearized Boltzmann equation for electrical transport is

$$\omega_c \frac{\partial \eta_E}{\partial \phi} = \left(-\frac{\partial f_0}{\partial \epsilon}\right) v_x + W(\eta_E) , \tag{4}$$

where $W(\eta)$ is the collision integral. For the scattering of an electron from the state $\vec{k}$ to the state $\vec{k}'$ by an impurity or a phonon, $W(\eta)$ is given by[10]

$$W(\eta) = -\frac{1}{k_B T}\frac{1}{4\pi^3}\int d\vec{k}' \, (\psi - \psi') P(\vec{k}, \vec{k}'), \tag{5}$$

where

$$\eta = \left(-\frac{\partial f_0}{\partial \epsilon}\right)\psi \tag{6}$$

and the integration extends over the Brillouin zone. $P(\vec{k}, \vec{k}')$ is related to the scattering rate $Q(\vec{k}, \vec{k}')$ through

$$P(\vec{k}, \vec{k}') = f_0(\epsilon)[1 - f_0(\epsilon')]\, Q(\vec{k}, \vec{k}') = P(\vec{k}', \vec{k}).$$

For the scattering of an electron from the state $\vec{k}_1$ to the state $\vec{k}_1'$ by an electron which is scattered from the state $\vec{k}_2$ to the state $\vec{k}_2'$, $W(\eta_1)$ is given by

$$W(\eta_1) = -\frac{1}{k_B T}\left(\frac{1}{4\pi^3}\right)^3 \int d\vec{k}_1' \int d\vec{k}_2' \int d\vec{k}_2 \times (\psi_1 + \psi_2 - \psi_1' - \psi_2')\, P(\vec{k}_1, \vec{k}_2;\ \vec{k}_1', \vec{k}_2') . \tag{7}$$

In this case $P(\vec{k}_1, \vec{k}_2;\ \vec{k}_1', \vec{k}_2')$ is related to the scattering rate $Q(\vec{k}_1, \vec{k}_2;\ \vec{k}_1', \vec{k}_2')$ through

$$P(\vec{k}_1, \vec{k}_2; \vec{k}_1', \vec{k}_2') = f_0(\epsilon_1) f_0(\epsilon_2)[1 - f_0(\epsilon_1')][1 - f_0(\epsilon_2')] \times Q(\vec{k}_1, \vec{k}_2;\ \vec{k}_1', \vec{k}_2') = P(\vec{k}_1, \vec{k}_2; \vec{k}_1', \vec{k}_2').$$

Following the treatment of LAK, at high fields the solution $\eta_E$ of Eq. (4) can be written as a series in powers of $1/H$:

$$\eta_E = \eta_E^{(0)} + (1/H)\,\eta_E^{(1)} + (1/H^2)\,\eta_E^{(2)} + \cdots . \tag{8}$$

Inserting this series in Eq. (4), assuming that the scattering rate $Q$ does not depend upon $H$, and equating coefficients of the same power of $1/H$, yield

$$\frac{\partial \eta_E^{(0)}}{\partial \phi} = 0 , \tag{9}$$

$$\left(\frac{\omega_c}{H}\right)\frac{\partial \eta_E^{(1)}}{\partial \phi} = \left(-\frac{\partial f_0}{\partial \epsilon}\right) v_x + W(\eta_E^{(0)}) , \tag{10}$$

$$\left(\frac{\omega_c}{H}\right)\frac{\partial \eta_E^{(2)}}{\partial \phi} = W(\eta_E^{(1)}), \text{ etc.} \tag{11}$$

Equation (9) has the solution $\eta_E^{(0)} = C_E^{(0)}(k_z, \epsilon)$. Each coefficient must be a single-valued function of $\phi$. Thus $\eta^{(1)}$ is single valued, provided that

$$\oint W(C_E^{(0)})\, d\phi = 0 , \tag{12}$$

where use has been made of the fact that $\oint v_x d\phi = 0$ for closed orbits. For either choice of the collision integral given by Eq. (5) or (7), the solution to Eq. (12) is

$$C_E^{(0)} = \left(-\frac{\partial f_0}{\partial \epsilon}\right) \times \text{const}$$

(see Appendix A). The constant is determined by the condition that the total number of electrons be conserved. This condition is satisfied only if the constant is zero. Thus $C_E^{(0)} = 0$.

The solution of Eq. (10) is

$$\eta_E^{(1)} = \frac{H}{\omega_c}\left(-\frac{\partial f_0}{\partial \epsilon}\right)\left(C_E^{(1)}(k_z, \epsilon) + \int_0^{\phi} v_x d\phi'\right) . \tag{13}$$

The function $C_E^{(1)}$ is determined within a constant by the condition that $\eta_E^{(2)}$ be single valued, that is

$$\oint W(\eta_E^{(1)})\, d\phi = 0 . \tag{14}$$

In general, this equation has no simple solution. Equation (14) and the condition that the total number of electrons be conserved determine $\eta_E^{(1)}$ uniquely. Finally, the solution of Eq. (11) is

$$\eta_E^{(2)} = \frac{H}{\omega_c}\left(C_E^{(2)}(k_z, \epsilon) + \int_0^{\phi} W(\eta_E^{(1)})\, d\phi'\right) . \tag{15}$$

The function $C_E^{(2)}$ can be determined by following the same procedure used to determine $C_E^{(1)}$, but is not needed in this treatment.

Once $\eta_E$ has been determined, the conductivity tensor element $\sigma_{xx}$ can be calculated from

$$\sigma_{xx} = (e^2/4\pi^3) \int d\vec{k}\, v_x \eta_E , \tag{16}$$

where the integration extends over the Brillouin zone. If several bands are present, the results for each band should be summed. Using Eq. (8) for $\eta_E$, Eq. (16) for $\sigma_{xx}$, and the fact that $d\vec{k} = (-eH/\hbar^2 c\omega_c) \times dk_z d\epsilon\, d\phi$, one obtains a series expression for $\sigma_{xx}$ in powers of $1/H$. The term proportional to $1/H$ vanishes by virtue of the fact that $\oint v_x d\phi \int_0^{\phi} v_x d\phi' = 0$. The term proportional to $1/H^2$ does not vanish and is given by

$$\sigma_{xx} \sim -\frac{e^3}{4\pi^3\hbar^2 c}\int \frac{dk_z}{\omega_c}\int d\epsilon \oint d\phi\, v_x$$

$$\times \frac{1}{\omega_c}\int_0^{\phi} W(\eta_E^{(1)})\,d\phi'. \quad (17)$$

The integral over $\phi$ can be simplified by integrating by parts:

$$\oint d\phi\, v_x \frac{1}{\omega_c}\int_0^{\phi} W(\eta_E^{(1)})d\phi'$$
$$=\left(\oint W(\eta_E^{(1)})\,d\phi'\right)\left[\frac{1}{\omega_c}\left(C_E^{(1)}+\oint v_x d\phi\right)\right]$$
$$-\oint d\phi'\, W(\eta_E^{(1)})\,\frac{1}{\omega_c}\left(C_E^{(1)}+\int_0^{\phi'} v_x d\phi\right). \quad (18)$$

The first term on the right-hand side of Eq. (18) vanishes by virtue of Eq. (14). Direct substitution of Eq. (18) into (17) and a change of variables back to Cartesian coordinates yields the simple result

$$a_{xx}=(e^2/4\pi^3)\int d\vec{\mathrm{k}}\,[-W(\eta_E^{(1)})\psi_E^{(1)}], \quad (19)$$

where $a_{xx}$ is the coefficient of the $1/H^2$ term in $\sigma_{xx}$, and $\psi_E^{(1)}$ is related to $\eta_E^{(1)}$ by Eq. (6).

A similar expression for $A_{xx}/T$ can be derived in the same manner. With a temperature gradient $(-\nabla_x T)$ and zero electric field, the linearized Boltzmann equation is

$$\omega_c \frac{\partial \eta_T}{\partial \phi}=\left(-\frac{\partial f_0}{\partial \epsilon}\right)uv_x+W(\eta_T)\,, \quad (20)$$

where $u=(\epsilon-\mu)/k_BT$ and $(-k_B\nabla_x T)\eta_T=f-f_0$. Using the defining relation

$$(\kappa_e)_{xx}=(k_B^2T/4\pi^3)\int d\vec{\mathrm{k}}\,uv_x\eta_T\,, \quad (21)$$

one obtains

$$\frac{A_{xx}}{T}=\frac{k_B^2}{4\pi^3}\int d\vec{\mathrm{k}}\,[-W(\eta_T^{(1)})\psi_T^{(1)}]\,, \quad (22)$$

where $A_{xx}$ is the coefficient of the $1/H^2$ term in $(\kappa_e)_{xx}$, and $\psi_T^{(1)}$ is related to $\eta_T^{(1)}$ by Eq. (6). The quantity $\eta_T^{(1)}$ is given by

$$\eta_T^{(1)}=\frac{H}{\omega_c}\left(-\frac{\partial f_0}{\partial \epsilon}\right)u\left(C_T^{(1)}(k_z,\epsilon)+\int_0^{\phi} v_x d\phi'\right), \quad (23)$$

and the function $C_T^{(1)}$ is determined by the condition

$$\oint W(\eta_T^{(1)})\,d\phi=0\,. \quad (24)$$

To calculate $a_{xx}$ and $A_{xx}/T$ from Eqs. (19) and (22) for a specific scattering mechanism, it is first necessary to determine $\eta_E^{(1)}$ and $\eta_T^{(1)}$. In general, both quantities will depend upon the scattering mechanism through $C_E^{(1)}$ and $C_T^{(1)}$. However, if the magnetic field is oriented along a twofold or higher-symmetry axis, and each orbit possesses this same symmetry with respect to the rest of the zone, then $\eta_E^{(1)}$ and $\eta_T^{(1)}$ are determined uniquely by symmetry and do not depend upon the scattering mechanism, as shown below.

In the remainder of this section, it is assumed that $\vec{\mathrm{H}}$ is directed along a twofold symmetry axis. This will naturally also include the case in which $\vec{\mathrm{H}}$ is directed along a fourfold symmetry axis, corresponding to the experimental situation considered in this paper. The $E$ and $T$ subscripts on $\eta$ can be dropped temporarily, for the arguments given here are identical for electrical and thermal transport. With each orbit possessing twofold symmetry about $\vec{\mathrm{H}}$, it follows that $v_x(\phi+\pi)=-v_x(\phi)$, and it can be shown (Appendix B) that the solution to the Boltzmann equation must also have the same property; in particular, $\eta^{(1)}(\phi+\pi)=-\eta^{(1)}(\phi)$, or equivalently,

$$\oint \eta^{(1)}(\phi)\,d\phi=0\,. \quad (25)$$

This condition determines $\eta^{(1)}$ uniquely. Since

$$W(\eta^{(1)}(\phi+\pi))=-W(\eta^{(1)}(\phi))\,,$$

Eqs. (14) or (24) will be automatically satisfied by the choice of $\eta^{(1)}$ that satisfies Eq. (25).[11] Using Eqs. (13) and (23) for $\eta_E^{(1)}$ and $\eta_T^{(1)}$, and the fact that $v_x d\phi=(-\hbar c\omega_c/eH)\,dk_y$, one can readily show that

$$\eta_T^{(1)}=u\eta_E^{(1)}=\left(-\frac{\partial f_0}{\partial \epsilon}\right)\left(\frac{\hbar c}{e}\right)(-u\tilde{k}_y)\,, \quad (26)$$

where $\tilde{k}_y$ is the $y$ component of wave vector measured from the center of symmetry of the orbit.

If several scattering processes are present simultaneously, the collision integral $W(\eta)$ in the Boltzmann equation should be replaced by the sum of the collision integrals for each scattering mechanism. Since $\eta_E^{(1)}$ and $\eta_T^{(1)}$ given by Eq. (26) do not depend upon the scattering mechanism, it is clear that the high-field conductivities, given by Eqs. (19) and (22), will be the sum of the conductivities for each scattering mechanism, as if each acted separately.

Having established the additivity of the high-field conductivities for different scattering mechanisms, it is possible to consider the contribution of each mechanism separately. Using Eqs. (19) and (22) for $a_{xx}$ and $A_{xx}/T$, Eq. (26) for $\eta_E^{(1)}$ and $\eta_T^{(1)}$, and Eq. (5) for $W(\eta)$, one obtains

$$a_{xx}=\left(\frac{\hbar c}{4\pi^3}\right)^2\frac{1}{2k_BT}\int d\vec{\mathrm{k}}\int d\vec{\mathrm{k}}'\,(\tilde{k}_y-\tilde{k}'_y)^2P(\vec{\mathrm{k}},\vec{\mathrm{k}}') \quad (27)$$

and

$$\frac{A_{xx}}{T}=\left(\frac{k_B}{e}\right)^2\left(\frac{\hbar c}{4\pi^3}\right)^2\frac{1}{2k_BT}\int d\vec{\mathrm{k}}\int d\vec{\mathrm{k}}'$$
$$\times(u\tilde{k}_y-u'\tilde{k}'_y)^2P(\vec{\mathrm{k}},\vec{\mathrm{k}}')\,. \quad (28)$$

If Eq. (7) is used for the collision integral $W(\eta)$, one obtains

$$a_{xx}=\frac{(\hbar c)^2}{(4\pi^3)^4}\frac{1}{4k_BT}\int d\vec{\mathrm{k}}_1\int d\vec{\mathrm{k}}_2\int d\vec{\mathrm{k}}'_1\int d\vec{\mathrm{k}}'_2$$
$$\times(\tilde{k}_{1y}+\tilde{k}_{2y}-\tilde{k}'_{1y}-\tilde{k}'_{2y})^2\,P(\vec{\mathrm{k}}_1,\vec{\mathrm{k}}_2;\,\vec{\mathrm{k}}'_1,\vec{\mathrm{k}}'_2) \quad (29)$$

and

$$\frac{A_{xx}}{T}=\left(\frac{k_B}{e}\right)^2\frac{(\hbar c)^2}{(4\pi^3)^4}\frac{1}{4k_BT}\int d\vec{k}_1\int d\vec{k}_2\int d\vec{k}_1'\int d\vec{k}_2'$$
$$\times(u_1\tilde{k}_{1y}+u_2\tilde{k}_{2y}-u_1'\tilde{k}_{1y}'-u_2'\tilde{k}_{2y}')^2\,P(\vec{k}_1,\vec{k}_2;\ \vec{k}_1',\vec{k}_2')\ . \tag{30}$$

To this point, only intraband scattering has been considered. To illustrate the considerations involved when interband scattering is present, consider two bands, denoted by $\alpha$ and $\beta$, each containing a single closed sheet of the Fermi surface and possessing twofold symmetry about $\vec{H}$. The Boltzmann equations for electrical transport are given by

$$\omega_{c\alpha}\frac{\partial\eta_{E\alpha}}{\partial\phi_\alpha}=\left(-\frac{\partial f_0}{\partial\epsilon_\alpha}\right)v_{x\alpha}+W_\alpha(\eta_{E\alpha};\ \eta_{E\beta}), \tag{31}$$

$$\omega_{c\beta}\frac{\partial\eta_{E\beta}}{\partial\phi_\beta}=\left(-\frac{\partial f_0}{\partial\epsilon_\beta}\right)v_{x\beta}+W_\beta(\eta_{E\beta};\ \eta_{E\alpha}). \tag{32}$$

For the scattering of an electron by an impurity or phonon, $W_\alpha(\eta_\alpha;\ \eta_\beta)$ is given by

$$W_\alpha(\eta_\alpha;\eta_\beta)=-\frac{1}{k_BT}\frac{1}{4\pi^3}\left(\int_\alpha d\vec{k}'\,(\psi_\alpha-\psi_\alpha')\,P_{\alpha\alpha}(\vec{k},\vec{k}')\right.$$
$$\left.+\int_\beta d\vec{k}'\,(\psi_\alpha-\psi_\beta')\,P_{\alpha\beta}(\vec{k},\vec{k}')\right). \tag{33}$$

The first term represents intraband scattering of the type $\vec{k}_\alpha\rightarrow\vec{k}_\alpha'$, and the second term represents interband scattering of the type $\vec{k}_\alpha\rightarrow\vec{k}_\beta'$. The expression for $W_\beta(\eta_\beta;\ \eta_\alpha)$ can be obtained by permuting $\alpha$ and $\beta$ in Eq. (33). An expression analogous to Eq. (33) can be obtained for electron-electron scattering.

The results derived earlier in this section for intraband scattering hold also for combined intra- and interband scattering. In particular, it is shown in Appendix A that $C_\alpha^{(0)}=C_\beta^{(0)}=0$. Also $\eta_{E\alpha}^{(1)}$ and $\eta_{E\beta}^{(1)}$ are determined by Eq. (25) as before, and therefore are given by Eq. (26). Using Eq. (19) to calculate the contribution to $a_{xx}$ from each band, one obtains

$$a_{xx}=\frac{e^2}{4\pi^3}\left(\int_\alpha d\vec{k}\{-W_\alpha(\eta_{E\alpha}^{(1)};\ \eta_{E\beta}^{(1)})\psi_{E\alpha}^{(1)}\}\right.$$
$$\left.+\int_\beta d\vec{k}\{-W_\beta(\eta_{E\beta}^{(1)};\ \eta_{E\alpha}^{(1)})\,\psi_{E\beta}^{(1)}\}\right)\ . \tag{34}$$

In the case of the scattering of an electron by an impurity or phonon, Eq. (34) becomes

$$a_{xx}=\left(\frac{\hbar c}{4\pi^3}\right)^2\frac{1}{2k_BT}\left(\int_\alpha d\vec{k}\int_\alpha d\vec{k}'(\tilde{k}_y-\tilde{k}_y')^2\,P_{\alpha\alpha}(\vec{k},\vec{k}')\right.$$
$$+\int_\beta d\vec{k}\int_\beta d\vec{k}'\,(\tilde{k}_y-\tilde{k}_y')^2\,P_{\beta\beta}(\vec{k},\vec{k}')$$
$$\left.+2\int_\alpha d\vec{k}\int_\beta d\vec{k}'(\tilde{k}_y-\tilde{k}_y')^2\,P_{\alpha\beta}(\vec{k},\vec{k}')\right)\ , \tag{35}$$

where use has been made of the symmetry properties of the $P$'s:

$$P_{\alpha\alpha}(\vec{k},\vec{k}')=P_{\alpha\alpha}(\vec{k}',\vec{k}),$$

$$P_{\beta\beta}(\vec{k},\vec{k}')=P_{\beta\beta}(\vec{k}',\vec{k}),$$

and

$$P_{\alpha\beta}(\vec{k},\vec{k}')=P_{\beta\alpha}(\vec{k}',\vec{k}).$$

An analogous expression for $A_{xx}/T$ can be derived in the same manner. The argument can also be extended to include interband electron-electron scattering, although the expressions obtained are considerably more cumbersome.

### C. High-Field Conductivities: Relationship to Zero-Field Resistivities and Their Temperature Dependence

In the first part of this section, we explore the relationship between the high-field conductivity formulas derived in Sec. II B and the zero-field resistivities given by the Kohler variational principle. We consider initially a system whose Fermi surface is a single closed sheet, belonging to a single band. According to the Kohler variational principle,[12] the zero-field resistivity $\rho$ can be written

$$\rho\le\rho_v\{\Phi\}\equiv\frac{1}{J_x^2}\frac{1}{4\pi^3}\int d\vec{k}\left[-W\left\{\left(-\frac{\partial f_0}{\partial\epsilon}\right)\Phi\right\}\Phi\right], \tag{36}$$

with

$$J_x=\frac{1}{4\pi^3}\int d\vec{k}\,ev_x\left(-\frac{\partial f_0}{\partial\epsilon}\right)\Phi\ .$$

The trial function $\Phi$ is chosen to minimize the right-hand side of Eq. (36), reducing the equation to an equality. In this case $(-\partial f_0/\partial\epsilon)\Phi$ represents (within a constant factor) the deviation of the distribution function $f$ from its equilibrium value $f_0$. The quantity $J_x$ represents the $x$ component of the electrical current density associated with the distribution function $f_0+(-\partial f_0/\partial\epsilon)\Phi$. (The electric field is taken to be along the $x$ axis.) Generally, the proper choice of $\Phi$ will depend upon the scattering mechanism, and the additivity of the resistivities for different scattering mechanisms (Matthiessen's rule) will be only approximate.

We suppose that the system has cubic symmetry and that the $z$ axis is directed along a fourfold symmetry axis. A possible choice for $\Phi$ is $\hbar c\tilde{k}_x$, where $\tilde{k}_x$ is measured from the center of symmetry of the sheet. In almost every case,[13] this will not be the best choice for $\Phi$, and Eq. (36) will be an inequality. However, with this choice of $\Phi$, the denominator of Eq. (36) can be written

$$J_x^2=\left[\frac{1}{4\pi^3}\int\frac{dS}{\hbar v}\,ev_x(\hbar ck_x)\right]^2$$

$$= \left[\frac{ec}{4\pi^3}\ \frac{1}{3}\int dS\,\hat{n}\cdot\vec{k}\right]^2 , \tag{37}$$

where $dS$ is an element of the Fermi surface, and $\hat{n}$ is a unit vector normal to the surface, directed outward from the surface if the sheet is an electron surface, and inward if the sheet is a hole surface. The integral $|\frac{1}{3}\int dS\,\hat{n}\cdot\vec{k}|$ is just the volume enclosed by the Fermi surface, which is $4\pi^3$ times the number of states $n$ enclosed. Consequently, the denominator has the value $(nec)^2$. Since the $z$ axis is assumed to be a fourfold symmetry axis, the trial function $\Phi = \hbar c\tilde{k}_x$ appearing in the numerator of Eq. (36) can be replaced by $-\hbar c\tilde{k}_y$ without changing the value of the numerator. Then it follows from Eqs. (19) and (26) that the numerator is identically equal to $a_{xx}$. Consequently, $\rho \leq \rho_v\{\hbar c\tilde{k}_x\} \equiv a_{xx}/(nec)^2$. However, $a_{xx}/(nec)^2$ is just the high-field magnetoresistivity $\rho_{xx}$ of the material, so that $\rho \leq \rho_v\{\hbar c\tilde{k}_x\} \sim \rho_{xx}$.

This treatment can be easily generalized to systems in which portions of the Fermi surface are distributed among several bands. For example, if there are two bands $\alpha$ and $\beta$, the numerator of Eq. (36) is identical to Eq. (34), if $e\psi^{(1)}_{E\alpha}$ and $e\psi^{(1)}_{E\beta}$ appearing in Eq. (34) are replaced by the trial functions $\Phi_\alpha$ and $\Phi_\beta$, respectively. The quantity $J_x$ in Eq. (36) represents the total $x$ component of the current density, corresponding to the distribution function $f_0 + (-\partial f_0/\partial\epsilon_\alpha)\Phi_\alpha$ for the $\alpha$ band and $f_0 + (-\partial f_0/\partial\epsilon_\beta)\Phi_\beta$ for the $\beta$ band. Following the earlier arguments, one obtains

$$\rho \leq a_{xx}/[(n_e - n_h)ec]^2 \sim \rho_{xx} , \tag{38}$$

and for the thermal case,

$$WT \leq \frac{A_{xx}/T}{[L_0(n_e - n_h)ec]^2} \sim W_{xx}T , \tag{39}$$

where $n_e$ and $n_h$ are the numbers of electrons and holes, respectively.

While these results by themselves may not be very surprising, the derivation is interesting because it indicates that in an uncompensated $(n_e \neq n_h)$ metal, the difference between the high-field magnetoresistivity and the zero-field resistivity is a measure of the error produced when the trial function $\Phi = \hbar c\tilde{k}_x$ ($\Phi = \hbar cu\tilde{k}_x$ in the thermal case) for each sheet is used to calculate the zero-field resistivity by the Kohler variational principle. If the system is compensated $(n_e = n_h)$, the right-hand sides of Eqs. (38) and (39) diverge. This reflects the fact that the trial function $\hbar c\tilde{k}_x$ for each sheet yields a vanishing total current, while the correct trial function for each sheet would give a finite total current.

In view of the similarity of the variational expression for $\rho$ and the expression for $a_{xx}$ [for example, compare Eqs. (36) and (19)], it *may* be reasonable to expect that $\rho$ and $a_{xx}$ (or $WT$ and $A_{xx}/T$ in the thermal case) would have similar temperature dependences. This is especially evident for the simple case in which a relaxation time $\tau$ exists. In this case $W(\eta) = -\eta/\tau$ and both $\rho$ and $a_{xx}$, as well as $WT$ and $A_{xx}/T$, depend upon $\tau$ in the same way: Each is proportional to $1/\tau$. In the remainder of this section we consider the temperature dependences of $a_{xx}$ and $A_{xx}/T$ expected for impurity, electron-electron, and electron-phonon scattering.

Of these scattering mechanisms, impurity scattering is the simplest to discuss. Such scattering is elastic and leads to values of $a_{xx}$ and $A_{xx}/T$ that are temperature independent and related by the Wiedemann-Franz law, i.e., $A_{xx}/Ta_{xx} = L_0$.

The temperature dependence of $a_{xx}$ and $A_{xx}/T$ for electron-electron scattering is determined mainly by the Fermi factors contained in $P(\vec{k}_1, \vec{k}_2; \vec{k}_1', \vec{k}_2')$. This reflects the operation of the Pauli exclusion principle, which confines the scattering to a region within $k_BT$ of the Fermi energy, resulting in a characteristic $T^2$ temperature dependence for the high field conductivities $a_{xx}$ and $A_{xx}/T$. This $T^2$ behavior should be largely independent of the Fermi surface topology,[14] and should occur for interband as well as intraband processes.

The situation for electron-phonon scattering is much less clear. In the simple metals in the absence of a magnetic field, the Bloch theory,[15] which treats only normal processes, predicts a $T^5$ dependence of $\rho$ and a $T^3$ dependence of $WT$ at low temperatures. Umklapp processes,[16] not treated in the Bloch theory, must be dealt with on a metal-to-metal basis, since the contribution of these processes to the resistivities are sensitive to the positions of the Bragg planes relative to the Fermi surface and to the form of the pseudopotential. Theoretical calculations have indicated for some time that umklapp scattering plays a dominant role in determining the electrical resistivity,[17] and there is growing evidence that umklapp processes do, in fact, cause serious deviations from a simple low-temperature $T^5$ behavior in some metals.[18]

While the role of electron-phonon scattering is still not completely understood in the simple metals, it is even less well understood in the transition metals. The problem is difficult for several reasons. First, the Fermi surface of a typical transition metal does not generally even resemble the surface that would be derived from a nearly-free-electron model. Second, the wave functions generally have significant $s$-$d$ character and the pseudopotential cannot yet be calculated reliably, so that the matrix element for electron-phonon scattering is uncertain. Consequently, for lack of a better alternative, experimentalists have had to assume, without much theoretical justification, that the simple $T^5$ and $T^3$ behavior for $\rho$ and $WT$ can be used to

describe electron-phonon scattering at low temperatures in transition metals. One might expect by analogy that the contribution of electron-phonon scattering to $a_{xx}$ and $A_{xx}/T$ would be proportional to $T^5$ and $T^3$, respectively, but, as far as the transition metals are concerned, it is probably wise to apply these results with considerable caution.

### III. EXPERIMENTAL RESULTS

The tungsten sample used in the experiment was spark-cut from a longer 3-mm-diam zone-refined crystal, which had a residual resistivity ratio $\rho(299\,\mathrm{K})/\rho(0\,\mathrm{K})$ of 63 000.[19] The rod axis was parallel to the [110] direction (taken as the $x$ axis), and the magnetic field $\vec{H}$ was oriented normal to the rod axis along the [001] direction ($z$ axis). Measurements of the transverse electrical and thermal magnetoresistivities $\rho_{xx}$ and $W_{xx}$ were made in the conventional potentiometric manner in separate experiments.

The cryostat used was identical to the one described in a previous paper,[1] except that the sample was mounted horizontally rather than vertically, so that a superconducting solenoid could be used to produce a magnetic field transverse to the sample. In the thermal measurements, the temperature difference created between two points on the sample by a heat current was measured by means of two matched Allen Bradley 56-Ω $\frac{1}{10}$-W carbon resistors soldered to copper rings which had been electroplated to the sample. For the electrical measurements, the thermometers were left in place, a current lead was attached to the free end of the sample, and potential leads were attached to the sample at the thermometers.

It is worth mentioning that at high magnetic fields the use of ring contacts rather than point contacts will not seriously perturb the current distribution in the sample, *provided* that the sample is compensated and the magnetic field is oriented along a high-symmetry direction. Although the rings will short-circuit the Hall electric field locally, under these circumstances the Hall field is so small compared to the electric field along the specimen axis that the effect is negligible. On the other hand, if the specimen were uncompensated, just the reverse would be true, and the current distribution would be seriously perturbed.

The thermometers were calibrated during each experiment (in zero magnetic field) against a standard germanium resistance thermometer. Small corrections for the magnetoresistance of the carbon resistors[20] were made, but were so small for the field strengths used that they were hardly necessary. During the thermal measurements, heat currents of 0.1 to 1 mW were used to generate temperature differences of about 200 mK at each field, allowing measurement of $W_{xx}$ to a precision of better than 2%. All thermometer and heater leads to the sample were constantan, and the heat leak through these leads was negligible. It was verified that the heat lost through the vacuum surrounding the sample was also negligible by comparing measurements taken with the sample at different temperatures relative to its environment.

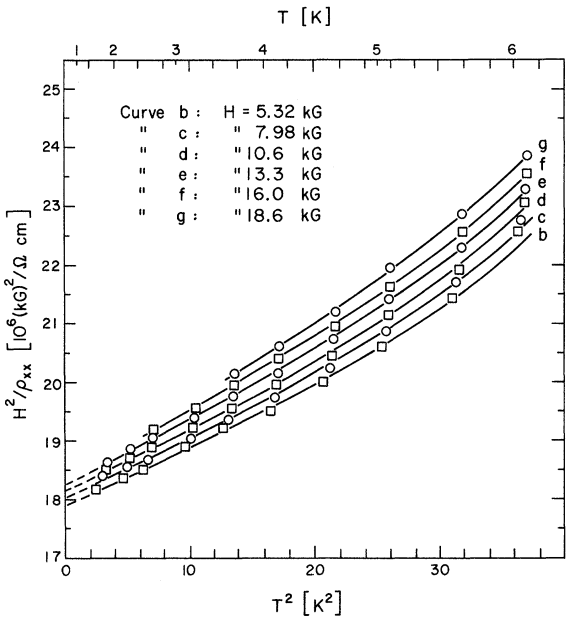


FIG. 1. Variation of $H^2/\rho_{xx}$ with $T^2$ for various values of the magnetic field $H$.

The results of the electrical measurements are summarized in Fig. 1. In this figure, $H^2/\rho_{xx}$ is plotted against $T^2$ for several values of the magnetic field. The purpose of plotting the data in this manner is first to show the field dependence of $\rho_{xx}$, and second, to show the temperature dependence of $H^2/\rho_{xx} \sim a_{xx}$ [Eq. (1)] in the high-field limit. It is clear from the figure that $\rho_{xx}$ is nearly proportional to $H^2$, the exponent being approximately 1.96. Small deviations from the $H^2$ law were observed in earlier work by Fawcett,[21] but were not observed in a more recent study by Long.[22] It is not certain whether the deviations observed in this study are an intrinsic effect, or merely an artifact caused by the finite width of the electroplated copper rings which were used as potential contacts. In any case, the deviations pose no serious problem in the interpretation of the measurement, and will not be pursued further.

The temperature dependence of $H^2/\rho_{xx} \sim a_{xx}$ is nearly quadratic, but increases at a somewhat faster rate at higher temperatures. For this sample, the temperature-dependent part of $a_{xx}$ is roughly 30% of the residual part of 6 K. It is interesting to note that the zero-field resistivity $\rho$ of the same sample measured in an earlier experiment showed

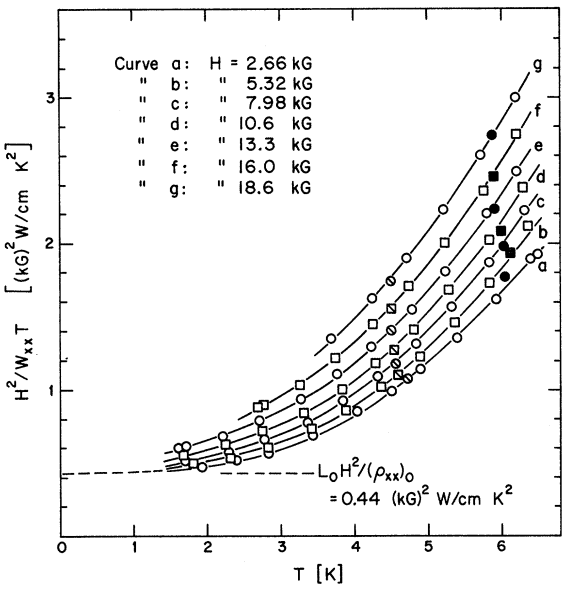


FIG. 2. Variation of $H^2/W_{xx}T$ with temperature for various values of the magnetic field $H$. The slashed data points were taken in a separate experiment with no change in the cryostat. The solid data points were taken after changing the separation of the thermometers on the sample. The dashed line indicates the residual value of $H^2/W_{xx}T$ obtained by the Wiedemann-Franz law.

a similar temperature dependence.[1] Furthermore, the temperature-dependent part of the zero-field resistivity was also about 30% of the residual part at 6 K.

There is little doubt that the high-field regime has been reached in this sample at the magnetic fields used. At 13.3 kG and 4 K the resistance had increased by nearly five orders of magnitude over its value at zero field. Furthermore, a measurement of $\rho_{xy}$ indicated that it was about 300 times smaller than $\rho_{xx}$ at 13.3 kG and 4 K, so that Eq. (1) was valid for all field strengths used in this study.

The results of the thermal measurements are summarized in Fig. 2, in which $H^2/W_{xx}T$ is plotted against temperature for several values of the magnetic field. To compare these results with Eq. (2), these data have been replotted in Figs. 3(a) and 3(b) as $1/W_{xx}T$ vs $1/H^2$ for several temperatures. The agreement with Eq. (2) is remarkable. In particular, the intercepts give the values of $\kappa_g/T$ at each temperature, and these have been plotted against temperature in Fig. 4. As can be seen from this figure, $\kappa_g/T$ has a very plausible linear temperature dependence given by $\kappa_g/T \approx 0.5T$ mW/cm K$^2$. According to Eq. (2), the slope of each line in Fig. 3 is the value of $A_{xx}/T$ for that temperature. These slopes have been plotted against $T^3$ in Fig. 5, demonstrating that the temperature-dependent part of $A_{xx}/T$ has an almost precisely $T^3$ behavior below 6 K. In contrast, the zero-field measurements of $WT$ showed an almost precisely $T^2$ behavior below 6 K.[1]

In the derivation of Eq. (2) we neglected thermoelectric effects which arise due to the fact that the thermal magnetoresistivity $W_{xx}$ was measured under

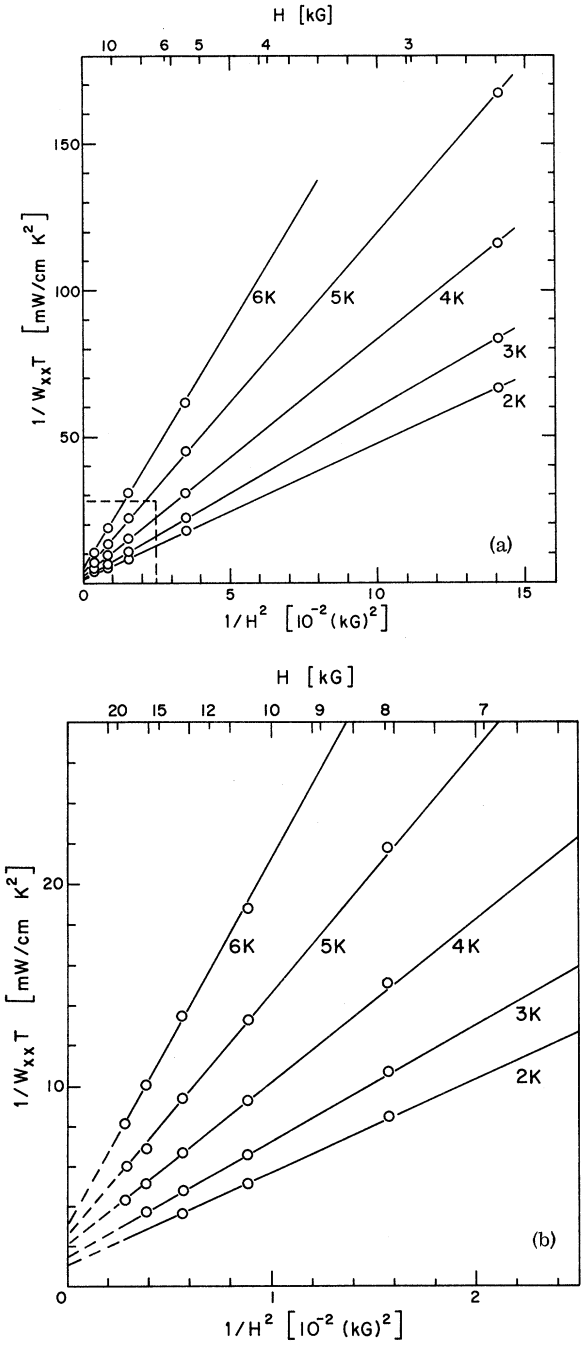


FIG. 3. Plot of $1/W_{xx}T$ as a function of $1/H^2$ for several temperatures. In (a) some of the data within the dashed box has been omitted for clarity. The data falling within the dashed box has been plotted in (b).

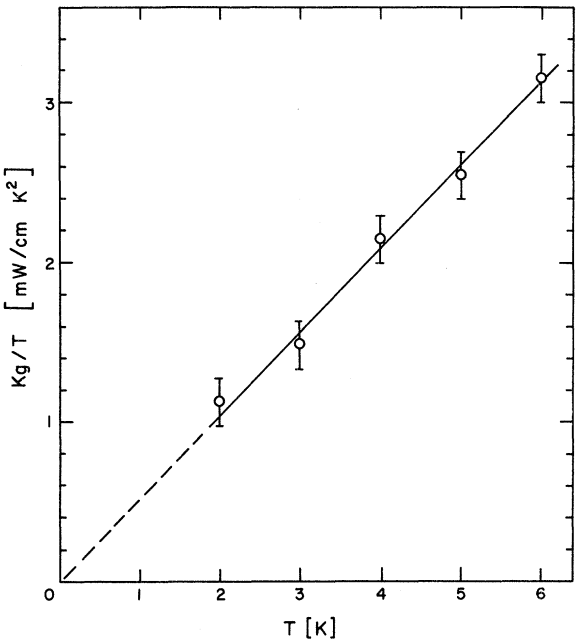


FIG. 4. Lattice thermal conductivity divided by temperature as a function of temperature.

the condition of zero electric current rather than zero electric field in the sample. The dominant correction due to these effects is a term $\rho_{xx}(\epsilon''_{xy})^2$ that should be added to the right-hand side of Eq. (2). The quantities $\rho_{xx}$ and $\epsilon''_{xy}$ are elements of the electrical resistivity and thermoelectric power tensors, respectively.[23] At high fields this term becomes independent of magnetic field and has the value $\rho_{xx}(c\gamma T/H)^2$, where $\gamma$ is the coefficient of the electronic specific heat per cm$^3$ of electrons. For the tungsten sample used in this study, this term has the value 0.1 mW/cm K$^2$ at 4 K.[24] Since this is *at least* 20 times smaller than $1/W_{xx}T$ for all values of the magnetic field at this temperature, the correction is of little consequence and can be ignored.

## IV. DISCUSSION

One of the principal findings of this investigation is that the magnetic field dependences of both $\sigma_{xx}$ and $\kappa_{xx}$ closely follow the predictions of the high-field semiclassical theory, provided that thermal conduction by the lattice is taken into account. The measurements indicate that the lattice conductivity $\kappa_g$ is proportional to $T^2$, as expected for a pure metal in which the phonons are scattered principally by the conduction electrons.[25] Furthermore, the temperature dependence, as well as the magnitude of $\kappa_g$ is in good agreement with measurements made on transition-metal alloys.[26] The first attempts to measure the lattice conductivity of tungsten were made over three decades ago by de Haas and de Nobel and, subsequently, by de Nobel, but were hampered by the lack of specimens of sufficient purity.[27] Very recently, a successful measurement of the lattice conductivity has been made by Long[28] in a tungsten crystal of higher purity than was available to de Haas and de Nobel. The measurements of $\kappa_g$ reported here and reported by Long are in substantial agreement, and, as shown by Long, agree within better than a factor of 2 with reasonable theoretical estimates of $\kappa_g$.

The temperature dependences of the high-field electrical conductivity $\sigma_{xx} \sim a_{xx}/H^2$ and the electronic contribution to the thermal conductivity $(\kappa_e)_{xx}/T \sim (A_{xx}/T)/H^2$ should reflect the scattering mechanisms present in tungsten at low temperatures. It was shown in Sec. IIB that with the magnetic field $\vec{H}$ along a twofold symmetry axis, $a_{xx}$ and $A_{xx}/T$ are composed of the sum of the contributions of the various scattering mechanisms present in the metal, as if each mechanism acted separately. Furthermore, the expressions for $a_{xx}$ and $A_{xx}/T$ obtained in Sec. IIB were, apart from the choice of trial function, similar to the variational expressions for $\rho$ and $WT$, respectively. For this reason it was argued that the temperature dependence of $a_{xx}$ and $\rho$, as well as $A_{xx}/T$ and $WT$, would probably be similar. This appears to be the case for the electrical quantities $a_{xx}$ and $\rho$ but not for their thermal counterparts $A_{xx}/T$ and $WT$. In this study, $a_{xx}$ appears to increase nearly quadratically with temperature at low temperatures, but at a somewhat faster rate at higher temperatures approaching 6 K. In

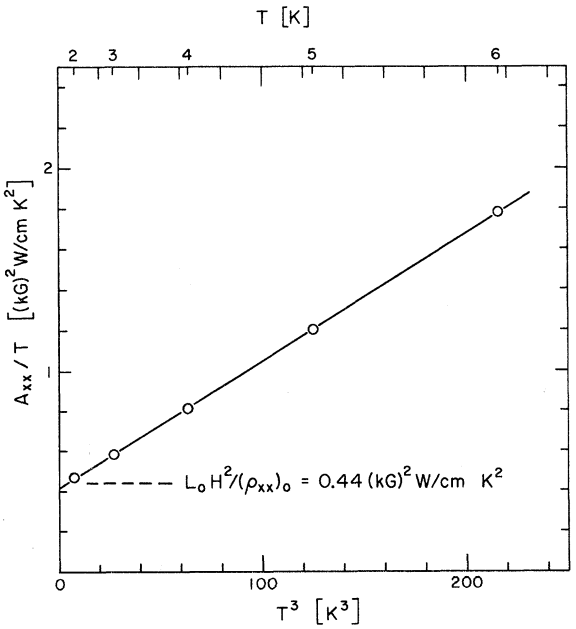


FIG. 5. Plot of $A_{xx}/T$ as a function of $T^3$. The values of $A_{xx}/T$ were obtained from the slopes of the lines of Fig. 4. The precision of each value is approximately 1%.

a previous study at zero magnetic field,[1] $\rho$ was found to behave in a similar fashion. On the other hand, $A_{xx}/T$ showed an almost precisely $T^3$ behavior in this study, while in the previous study $WT$ showed an almost precisely $T^2$ behavior. Although the zero-field results can be plausibly explained in terms of electron-electron scattering, it seems unlikely that this scattering could cause a $T^3$ dependence of $A_{xx}/T$. It may be possible to explain both the high-field and the zero-field results in terms of electron-phonon scattering, but the current understanding of the role of these processes in the transition metals is still too poor to assess this possibility properly.

We have assumed that the effect of boundary scattering on the temperature dependence of $a_{xx}$ and $A_{xx}/T$ can be neglected. Indeed, the previous measurements in zero magnetic field indicated that boundary scattering did not have a significant effect on the temperature dependences of $\rho$ and $WT$ for this sample. Moreover, it seems reasonable that the importance of boundary scattering should decrease as the magnetic field is increased, since the fraction of electrons striking the boundaries is reduced.

There is one assumption made in Sec. IIB that should be examined in the context of the tungsten Fermi surface. The tungsten Fermi surface has portions located in the second, third, and fourth Brillouin zones.[29] The second zone consists of an array of six small ellipsoidal hole pockets located at $N$ (at the centers of the $\langle 110\rangle$ zone faces), and the third zone consists of an octahedral-shaped hole surface located at $H$ (the $\langle 100\rangle$ zone vertex). The fourth zone contains the electron "jack" centered in the zone at $\Gamma$. In Sec. IIB it was assumed that all orbits have at least twofold symmetry about $\vec{H}$. Although $\vec{H}$ was directed along a fourfold symmetry axis in the experiment, there are a number of orbits that do not possess even twofold symmetry about this same axis. First, there are the orbits formed by the intersection of the "knobs" (but not the "necks") of the electron "jack" with a plane perpendicular to the magnetic field. These orbits occur on only a small fraction of the Fermi surface, so their contribution to the conductivities can be neglected without much error.[30] Second, four of the six hole ellipsoids taken individually do not have any special symmetry with respect to the rest of the zone. However, the surfaces are ellipsoidal, so that $v_x(\phi+\pi) = -v_x(\phi)$ for orbits on these surfaces. Although it does not follow that Eq. (25) will hold for an arbitrary scattering mechanism, Eq. (25) does hold in the relaxation-time approximation. Consequently, the assumption that $\eta^{(1)}$ for these surfaces is given by Eq. (26) should be a good starting point for calculating the contribution of these sheets to the conductivities.

We now turn to the experimental results reported by Long.[28] In that study the extraction of the lattice thermal conductivity from the measurements was based on the phenomenological expression

$$\kappa_{xx}(H, T) = TL_1(T)\,\sigma_{xx}(H, T) + \kappa_g(T)\ . \tag{40}$$

This equation can be shown to be a simple consequence of the high-field theory. Long's crystal was oriented with the magnetic field along the [001] direction, so that all the arguments given in Sec. II apply. Equation (40) is equivalent to Eq. (2), which can be written as

$$\kappa_{xx}(H, T) = T\left(\frac{A_{xx}(T)}{Ta_{xx}(T)}\right)\sigma_{xx}(H, T) + \kappa_g(T)\ , \tag{41}$$

showing that $L_1(T) = A_{xx}(T)/T\,a_{xx}(T)$. Thus the assumption made by Long that $L_1$ does not depend upon $H$ is well justified at high fields. In Long's sample, the temperature-dependent part of $a_{xx}$ was only a small fraction of the residual part; consequently, the temperature dependence of $L_1$ was determined by that of $A_{xx}/T$. It was found in this study that $A_{xx}/T$ varies almost precisely as $T^3$, while Long has fit his values of $L_1$ to a $T^2$ dependence. However, a $T^3$ dependence is not inconsistent with the data when the scatter is taken into account. Long's measurement of $L_2(T) = A_{xy}(T)/T\,a_{xy}(T)$ can also be understood in terms of semiclassical theory. As remarked in Sec. IIA, $L_2$ will be equal to $L_0$ only if the scattering is elastic and will generally be greater than $L_0$ if the scattering is inelastic. This is borne out by his measurements, which show that $L_2$ increases from a value near $L_0$ at 1.5 K to a value appreciably greater than $L_0$ at 4.2 K.

In summary, it is found that the magnetic field dependence of the high-field conductivities $\sigma_{xx}$ and $\kappa_{xx}$ can be understood excellently in the context of semiclassical magnetoresistance theory, provided that the thermal conductivity of the lattice $\kappa_g$ is taken into account. A lattice thermal conductivity $\kappa_g$ given by $\kappa_g \approx 0.5\ T^2$ mW/cm K is found experimentally, consistent with the view that the phonons are scattered mainly by the conduction electrons. The temperature dependence of the high-field conductivities is shown to yield information about low-temperature scattering mechanisms. However, when both zero-field and high-field measurements are considered together, it is found that they cannot be explained plausibly in terms of electron-electron scattering, as was previously concluded from measurements of the zero-field resistivities.[1] Electron-phonon scattering may be responsible, but the current understanding of such scattering in the transition metals is still too poor to explore this possibility in greater detail.

## ACKNOWLEDGMENTS

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## APPENDIX A

In this appendix we wish to show that

$$C_E^{(0)}(k_z, \epsilon) = \left(-\frac{\partial f_0}{\partial \epsilon}\right) \times \text{const.}$$

Since the distinction between electrical and thermal conduction is unimportant, we drop the $E$ and $T$ subscripts on $\eta$ and $\psi$. First, we prove the result for only intraband scattering. Next, we indicate how the argument is extended to two bands with combined intra- and interband scattering.

(i) Multiplying Eq. (12) by $(-eH/\hbar^2 c\omega_c)\, dk_z d\epsilon\, \psi^{(0)}$, integrating over $k_z$ and $\epsilon$, and noting that $\psi^{(0)}$ is *not* a function of $\phi$ gives

$$-\frac{eH}{\hbar^2 c} \int \frac{dk_z}{\omega_c} \int d\epsilon \oint d\phi\, W(C^{(0)})\, \psi^{(0)} = 0 \,. \tag{A1}$$

However, $d\vec{k} = (-eH/\hbar^2 c\omega_c)\, dk_z d\epsilon\, d\phi$, so that the integration may be changed back to an integration over wave vector. Using Eq. (5) for the collision integral, one obtains

$$\int d\vec{k} \int d\vec{k}'\ (\psi^{(0)} - \psi^{(0)\prime})^2\, P(\vec{k}, \vec{k}') = 0 \,. \tag{A2}$$

Since the integrand is always positive, this condition can be met only if $\psi^{(0)} = \text{const}$ or $C^{(0)}(k_z, \epsilon) = (-\partial f_0/d\epsilon) \times \text{const}$. A similar proof can be constructed for electron-electron scattering by using Eq. (7) for the collision integral.

(ii) Following the discussion in Sec. IIB, we denote the two bands by $\alpha$ and $\beta$. The quantities $C_\alpha^{(0)}$ and $C_\beta^{(0)}$ are determined by the two conditions

$$\begin{aligned} &\oint d\phi_\alpha\, W_\alpha(C_\alpha^{(0)};\ C_\beta^{(0)}) = 0 \,, \\ &\oint d\phi_\beta\, W_\beta(C_\beta^{(0)};\ C_\alpha^{(0)}) = 0 \,. \end{aligned} \tag{A3}$$

Multiplying the first equation by $(-eH/\hbar^2 c\omega_{c\alpha}) \times dk_{z\alpha} d\epsilon_\alpha\, \psi_\alpha^{(0)}$, the second by $(-eH/\hbar^2 c\omega_\beta)\, dk_{z\beta} d\epsilon_\beta \psi_\beta^{(0)}$, adding the resulting equations, and changing the integrations back to integrations over wave vector, one obtains for impurity or electron-phonon scattering

$$\begin{aligned} &\tfrac{1}{2} \int_\alpha d\vec{k} \int_\alpha d\vec{k}'\ (\psi_\alpha^{(0)} - \psi_\alpha^{(0)\prime})^2\, P_{\alpha\alpha}(\vec{k}, \vec{k}') \\ &+ \tfrac{1}{2} \int_\beta d\vec{k} \int_\beta d\vec{k}'\ (\psi_\beta^{(0)} - \psi_\beta^{(0)\prime})^2\, P_{\beta\beta}(\vec{k}, \vec{k}') \\ &+ \int_\alpha d\vec{k} \int_\beta d\vec{k}'\ (\psi_\alpha^{(0)} - \psi_\beta^{(0)\prime})^2\, P_{\alpha\beta}(\vec{k}, \vec{k}') = 0 \,, \end{aligned} \tag{A4}$$

where use has been made of the symmetry properties of the $P$'s. Since each term is positive definite, this condition can be satisfied only if each term vanishes separately. Therefore, $\psi_\alpha^{(0)} = \psi_\beta^{(0)} = \text{const}$. The extension to electron-electron scattering is straightforward, though tedious.

## APPENDIX B

We wish to show that if the magnetic field $\vec{H}$ is directed along a twofold symmetry axis and the orbits have this same symmetry, that the solution $\eta$ to the Boltzmann equation has the property $\eta(\phi + \pi) = -\eta(\phi)$. For simplicity consider electrical transport and intraband scattering given by the collision integral of Eq. (5). Let $\vec{k}_r$ and $\vec{k}_r'$ be the vectors corresponding to a rotation of $\pi$ about $\vec{H}$ of $\vec{k}$ and $\vec{k}'$. At $\vec{k}_r$, corresponding to the point $(\phi + \pi)$ on the orbit, the collision integral $W_r(\eta(\phi + \pi))$ is

$$-\frac{1}{k_B T}\ \frac{1}{4\pi^3} \int d\vec{k}_r' \left[\psi(\phi + \pi) - \psi'(\phi + \pi)\right] P(\vec{k}_r, \vec{k}_r') \,. \tag{B1}$$

However, $P(\vec{k}_r, \vec{k}_r') = P(\vec{k}, \vec{k}')$ by virtue of the twofold rotational symmetry and $d\vec{k}_r' = d\vec{k}'$. Consequently, $W_r(\eta(\phi + \pi)) = W(\eta(\phi + \pi))$. This last result also holds for electron-electron scattering, which can be verified by repeating the same argument using Eq. (7) for the collision integral. Utilizing the fact that $v_x(\phi + \pi) = -v_x(\phi)$, the Boltzmann equation for $\eta(\phi + \pi)$ can be written

$$\omega_c\, \frac{\partial \eta(\phi + \pi)}{\partial \phi} = -\left(-\frac{\partial f_0}{\partial \epsilon}\right) v_x(\phi) + W(\eta(\phi + \pi)) \,. \tag{B2}$$

Adding this equation to Eq. (4) for $\eta(\phi)$, and letting $\eta_+ = \tfrac{1}{2}[\eta(\phi) + \eta(\phi + \pi)]$, one obtains

$$\omega_c\, \frac{\partial \eta_+}{\partial \phi} = W(\eta_+) \,, \tag{B3}$$

subject to the condition that $\eta_+$ be a single-valued function of $\phi$. Note that $\eta_+$ satisfies the same equation, subject to the same boundary condition, that is satisfied by $f - f_0$ *in the absence of an electric field*. Since $f - f_0$ is zero in this situation, and this solution is unique, $\eta_+$ must be zero. Therefore, $\eta(\phi + \pi) = -\eta(\phi)$. This property depends only upon the symmetry properties of the collision integral and is valid for interband as well as intraband scattering. For thermal transport a similar argument can be constructed.

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[6]The symbol ~ is used to mean "equals asymptotically in the high-field limit."

[7]By thermal-conductivity tensor it is meant the tensor that relates the heat current and temperature gradient in the absence of an electric field.

[8]The thermal-resistivity tensor is commonly measured in the absence of an electrical current, rather than in the absence of an electric field in the sample. There are therefore small corrections to Eq. (2) arising from the thermoelectric effects, which we have ignored. In Sec. III this correction is estimated and is shown to be negligible.

[9]An example of the failure of this assumption is magnetic breakdown.

[10]A discussion of the collision integral can be found in J. M. Ziman, *Electrons and Phonons* (Oxford U.P., Oxford, England, 1960), Sec. 7.7.

[11]I am grateful to C. Herring for pointing this out.

[12]For a discussion of the Kohler variational principle for the electrical and thermal resistivities, see F. J. Blatt, *Physics of Electronic Conduction in Solids* (McGraw-Hill, New York, 1968), Sec. 5.5.

[13]The exception is the case in which the sheet is spherical and a relaxation time exists. This situation might be expected to exist in a metal such as potassium when the scattering is dominated by impurity scattering. For the application to potassium and a comparison with experimental results, see H. Taub and D. K. Wagner (unpublished).

[14]For an illuminating discussion of this point, see C. Hodges, H. Smith, and J. W. Wilkins, Phys. Rev. B 4, 302 (1971).

[15]See Ref. 10, Sec. 9.5.

[16]In discussions of the simple metals it is fashionable to use the extended zone scheme, in which the Fermi surface appears as a sphere. In this scheme, the meaning of "umklapp" is different from that in the reduced zone scheme, which is the scheme used in this paper.

[17]See, for example, M. Bailyn, Phys. Rev. 120, 381 (1960); W. E. Lawrence and J. W. Wilkins (unpublished).

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[24]In mksA units this correction term is $\rho_{xx}(\gamma T/B)^2$, where $B$ is the magnetic induction. At 10 kG the measured value of $\rho_{xx}$ is about $5\times10^{-6}$ Ω cm and for tungsten $\gamma$ is about $10^{-4}$ J/cm$^3$ K$^2$. Therefore, at 4 K, $\rho_{xx}(\gamma T/B)^2$ is 0.1 mW/cm K$^2$.

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[29]For a detailed discussion of the tungsten Fermi surface and a listing of references, see R. F. Girvan, A. V. Gold, and R. A. Phillips, J. Phys. Chem. Solids 29, 1485 (1968).

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